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# Passivity-based Control of Implicit Port-Hamiltonian Systems

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*Abstract*— **The main contribution of this paper is the generalisation of well-known energy-based control techniques (i.e., energy-balancing passivity-based control and passivity-based control with state modulated source), to the case in which the plant is a port-Hamiltonian system in implicit form. A typical situation is when (part of) the system is obtained from the spatial discretization of an infinite dimensional port-Hamiltonian system: in this case, the dynamics is not given in standard input-state-output form, but as a set of DAEs. Consequently, the control by energy-shaping has to be extended to deal with dynamical systems with constraints. The general methodology is discussed with the help of a simple but illustrative example, i.e. a transmission line interconnected with an RLC circuit.**

## I. INTRODUCTION

This paper deals with the extension of classical energy based control techniques (energy-balancing passivity-based control and passivity-based control with state-modulated source, [1], [2]) to port-Hamiltonian systems [3] in implicit form, i.e. not written in standard input-state-output form but as a set of DAEs, [4], [5]. The motivating application is when the port-Hamiltonian dynamics follows from the spatial discretization of a distributed port-Hamiltonian system carried out according to the technique proposed in [6].

In a recent work [7], the control by interconnection and energy shaping via Casimir generation [2], [5], [8] has been extended to this scenario, and the stabilization of the system in a non-zero equilibrium is accomplished by looking or generating a set of Casimir functions in the closed-loop system that robustly (i.e. independently from the Hamiltonian function) relates the state of the infinite dimensional port Hamiltonian system with the state of the controller. The shape of the energy function of the closed-loop system can be changed by properly choosing the Hamiltonian function of the controller in order to introduce a (possibly global) minimum in a desired configuration. This approach has shown its potentialities in the stabilisation of distributed port-Hamiltonian systems [9]–[12], and of their finite-element approximations, [7], [13].

In this paper, energy-balancing passivity-based control and passivity-based control via state-modulated source are extended to implicit port-Hamiltonian systems, with the final goal of being applied to a finite element approximation of the distributed parameter plant. In this way, standard tools for studying the stability of finite dimensional port Hamiltonian systems can be used to prove the validity of the boundary controller. Implicit port-Hamiltonian systems have been introduced at the very beginning of the port-Hamiltonian theory, but not so many results have been presented as far as their control is concerned, with the noticeable exceptions of [14], where the energy-shaping control via Casimir generation has been approached by starting from the properties of the Dirac structure of the plant both in the finite and infinite dimensional case, and of [15], [16]. In case of input-state-output port-Hamiltonian systems, a state-modulated source is the simplest way for dealing with the stabilisation of equilibria that require an infinite amount of supplied energy, i.e. with the so-called dissipation obstacle that limits the applicability of the control by interconnection via Casimir generation and of energy-balancing passivity-based control. The general methodology is illustrated with reference to a particular example, i.e. a trasmission line with RLC load, both in the series and parallel configuration.

The paper is organized as follows. In Sect. II, Dirac structures and associated port-Hamiltonian systems are briefly presented. Then, Sect. III contains the main theoretical contributions. More precisely, in Sect. III-A, the energy-balancing passivity-based control, and in Sect. III-B, the control with state-modulated source are extended to implicit port-Hamiltonian systems. The examples is reported in Sect. IV, while conclusions and a discussion about future activities are in Sect. V.

## II. BACKGROUND

### A. Dirac structures

A Dirac structure is a linear space which describes internal power flows and the power exchange between the system and the environment. Denote by $\mathcal{F} \times \mathcal{E}$ the space of power variables, with $\mathcal{F}$ an $n$-dimensional linear space, the space of flows (e.g. velocities and currents) and $\mathcal{E} \equiv \mathcal{F}^*$ its dual, the space of efforts (e.g. forces and voltages), and by $\langle e, f\rangle$ the power associated to the *port* $(f, e) \in \mathcal{F} \times \mathcal{E}$, where $\langle \cdot, \cdot\rangle$ is the dual product between $f$ and $e$.

*Definition 2.1:* Consider the space of power variables $\mathcal{F} \times \mathcal{E}$. A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\dim \mathcal{D} = \dim \mathcal{F}$ and $\langle e, f\rangle = 0$, $\forall (f, e) \in \mathcal{D}$.

A Dirac structure defines a power-conserving relation on $\mathcal{F} \times \mathcal{E}$, and different representations in coordinates are possible, [17]. For example, every Dirac structure $\mathcal{D}$ can be given in *kernel representation* as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0 \right\} \tag{1}$$

or in *image representation* as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^{\mathrm{T}}\lambda,\ e = F^{\mathrm{T}}\lambda,\ \lambda \in \mathbb{R}^n \right\} \tag{2}$$

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where $F$ and $E$ are $n \times n$ matrices such that

$$EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0 \qquad \operatorname{rank}\big(F \mid E\big) = n \tag{3}$$

and, in this case, $\langle e, f\rangle = e^{\mathrm{T}} f$.

### *B. Port-Hamiltonian systems*

In case of finite dimensional port-Hamiltonian systems, once the Dirac structure is given, the dynamics follows from the *port behavior* of the energy storing elements. Denote by $\mathcal{X}$ the space of energy variables and by $H : \mathcal{X} \to \mathbb{R}$ the energy function. Then, the port behavior is:

$$f = -\dot{x} \qquad e = \frac{\partial H}{\partial x}(x) \tag{4}$$

and, if the kernel representation (1) is adopted, the associated dynamics is expressed by $-F\dot{x} + E\frac{\partial H}{\partial x}(x) = 0$, with $x(0) = x_0 \in \mathcal{X}$. Note that, $\dot{H} = 0$, i.e. energy is conserved, which is coherent with the fact that no external ports and dissipative effects have been modelled.

In the general case, the Dirac structure $\mathcal{D}$ associated to the port-Hamiltonian system defines a power conserving relation between several port variables, e.g. two internal ports $\mathcal{S}$ and $\mathcal{R}$, which correspond to energy-storage and dissipation respectively, and two external ports $\mathcal{C}$ and $\mathcal{I}$, which are devoted to an exchange of energy with a controller and the *environment* respectively. If $(f_S, e_S) \in \mathcal{F}_S \times \mathcal{E}_S$, $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$, $(f_C, e_C) \in \mathcal{F}_C \times \mathcal{E}_C$ and $(f_I, e_I) \in \mathcal{F}_I \times \mathcal{E}_I$ denote the power variables of the energy-storage, dissipative, control and interaction ports respectively, in the kernel representation (1) the Dirac structure $\mathcal{D}$ is given by the following subset of $\mathcal{F} \times \mathcal{E}$, with $\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_C \times \mathcal{F}_I$ and $\mathcal{E} = \mathcal{E}_S \times \mathcal{E}_R \times \mathcal{E}_C \times \mathcal{E}_I$:

$$\begin{aligned}\mathcal{D} = \Big\{ &(f_S, f_R, f_C, f_I, e_S, e_R, e_C, e_I) \in \mathcal{F} \times \mathcal{E} \;\mid \\ &F_S f_S + F_R f_R + F_C f_C + F_I f_I + \\ &+ E_S e_S + E_R e_R + E_C e_C + E_I e_I = 0 \Big\}\end{aligned} \tag{5}$$

where the matrices $(F_i, E_i)$, with $i = S, R, C, I$, satisfy a set of conditions similar to (3). If the behavior at the energy storing port is given as in (4) and the dissipative port satisfies the (linear) resistive relation

$$R_f f_R + R_e e_R = 0 \tag{6}$$

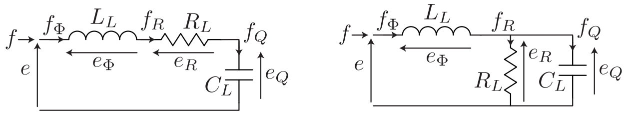
where $R_f$ and $R_e$ are square matrices such that $R_f R_e^{\mathrm{T}} = R_e R_f^{\mathrm{T}} > 0$, and $\operatorname{rank}(R_f \mid R_e) = \dim \mathcal{F}_R$, then the port-Hamiltonian dynamics results into the following set of DAEs:

$$\begin{aligned}-F_S\dot{x} + E_S\frac{\partial H}{\partial x}(x) + F_R f_R + E_R e_R + & \\ + F_C f_C + E_C e_C + F_I f_I + E_I e_I &= 0 \\ R_f f_R + R_e e_R &= 0\end{aligned} \tag{7}$$

with $x(0) = x_0 \in \mathcal{X}$. Note that, in this case,

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) \leq e_C^{\mathrm{T}}(t) f_C(t) + e_I^{\mathrm{T}}(t) f_I(t) \tag{8}$$

which means that the variation of internal energy is bounded by the incoming power flows through the control and interaction ports.



(a) Series configuration. (b) Parallel configuration.

Fig. 1. RLC circuits.

*Example 2.1 (RLC circuits):* The series RLC circuit of Fig. 1a is characterised by the Dirac structure

$$\begin{aligned}&\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}}_{=:F_{S,s}}\begin{pmatrix}f_Q\\ f_\Phi\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 1\\ -1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix}}_{=:E_{S,s}}\begin{pmatrix}e_Q\\ e_\Phi\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ 0\\ -1\\ 0\end{pmatrix}}_{=:F_{R,s}} f_R + \\ &\qquad + \underbrace{\begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}}_{=:E_{R,s}} e_R + \begin{pmatrix}0\\ 0\\ 0\\ -1\end{pmatrix} f + \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix} e = 0\end{aligned} \tag{9}$$

On the other hand, the Dirac stricture of the RLC circuit in parallel configuration of Fig. 1b is defined by

$$\begin{aligned}&\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}}_{=:F_{S,p}}\begin{pmatrix}f_Q\\ f_\Phi\end{pmatrix} + \underbrace{\begin{pmatrix}0 & 1\\ -1 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}}_{=:E_{S,p}}\begin{pmatrix}e_Q\\ e_\Phi\end{pmatrix} + \underbrace{\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}}_{=:F_{R,p}} f_R + \\ &\qquad + \underbrace{\begin{pmatrix}0\\ 0\\ -1\\ 0\end{pmatrix}}_{=:E_{R,p}} e_R + \begin{pmatrix}0\\ 0\\ 0\\ -1\end{pmatrix} f + \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix} e = 0\end{aligned} \tag{10}$$

In both cases, the Hamiltonian is given by

$$H_L(x_Q, x_\Phi) = \frac{1}{2}\left(\frac{x_Q^2}{C_L} + \frac{x_\Phi^2}{L_L}\right) \tag{11}$$

where $x_Q$ and $x_\Phi$ are the charge in the capacitor and the magnetic field in the inductor, respectively. Moreover, the resistive relation (6) takes the form

$$R_L f_R + e_R = 0 \tag{12}$$

Finally, the dynamics follows from (4), that now reads $f_Q = -\dot{x}_Q$, $f_\Phi = -\dot{x}_\Phi$, $e_Q = \frac{\partial H_L}{\partial x_Q}$, and $e_\Phi = \frac{\partial H_L}{\partial x_\Phi}$.

*Example 2.2 (Trasmission line):* The distributed port-Hamiltonian description of the lossless transmission line has been given in [18], but in this paper its finite element approximation discussed in [6] is adopted for control purposes. It is beyond the scope of this paper to provide a detailed description on how this approximation is obtained. Roughly speaking, the spatial domain $Z = [0, L]$ of the trasmission line is divided into $N$ segments, and on each segment the dynamics is approximated by a

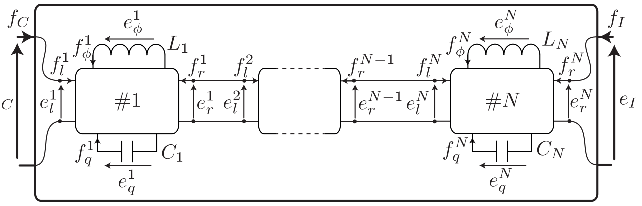


Fig. 2. Finite element model of a lossless transmission line.

finite dimensional port-Hamiltonian system. Denote by $Z_i = [l_i, r_i]$, with $i = 1, \ldots, N$, one segment. Clearly, $r_i = l_{i+1}$ for $i = 1, \ldots, N-1$, and $l_1 = 0$ and $r_N = L$. Then, a port-Hamiltonian system that approximates the infinite dimensional dynamics on $Z_i$ is characterized by a Dirac structure for which $e_l^i = e_q^i + (1-\alpha) f_\phi^i$, and $f_l^i = -e_\phi^i - (1-\alpha) f_q^i$ show the effect on the internal "energy port" of the boundary conditions in $z = l_i$, while $e_r^i = e_q^i - \alpha f_\phi^i$, and $f_r^i = -\alpha f_q^i + e_\phi^i$, in $z = r_i$, where $0 < \alpha < 1$ is a free parameter.

With reference to Fig. 2, $f_q^i$ and $f_\phi^i$ represent (minus) the currents flowing through the capacitor $C_i$ and the inductor $L_i$ that approximate the dynamics of electrical and magnetic fields on $Z_i$, while $e_q^i$ and $e_\phi^i$ are the voltages at the same components. Moreover, $(f_l^i, e_l^i)$ and $(f_r^i, e_r^i)$ define a pair of ports that are the discrete counterpart of the boundary conditions for the spatial domain $Z_i$. The complete Dirac structure of the trasmission line is obtained by interconnecting in power-conserving way all the Dirac structure defined on each $Z_i$, by imposing that $f_r^i = -f_l^{i+1}$ and $e_r^i = e_l^{i+1}$, for $i = 1, \ldots, N-1$. The result is a Dirac structure in the form (5) with $\mathcal{F}_R = \emptyset$, defined by the following relation:

$$\left(\begin{array}{ccccc} 1-\alpha & 0 & \cdots & 0 & 0 \\ 0 & 1-\alpha & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \alpha & 0 \\ 0 & 0 & \cdots & 0 & \alpha \\ \hline \multicolumn{5}{c}{\bar{F}_{S,\infty}} \end{array}\right) f_{S,\infty} + \\ + \left(\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \hline \multicolumn{5}{c}{\bar{E}_{S,\infty}} \end{array}\right) e_{S,\infty} + \\ + \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right) f_C + \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ \hline 0 \end{array}\right) e_C + \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ \hline 0 \end{array}\right) f_I + \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \end{array}\right) e_I = 0 \qquad (13)$$

where

$$\bar{F}_{S,\infty} = \begin{pmatrix} \tilde{F}_{S,\infty} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{F}_{S,\infty} \end{pmatrix}$$

$$\bar{E}_{S,\infty} = \begin{pmatrix} \tilde{E}_{S,\infty} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{E}_{S,\infty} \end{pmatrix}$$

with

$$\tilde{F}_{S,\infty} = \begin{pmatrix} 0 & -\alpha & 0 & -(1-\alpha) \\ \alpha & 0 & 1-\alpha & 0 \end{pmatrix}$$

$$\tilde{E}_{S,\infty} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

and the storage, control and interaction ports given by

$$\begin{aligned} f_{S,\infty} &= \begin{pmatrix} f_q^1 & f_\phi^1 & \cdots & f_q^N & f_\phi^N \end{pmatrix}^{\mathrm{T}} \\ e_{S,\infty} &= \begin{pmatrix} e_q^1 & e_\phi^1 & \cdots & e_q^N & e_\phi^N \end{pmatrix}^{\mathrm{T}} \\ f_C &= f_l^1 \qquad e_C = e_l^1 \\ f_I &= f_r^N \qquad e_I = e_r^N \end{aligned}$$

If $x_\infty = \begin{pmatrix} x_q^1 & x_\phi^1 & \cdots & x_q^N & x_\phi^N \end{pmatrix}^{\mathrm{T}}$ denotes the state variable, the total Hamiltonian is given by

$$H_\infty(x_\infty) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{{x_q^i}^2}{C_i} + \frac{{x_\phi^i}^2}{L_i} \right) \qquad (14)$$

and the dynamics follows from the port behaviour (4), i.e. $f_{S,\infty} = -\dot{x}_\infty$ and $e_{S,\infty} = \frac{\partial H_\infty}{\partial x_\infty}$.

## III. ENERGY-BASED CONTROL

To simplify the notation, in (5) the "interaction port" is not taken into account, i.e. $\mathcal{F}_I = \emptyset$. However, all the result that are presented in this section can be easily extended to the most general case.

### A. *Energy-balancing passivity-based control*

The energy-balance relation (8) can be equivalently rewritten in integral form

$$H(x(t)) - H(x(0)) = \int_0^t e_C^{\mathrm{T}}(\tau) f_C(\tau) \mathrm{d}\tau - d(t) \qquad (15)$$

where $d(t) \geq 0$ takes into account the dissipated energy. It is well known that, under the hypothesis that (7) as an effort-in causality at the control port, the standard formulation of passivity-based control requires to determine a control action

$$e_C = \beta(x) + e_C' \qquad (16)$$

such that the closed-loop dynamics satisfies the following *new* energy-balance relation:

$$H_d(x(t)) - H_d(x(0)) = \int_0^t {e_C'}^{\mathrm{T}}(\tau) f_C'(\tau) \mathrm{d}\tau - d_d(t) \qquad (17)$$

Here, $H_d$ is a desired energy function that has a strict minimum at $x^\star$, $f_C'$ is the new passive "output," while $d_d(t) \geq 0$ replaces the natural dissipation, that is usually increased to improve the convergence rate. So, a direct comparison between (15) and (17) clearly shows the main step of this control technique, i.e. the energy shaping *plus* the damping injection, [1], [2], [5].

A large class of dynamical systems can be stabilized by further requiring to find a function $\beta(x)$ such that the energy supplied by the controller is a function $H_a$ of the state of the plant, i.e. if

$$-\int_0^t \beta^{\mathrm{T}}(x(\tau)) f_C(\tau) \mathrm{d}\tau = H_a(x(t)) + \kappa \qquad (18)$$

with $\kappa \in \mathbb{R}$ some constant, [1], [2], [5]. Clearly, (18) is a particular case of (17). In this respect, let us write the "desired" closed-loop Hamiltonian as follows:

$$H_d(x) = H(x) + H_a(x) \tag{19}$$

Given a desired equilibrium configuration $x^\star$, the idea is to select $H_a$ in such a way that $H_d$ has a minimum in $x^\star$, that is made asymptotically stable by damping injection.

Having in mind the image representation (2) of a Dirac structure in the form (5), the differential formulation of (18) is equivalent to require that $\left(-\frac{\partial^{\mathrm{T}} H_a}{\partial x} E_S^{\mathrm{T}} + \beta^{\mathrm{T}} E_C^{\mathrm{T}}\right)\lambda = 0$, for all $\lambda \in \mathbb{R}^{n_S+n_R+n_C}$, or equivalently that

$$-E_S \frac{\partial H_a}{\partial x}(x) + E_C \beta(x) = 0 \tag{20}$$

The PDE (20) provides the family of Hamiltonian $H_a$ that can be used to shape the energy function in closed-loop, and the control action that realizes it. This equation is determined by the Dirac structure $\mathcal{D}$ of the plant, and it is independent from the resistive relation: an equivalent way to re-write (20) is in fact

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} E_S^{\mathrm{T}} \\ E_R^{\mathrm{T}} \\ E_C^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix} \equiv \mathcal{D} \tag{21}$$

It is easy to verify that, thanks to (16), the port-Hamiltonian system (7) is transformed into another port-Hamiltonian system with Hamiltonian $H_d$ given by (19) that satisfies the following energy-balance relation:

$$\begin{aligned} \frac{\mathrm{d}H_d}{\mathrm{d}t}(x(t)) &= e_R^{\mathrm{T}}(t) f_R(t) + {e'}_C^{\mathrm{T}}(t) f_C(t) \\ &\le {e'}_C^{\mathrm{T}}(t) f_C(t) \end{aligned} \tag{22}$$

Clearly, among all the possible choices compatible with (20) or (21), $H_a$ will be selected in such a way that $H_d$ is a candidate Lyapunov function with a minimum at the desired equilibrium $x^\star$. Then, (22) can be used in Lyapunov analysis to deduce stability of $x^\star$ by taking, for example, $e'_C = 0$. The equilibrium turns out to be asymptotically stable if the largest invariant set under the closed-loop dynamics contained in

$$\left\{ x \in \mathcal{X} \cap \mathcal{B} \mid e_R^{\mathrm{T}} f_R = 0 \right\} \tag{23}$$

equals $\{x^\star\}$, being $\mathcal{B}$ an open neighbourhood of $x^\star$.

Furthermore, from the linearity properties of the Dirac structure, the open-loop system with control input (16) and the target dynamics resulting from the Hamiltonian (19) share the *same behaviour* at the storage, resistive and control ports, independently from the resistive relation. This means that (20) or (21) impose a strong link between open and closed-loop dynamics, that is based only on the property of the Dirac structure. This is somehow related to the so-called "dissipation obstacle," that prevents energy-balancing passivity-based control schemes to stabilise equilibria that require an infinite amount of supplied energy. A possible solution to this problem is illustrated in the next section.

### B. *Passivity-based control with state-modulated source*

The solution of (21) can be stated as follows: find a state dependent control action $\beta$ that is able to shape the open-loop Hamiltonian thanks to $H_a$, and in such a way that closed-loop dynamics i.e., (7) with (16), and target dynamics i.e., the port-Hamiltonian system with the same Dirac structure (5) and resistive relation (6), but with Hamiltonian (19), have the *same behaviour* at the storage, resistive and control ports. This requirement is quite strong, and it can be relaxed by requiring that the control input $\beta(x)$ is able to map the trajectories of the open-loop system (7) into the trajectories of another port-Hamiltonian system with Hamiltonian (19), and characterised by the same Dirac structure (5) and resistive relation (6).

From the image representation (2) of a Dirac structure, as far as the open-loop dynamics is concerned, it exists $\lambda \in \mathbb{R}^{n_S+n_R+n_C}$ that depends on $x$, such that $-\dot{x} = E_S^{\mathrm{T}}\lambda$, $\frac{\partial H}{\partial x} = F_S^{\mathrm{T}}\lambda$, $0 = \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right)\lambda$, and $e_C = F_C^{\mathrm{T}}\lambda$. Similarly, as far as the "desired dynamics" is concerned, there exists $\lambda' \in \mathbb{R}^{n_S+n_R+n_C}$ such that $-\dot{x} = E_S^{\mathrm{T}}\lambda'$, $\frac{\partial H_d}{\partial x} = F_S^{\mathrm{T}}\lambda'$, $0 = \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right)\lambda'$, and $e'_C = F_C^{\mathrm{T}}\lambda'$. Since the trajectories are required to be the same, and in spite of (16) and (19), we have that

$$\begin{pmatrix} 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} E_S^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix} \tag{24}$$

Note that if (21) holds, then also (24) is satisfied.

It is easy to verify that, if (24) holds, then the open-loop system is mapped into the desired closed-loop one, for which the Hamiltonian function $H_d$ is selected so that "nice" stability properties are satisfied: the same consideration about the proof of asymptotic stability drawn for the energy-balancing passivity-based control discussed in the previous section are still valid here. More details on this point in the next session.

## IV. EXAMPLE: TRANSMISSION LINE WITH RLC LOAD

The scope of this section is to verify the applicability of the previously introduced techniques to a simple but illustrative example. The plant consists of the power conserving interconnection of the spatially discretized transmission line of Example 2.2 with the series and parallel RLC circuits presented in Example 2.1. In both cases, we impose that $f_I = -f$ and $e_I = e$.

In case the load is the RLC circuit in the series configuration, from (9) and (13), the port-Hamiltonian model is characterised by a Dirac structure defined by (5), where

$$F_S = \left(\begin{array}{c|ccccc} & 0 & 0 & \cdots & 0 & 0 \\ F_{S,s} & 0 & 0 & \cdots & 0 & -\alpha \\ & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & -\alpha & 0 \\ \hline 0 & 0 & -(1-\alpha) & \cdots & 0 & 0 \\ & 1-\alpha & 0 & \cdots & 0 & 0 \\ \hline 0 & \multicolumn{5}{c}{\bar{F}_{S,\infty}} \end{array}\right)$$

and

$$E_S = \left(\begin{array}{c|ccccc} & 0 & 0 & \cdots & 0 & 0 \\ E_{S,s} & 0 & 0 & \cdots & 1 & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & 0 & 1 \\ \hline \multirow{2}{*}{0} & -1 & 0 & \cdots & 0 & 0 \\ & 0 & 1 & \cdots & 0 & 0 \\ \hline 0 & \multicolumn{5}{c}{\bar{E}_{S,\infty}} \end{array}\right) \quad F_R = \left(\begin{array}{c} F_{R,s} \\ \hline 0 \\ \hline 0 \end{array}\right)$$

$$E_R = \left(\begin{array}{c} E_{R,s} \\ \hline 0 \\ \hline 0 \end{array}\right) \quad F_C = \left(\begin{array}{c} 0 \\ \hline 0 \\ 1 \\ \hline 0 \end{array}\right) \quad E_C = \left(\begin{array}{c} 0 \\ \hline 1 \\ 0 \\ \hline 0 \end{array}\right)$$

with

$$f_S = \begin{pmatrix} f_Q & f_\Phi & f_{S,\infty} \end{pmatrix}^{\mathrm{T}} \quad e_S = \begin{pmatrix} e_Q & e_\Phi & e_{S,\infty} \end{pmatrix}^{\mathrm{T}} \tag{25}$$

Moreover, the resistive relation (12) holds. The state variable $x = \left(x_Q,\, x_\Phi,\, x_\infty\right)^{\mathrm{T}}$ is the collection of the state variables of the two main subsystems, and the total Hamiltonian the sum of the (11) and (14), i.e.:

$$H(x) = H_\infty(x_\infty) + H_L(x_Q, x_\Phi) \tag{26}$$

Simple physical considerations show that the desired equilibrium configuration is $\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star$, and $\frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = 0$, for $i = 1, \ldots, N$, that means zero flowing current and constant voltage $e^\star$ along the transmission line and on the load. As in the case in which the transmission line is not present, such equilibrium can be stabilised via energy-shaping by following the methodology discussed in Sect. III-A. In particular, the PDE (20) or its equivalent formulation (21) has a solution. With reference to (21), it is necessary to find $\lambda = \left(\lambda_1,\, \lambda_2,\, \lambda_3\right)^{\mathrm{T}}$, possibly dependent on $x$, such that

$$\begin{aligned} -\frac{\partial H_a}{\partial x} &= {F_S}^{\mathrm{T}}\lambda \\ \beta &= {F_C}^{\mathrm{T}}\lambda \\ 0 = E_S^{\mathrm{T}}\lambda &= E_R^{\mathrm{T}}\lambda = F_R^{\mathrm{T}}\lambda = E_C^{\mathrm{T}}\lambda \end{aligned} \tag{27}$$

Let assume $\lambda_1 = \left(\lambda_{1,2},\, \ldots,\, \lambda_{1,4}\right)^{\mathrm{T}}$, $\lambda_2 = \left(\lambda_{2,1}, \lambda_{2,2}\right)^{\mathrm{T}}$ and $\lambda_3 = \left(\lambda_{3,1},\, \ldots,\, \lambda_{3,2(N-1)}\right)^{\mathrm{T}}$. Then, with simple calculations, from the last set of relations in (27), it follows that

$$\begin{aligned} \lambda_{1,2} = \lambda_{1,3} = \lambda_{2,1} = \lambda_{3,1} = \lambda_{3,2i-1} &= 0 \\ \lambda_{1,1} = -\lambda_{1,4} = \lambda_{2,2} &= \lambda_{3,2i} \end{aligned}$$

with $i = 1, \ldots, N-1$. Since

$$\begin{aligned} -\frac{\partial H_a}{\partial x_Q} &= -\frac{\partial H_a}{\partial x_q^i} = \lambda_{1,1} & \beta &= \lambda_{2,2} = \lambda_{1,1} \\ -\frac{\partial H_a}{\partial x_\Phi} &= -\frac{\partial H_a}{\partial x_\phi^i} = \lambda_{1,2} = 0 \end{aligned}$$

we have that

$$\begin{aligned} H_a(x) &= H_a(\xi)\Big|_{\xi = x_Q + \sum_{i=1}^N x_q^i} \\ \beta(x) &= -\left.\frac{\partial H_a}{\partial \xi}(\xi)\right|_{\xi = x_Q + \sum_{i=1}^N x_q^i} \end{aligned} \tag{28}$$

A possible choice for $H_a$ is the following:

$$H_a(\xi) = \frac{1}{2}\frac{(\xi - \xi^\star)^2}{C_C} - e^\star\left(1 + \frac{C_L}{C_C} + \sum_{i=1}^N \frac{C_i}{C_C}\right)\xi + \kappa$$

where $C_C > 0$ is a design parameter, $\xi^\star$ is the value of $\xi$ at the equilibrium, and $\kappa$ a constant. This is the same result obtained in [7], where the controller has been developed by generating Casimir functions in closed-loop. The constant $\kappa$ can be selected to have the closed-loop Hamiltonian (19) quadratic in the increments, i.e.:

$$H_d(x) = \frac{1}{2}\begin{pmatrix} \frac{x_Q}{C_L} - e^\star \\ \frac{x_q^1}{C_1} - e^\star \\ \vdots \\ \frac{x_q^N}{C_N} - e^\star \\ x_\Phi \\ x_\phi^1 \\ \vdots \\ x_\phi^N \end{pmatrix}^{\mathrm{T}} \mathcal{H}_d \begin{pmatrix} \frac{x_Q}{C_L} - e^\star \\ \frac{x_q^1}{C_1} - e^\star \\ \vdots \\ \frac{x_q^N}{C_N} - e^\star \\ x_\Phi \\ x_\phi^1 \\ \vdots \\ x_\phi^N \end{pmatrix} \tag{29}$$

with

$$\mathcal{H}_d = \begin{pmatrix} \frac{C_L^2}{C_C} + C_L & \cdots & \frac{C_L C_N}{C_C} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ \frac{C_L C_N}{C_C} & \cdots & \frac{C_N^2}{C_C} + C_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{L_L} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{1}{L_N} \end{pmatrix}$$

Stability easily follows from (22) and from the fact that (29) is bounded from below. Asymptotic stability is proved by checking that under the closed-loop dynamics, the largest invariant solution contained in (23) equals the desired equilibrium. In fact, when $\dot{H}_d = e_R^{\mathrm{T}} f_R = 0$, we have that $x_\Phi = 0$ and $x_Q = \bar{x}_Q$ constant, to be determined later on. From the system dynamics we obtain that

$$\alpha\dot{x}_\phi^N + \frac{x_q^N}{C_N} = \frac{\bar{x}_Q}{C_L} \qquad\qquad \alpha\dot{x}_q^N + \frac{x_\phi^N}{L_N} = 0$$

The only invariant solution compatible with $\dot{H}_d = 0$ is $x_\phi^N = 0$ and $\frac{x_q^N}{C_N} = \frac{\bar{x}_Q}{C_L}$, and the iteration of this procedure for all the elements of the transmission line leads to $x_\phi^i = 0$ and $\frac{x_q^i}{C_i} = \frac{\bar{x}_Q}{C_L}$, with $i = 1, \ldots, N$. From (28) the value assumed by the control action $\beta$ in this steady state configuration can be computed, and then it follows that $\frac{\bar{x}_Q}{C_L} = e^\star$, and this completes the proof.

In case the load interconnected to the transmission line is the RLC circuit in parallel configuration, the resulting port-Hamiltonian model is characterised by a Dirac structure defined by (5), where the matrices $F_S$, $F_R$, $F_C$, $E_S$, $E_R$, and $E_C$ are not modified, but with now $F_{S,s}$, $E_{S,s}$, $F_{R,s}$, and $E_{R,s}$ replaced by $F_{S,p}$, $E_{S,p}$, $F_{R,p}$, and $E_{R,p}$ defined in (10). Moreover, the port variables are defined in (25), the resistive relation (12) holds, and Hamiltonian is given by (26). The desired equilibrium configuration is

$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = \frac{e^\star}{R_L} \tag{30}$$

with $i = 1, \dots, N$, which means constant voltage $e^\star$ and current $\frac{e^\star}{R_L}$ along the transmission line and on the load. Since as in [2] without the transmission line, an energy-balancing controller is not able to stabilise the system, it is preferable to rely on the method discussed in Sect. III-B, and look for solutions of (24), i.e. to find $\lambda = (\lambda_1, \lambda_2, \lambda_3)^{\mathrm{T}}$ such that

$$\begin{aligned} -\frac{\partial H_a}{\partial x} &= {F_S}^{\mathrm{T}} \lambda \\ \beta &= {F_C}^{\mathrm{T}} \lambda \\ 0 &= E_S^{\mathrm{T}} \lambda = (R_L E_R^{\mathrm{T}} + F_R^{\mathrm{T}}) \lambda \end{aligned} \tag{31}$$

As before, from the last set of relations in (31)

$$\begin{aligned} \lambda_{1,2} &= \lambda_{1,3} = \lambda_{2,1} = \lambda_{3,2i-1} \\ \lambda_{1,1} &= -\lambda_{1,4} = \lambda_{2,2} = \lambda_{3,2i} \\ -R_L \lambda_{1,2} &+ \lambda_{1,1} = 0 \end{aligned} \tag{32}$$

with $i = 1, \dots, N-1$. Since

$$-\frac{\partial H_a}{\partial x_Q} = -\frac{\partial H_a}{\partial x_q^i} = \lambda_{1,1} \qquad \beta = \lambda_{2,2} = \lambda_{1,1}$$
$$-\frac{\partial H_a}{\partial x_\Phi} = -\frac{\partial H_a}{\partial x_\phi^i} = \lambda_{1,2}$$

with $i = 1, \dots, N-1$, from (32), we have that

$$\begin{aligned} H_a(x) &= H_a(\xi) \Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)} \\ \beta(x) &= -R_L \, \frac{\partial H_a}{\partial \xi}(\xi) \Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)} \end{aligned} \tag{33}$$

A possible choice for $H_a$ is the following:

$$H_a(\xi) = \frac{1}{2} \frac{(\xi - \xi^\star)^2}{L_C} - \frac{e^\star}{R_L} \xi + \kappa \tag{34}$$

where $L_C > 0$ is a design parameter, $\kappa$ a constant and $\xi^\star$ the value of $\xi$ at the equilibrium (30). Then, the closed-loop Hamiltonian (19) is quadratic in the increments, and (30) is the unique minimum. Asymptotic stability is proved in the same way as before.

## V. CONCLUSIONS AND FUTURE WORK

In this paper, the energy-balancing passivity-based control and the control by state-modulated source control are generalised to the case in which the plant is a port-Hamiltonian system in implicit form. The final goal is to apply these control methodologies to finite dimensional port-Hamiltonian systems obtained from the spatial discretization of distributed parameter systems. This is the typical situation in which the plant cannot be generally written in standard input-state-output form, but as a set of DAEs. Consequently, classical energy-shaping control techniques have to be extended in order to deal with dynamical systems with constraints, usually appearing in the form of Lagrangian multipliers. The general methodologies have been illustrated with the help of a simple example, i.e. a trasmission line with RLC load in both series and parallel configuration. Future work deals with the rigorous treatment of the case in which the Dirac structure is not constant (i.e., state-modulated), the generalisation to implicit systems of IDA-PBC technique, and the formalization of these energy-based techniques directly in the distributed parameter scenario.

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