

Computing the distance to instability for large-scale nonlinear eigenvalue problems

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Abstract—A quadratically converging algorithm for the computation of the distance to instability of a broad class of nonlinear eigenvalue problems is presented, including the polynomial eigenvalue problem and the delay eigenvalue problem. The algorithm is grounded in a recently presented approach for computing the pseudospectral abscissa. The application of the algorithm only relies on the availability of a method to compute the rightmost eigenvalue of perturbed problems obtained by adding rank one perturbations to the coefficient matrices, for which, in case of large and sparse matrices, efficient iterative algorithms can be used.

I. INTRODUCTION

In order to analyze the robustness of dynamical systems with respect to uncertainty the stability radius (distance to instability) is an important concept. In this paper we consider the stability radius computation for a broad class of nonlinear eigenvalue problems of the form

$$\left(\sum_{i=0}^m A_i p_i(\lambda) \right) v = 0, \quad \lambda \in \mathbb{C}, v \in \mathbb{C}^n, \quad (1)$$

where $A_i \in \mathbb{C}^{n \times n}$, $i = 0, \dots, m$ and the functions $p_i : \mathbb{C} \rightarrow \mathbb{C}$, $i = 0, \dots, m$ are entire and satisfy $p_i(\bar{\lambda}) = \overline{p_i(\lambda)}$. This class includes polynomial eigenvalue problems and nonlinear eigenvalue problems arising in the analysis of linear delay differential equations as special cases. Although many of these problems can be reformulated as linear eigenvalue problems (for example, by a so-called linearization of a polynomial eigenvalue problem or by a transformation to an equivalent operator eigenvalue problem for the delay eigenvalue problem [15]) we will not consider the unstructured distance to instability of a particular type of linearization. Instead, as in the works [13], [16] on pseudospectra, we will explicitly take the structure of the original nonlinear eigenvalue problem into account in the definition of stability radii, which we relate to the effect of perturbations on the individual coefficient matrices A_i in (1).

For the linear eigenvalue problem level set methods are well established methods for computing the distance to instability and related quantities such as the pseudospectral abscissa and \mathcal{H}_∞ norms. The underlying idea is that the intersections between the singular value curves of a transfer function and a constant function can be computed from the solutions of a Hamiltonian eigenvalue problem. This

property lays at the basis of the bisection based algorithm for computing the stability radius in [4] and the quadratically converging algorithms for computing \mathcal{H}_∞ norms and pseudospectral abscissa in [1], [3]. Although these algorithms are very robust, they are computationally demanding, since they require in each iteration step the computation of *all* eigenvalues on the real or imaginary axis of a structured matrix of twice the size of the original problem, and they do not extend to nonlinear eigenvalue problems.

Recently in [8] an algorithm for computing the pseudospectral abscissa of a matrix has been proposed, which relies on the property that eigenvalues can be shifted to the boundary of the pseudospectrum by adding rank one perturbations to the matrix and on the characterization of extrema of the pseudospectra as fixed points of a nonlinear map. A related algorithm based on a steepest ascent differential equation on the manifold of normalized rank one matrices has been proposed in [7]. Using similar ideas the \mathcal{H}_∞ norm computation is addressed in [6]. These algorithm only require the computation of the rightmost eigenvalue of matrices obtained by adding rank one perturbations to the original matrix, for which -in case of large and sparse problems- efficient iterative solvers can be used. In [14] it is demonstrated how the method of [8] extends towards nonlinear eigenvalue problems, starting from the observation that critical perturbations on the different matrices A_i in (1), i.e., perturbations shifting eigenvalues to the boundary of the pseudospectra contours, can be chosen as multiples of the same rank-one matrix.

As we shall see in Section 2 the distance to instability can be characterized by the bound on the perturbations for which the corresponding pseudospectral abscissa is zero. This will lead us to an algorithm based on computing a zero of the spectral abscissa function. Every iteration of the algorithm involves the computation of the spectral abscissa, for which we use the approach of [14], and its derivative with respect to the perturbation bound ϵ , which can be obtained as a by-product. Therefore, the applicability to a particular class of nonlinear eigenvalue problems ultimately relies on a procedure to compute rightmost eigenvalues. An overview of methods and software for the quadratic eigenvalue problem can be found in, for instance, [17] and the references therein. For the delay eigenvalue problem we refer to [2], [5], [10]. General purpose methods for solving nonlinear eigenvalue problems of the form (1) are described in, e.g., [11], [19], [12], [20] and the references therein.

The structure of the paper is as follows. In Section II we introduce the problem and briefly describe the algorithm for

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the pseudospectral abscissa computation of [14]. Section III is devoted to the main result, the computation of stability radii. In Section IV the application to time-delay systems is briefly discussed. Finally, numerical experiments are described in Section V.

II. PRELIMINARIES

A. Pseudospectra and stability radii

We consider the nonlinear eigenvalue problem (1). In what follows we call

$$F(\lambda) := \sum_{i=0}^m A_i p_i(\lambda) \quad (2)$$

the characteristic matrix. We denote the spectrum by Λ and the spectral abscissa by α , i.e.

$$\begin{aligned} \Lambda(F) &:= \{\lambda \in \mathbb{C} : \det(\sum_{i=0}^m A_i p_i(\lambda)) = 0\}, \\ \alpha(F) &:= \sup \{\Re(\lambda) : \lambda \in \Lambda(F)\}. \end{aligned}$$

We are interested in the effect of bounded perturbations of the matrices A_i on the spectrum, which leads to the perturbed eigenvalue problem,

$$\left(\sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda) \right) v = 0, \quad \lambda \in \mathbb{C}, \quad v \in \mathbb{C}^n. \quad (3)$$

The first step in the robustness analysis is to define the class of perturbations under consideration, as well as a measure of the combined perturbation

$$\Delta := (\delta A_0, \dots, \delta A_m).$$

In analogy to the classical definition of ϵ -pseudospectrum of a matrix [18], we allow the perturbations δA_i , $i = 0, \dots, m$, to be *complex* matrices, i.e.,

$$\Delta \in \mathbb{C}^{n \times n \times (m+1)}. \quad (4)$$

Introducing weights $w_i \in \overline{\mathbb{R}}_0^+$, $i = 0, \dots, m$, where $\overline{\mathbb{R}}_0^+ = \mathbb{R}^+ \setminus \{0\} \cup \{\infty\}$, we define the following global measure of the perturbations:

$$\|\Delta\|_{\text{glob}} := \left\| \begin{bmatrix} w_0 \|\delta A_0\|_2 \\ \vdots \\ w_m \|\delta A_m\|_2 \end{bmatrix} \right\|_{\infty}. \quad (5)$$

In this way the condition

$$\|\Delta\|_{\text{glob}} \leq \epsilon$$

corresponds to the natural assumptions of taking perturbations satisfying

$$\|\delta A_i\|_2 \leq \epsilon/w_i, \quad i = 0, \dots, m.$$

This uncertainty bound is also used in [16] and fits within the general class considered in [13]. Taking a weight equal to infinity implies that the corresponding matrix is not perturbed.

With the above class of allowable perturbations and with the measure (5) we define the ϵ -pseudospectrum of (1) as the set

$$\Lambda_{\epsilon}(F) = \bigcup_{\|\Delta\|_{\text{glob}} \leq \epsilon} \{\lambda \in \mathbb{C} : \det(\sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda)) = 0\} \quad (6)$$

and we define the corresponding pseudospectral abscissa as

$$\alpha_{\epsilon}(F) := \sup \{\Re(\lambda) : \lambda \in \Lambda_{\epsilon}\}. \quad (7)$$

In [13] the following explicit expression for the pseudospectrum is obtained.

Proposition 2.1: For the perturbation class (4) and measure (5) the pseudospectrum Λ_{ϵ} of (2) satisfies

$$\Lambda_{\epsilon}(F) = \left\{ \lambda \in \mathbb{C} : \sigma_n \left(\sum_{i=0}^m A_i p_i(\lambda) \right) \leq \epsilon \|w(\lambda)\|_1, \right\} \quad (8)$$

where $\sigma_n(\cdot)$ denotes the smallest singular value and

$$w(\lambda) = \begin{bmatrix} p_0(\lambda) & \dots & p_m(\lambda) \\ w_0 & & w_m \end{bmatrix}^T. \quad (9)$$

We say that F is exponentially stable if all zeros are confined to the open left half plane and bounded away from the imaginary axis, i.e., $\alpha(F) < 0$. To assess the robustness of stability w.r.t. perturbations on the coefficient matrices A_i we introduce the concept of a stability radius.

Definition 2.2: The stability radius of (2) w.r.t. the perturbation class (4) and measure (5) is defined as

$$r(F) := \inf \{\epsilon \geq 0 : \alpha_{\epsilon} \geq 0\}.$$

Throughout the paper we make the following assumption.

Assumption 2.3: For all $r \in \mathbb{R}$ and $\epsilon \geq 0$ the set $\Lambda_{\epsilon} \cap \{\lambda \in \mathbb{C} : \Re(\lambda) \geq r\}$ is bounded.

This assumption implies that, by varying ϵ in a continuous way, a transition from the situation where $\alpha_{\epsilon} < 0$ to a situation where $\alpha_{\epsilon} \geq 0$ is characterized by eigenvalues moving from the open left half plane to the imaginary axis (i.e., right half plane eigenvalues coming from the point at infinity cannot occur). Combined with the characterization (8) this leads to the following expression for the stability radius [13, Corollary 3].

Proposition 2.4: If the zeros of F are confined to the open left half plane and Assumption 2.3 holds then we have

$$r(F) = \left\{ \sup_{\omega \in \mathbb{R}} \left\| \left(\sum_{i=0}^m A_i p_i(j\omega) \right)^{-1} \right\|_2 \|w(j\omega)\|_1 \right\}^{-1}, \quad (10)$$

with w given by (9).

Using (10) the computation of the stability radius can be turned into a frequency sweeping test. However, a gridding procedure, where for each grid point a singular value needs to be calculated, is computationally expensive, while making an appropriate choice of the grid points is a difficult task. Note also that gridding leads to an upper bound on the stability radius. In this work we will use Proposition 2.4 for analysis purposes only, while in Section III we will present

an alternative approach for determining the stability radius, based on computing a zero of the pseudospectral abscissa function.

B. Computing the pseudospectral abscissa of large-scale problems

In [14] a fast algorithm for computing the pseudospectral abscissa is proposed, which we briefly outline. Assumption 2.3 guarantees the presence of a globally rightmost point of the pseudospectrum Λ_ϵ , which we denote by λ_ϵ , i.e.,

$$\Re(\lambda_\epsilon) = \alpha_\epsilon.$$

Let (u_ϵ, v_ϵ) be a pair of normalized left and right singular vectors corresponding to

$$\sigma_n \left(\sum_{i=0}^m A_i p_i(\lambda_\epsilon) \right)$$

and let $c \in \mathbb{C}^n$ be such that u_ϵ satisfies the normalization constraint

$$c^* u_\epsilon \in \mathbb{R}_0^+.$$

We make some technical assumptions, on which the algorithm relies.

Assumption 2.5: The following conditions are assumed:

- 1) the smallest singular value of $\sum_{i=0}^m A_i p_i(\lambda_\epsilon)$ is simple;
- 2) the rightmost eigenvalue of

$$\left(\sum_{i=0}^m \left(A_i - \frac{\overline{p_i(\lambda_\epsilon)}}{w_i |p_i(\lambda_\epsilon)|} u_\epsilon v_\epsilon^* \epsilon \right) p_i(\lambda) \right) v = 0. \quad (11)$$

is simple;

- 3) we have $p_i(\lambda_\epsilon) \neq 0$ whenever w_i is finite, for $0 \leq i \leq m$.

The cases where Assumption 2.5 are not satisfied correspond to degenerate cases, see [14] for a detailed discussion. In Propositions 3.1 and 3.3 of this reference it has been shown that under Assumption 2.5 the triple $(\lambda, u, v) = (\lambda_\epsilon, u_\epsilon, v_\epsilon)$ is an isolated solution of the following system of equations and inequalities:

$$\left(\sum_{i=0}^m \left(A_i - \frac{\overline{p_i(\lambda)}}{w_i |p_i(\lambda)|} u v^* \epsilon \right) p_i(\lambda) \right) v = 0, \quad (12)$$

$$u^* \left(\sum_{i=0}^m \left(A_i - \frac{\overline{p_i(\lambda)}}{w_i |p_i(\lambda)|} u v^* \epsilon \right) p_i(\lambda) \right) = 0, \quad (13)$$

$$u^* u = v^* v = 1, \quad c^* u > 0, \quad (14)$$

$$u^* \left(\sum_{i=0}^m A_i p_i'(\lambda) \right) v - \epsilon \sum_{i=0}^m \frac{\overline{p_i(\lambda)} p_i'(\lambda)}{w_i |p_i(\lambda)|} > 0. \quad (15)$$

Furthermore, λ_ϵ is the rightmost eigenvalue of the perturbed eigenvalue problem (11). The meaning of (12)-(15) can be summarized as follows. Conditions (12)-(13) characterize points lying on curves described by

$$\sigma_k \left(\sum_{i=0}^m A_i p_i(\lambda) \right) = \epsilon \|w(\lambda)\|_1, \quad k \in \{1, \dots, n\},$$

which include the boundaries of the pseudospectrum (case where $k = n$). Condition (15) is an optimality condition characterizing that the outwards pointing normal vector to the set

$$\left\{ \lambda \in \mathbb{C} : \sigma_k \left(\sum_{i=0}^m A_i p_i(\lambda) \right) - \epsilon \|w(\lambda)\|_1 \leq 0 \right\}$$

lies in the direction of the positive real axis. Conditions (14) are normalization constraints.

The algorithms in [14] are based on turning (12)-(15) into a fixed-point iteration. The basic algorithm is as follows.

Algorithm 1 (fixed-point iteration):

Initialize (λ_0, u_0, v_0) and repeat for $k = 1, 2, \dots$

- 1) Compute λ_k as the rightmost eigenvalue of

$$\begin{aligned} \left(\sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda) \right) v &= 0, \\ \delta A_i &= -\frac{p_i(\lambda_{k-1})}{w_i |p_i(\lambda_{k-1})|} u_{k-1} v_{k-1}^* \epsilon, \quad 0 \leq i \leq m. \end{aligned} \quad (16)$$

- 2) Define u_k and v_k as the left and right eigenvectors of (16) associated with λ_k , which are scaled such that

$$u_k^* u_k = v_k^* v_k = 1, \quad c^* u_k > 0, \quad (17)$$

$$\begin{aligned} u_k^* \left(\sum_{i=0}^m A_i p_i'(\lambda_k) \right) v_k \\ - \epsilon \sum_{i=0}^m \frac{p_i(\lambda_k) p_i'(\lambda_k)}{w_i |p_i(\lambda_k)|} > 0. \end{aligned} \quad (18)$$

Refinements of Algorithm 1 to make the iteration well defined for all possible starting values¹ and to enforce global convergence to the globally rightmost point of the pseudospectrum can be found in [14]. They all share the property that their application only requires methods to compute the *rightmost* eigenvalues and the corresponding left and right eigenvectors, for which fast iterative solvers can be used if the system matrices are large and spares. This feature makes them applicable to large-scale problems.

III. COMPUTING THE STABILITY RADIUS

We start with a technical lemma about the *spectral abscissa function*

$$\mathbb{R}_+ \ni \epsilon \mapsto \alpha_\epsilon(F), \quad (19)$$

which is a consequence of Assumption 2.3 and the characterization (8).

Lemma 3.1: The function (19) is continuous, strictly increasing, and

$$\lim_{\epsilon \rightarrow \infty} \alpha_\epsilon = \infty.$$

Based on Definition 2.2 and Lemma 3.1 we can refine (for exponentially stable F)

$$r(F) = \{\epsilon > 0 : \alpha_\epsilon(F) = 0\},$$

i.e., we can compute the stability radius as the zero of the spectral abscissa function.

The following, main result shows that for almost all ϵ the derivative of the pseudospectral abscissa with respect to ϵ exists and can be obtained as a by-product from its computation using the method described Section II-B.

¹Away from the fixed point it could happen that the normalizing conditions (17)-(18) are too restrictive or allow multiple solutions [14].

Theorem 3.2: Let $\hat{\epsilon} > 0$ and assume that $\lambda_{\hat{\epsilon}}$ is a globally rightmost point of $\Lambda_{\hat{\epsilon}}$. Let $(u_{\hat{\epsilon}}, v_{\hat{\epsilon}})$ be a pair of (left, right) singular vectors corresponding to $\sigma_n(\sum_{i=0}^m A_i p_i(\lambda_{\hat{\epsilon}}))$. Let Assumption 2.5 be satisfied, with the triple $(\lambda_{\hat{\epsilon}}, u_{\hat{\epsilon}}, v_{\hat{\epsilon}})$ solving (12)-(15) for $\epsilon = \hat{\epsilon}$. There exists a constant $\delta > 0$ and a continuous function

$$(\hat{\epsilon} - \delta, \hat{\epsilon} + \delta) \ni \epsilon \mapsto (\lambda(\epsilon), u(\epsilon), v(\epsilon))$$

such that $(\lambda(\epsilon), u(\epsilon), v(\epsilon))$ satisfies (12)-(15) for all $\epsilon \in (\hat{\epsilon} - \delta, \hat{\epsilon} + \delta)$, such that $(\lambda(\hat{\epsilon}), u(\hat{\epsilon}), v(\hat{\epsilon})) = (\lambda_{\hat{\epsilon}}, u_{\hat{\epsilon}}, v_{\hat{\epsilon}})$ and

$$\Re \lambda'(\hat{\epsilon}) = \frac{\sum_{i=0}^m \frac{|p_i(\lambda_{\hat{\epsilon}})|}{w_i}}{u_{\hat{\epsilon}}^* \left(\sum_{i=0}^m A_i p_i'(\lambda_{\hat{\epsilon}}) \right) v_{\hat{\epsilon}} - \epsilon \sum_{i=0}^m \left(\frac{\bar{p}_i(\lambda_{\hat{\epsilon}}) p_i'(\lambda_{\hat{\epsilon}})}{w_i |p_i(\lambda_{\hat{\epsilon}})|} \right)} \quad (20)$$

Moreover, if the globally rightmost point is unique² then we have

$$\left. \frac{d\alpha_{\epsilon}}{d\epsilon} \right|_{\epsilon=\hat{\epsilon}} = \Re \lambda'(\hat{\epsilon}). \quad (21)$$

Proof: If δ is sufficiently small then for all $\epsilon \in (\hat{\epsilon} - \delta, \hat{\epsilon} + \delta)$, the equations and inequalities

$$\begin{aligned} & \left(\sum_{i=0}^m (A_i + \delta A_i(\epsilon)) p_i(\lambda(\epsilon)) \right) v(\epsilon) = 0, \\ & u(\epsilon)^* \left(\sum_{i=0}^m (A_i + \delta A_i(\epsilon)) p_i(\lambda(\epsilon)) \right) = 0, \\ & v(\epsilon)^* v(\epsilon) = 1, \quad u(\epsilon)^* u(\epsilon) = 1, \\ & \Im(c^* u(\epsilon)) = 0, \quad \Re(c^* u(\epsilon)) > 0, \\ & \Im(u(\epsilon)^* \left(\sum_{i=0}^m A_i p_i'(\lambda(\epsilon)) \right) v(\epsilon)) \\ & - \epsilon \sum_{i=0}^m \frac{\Im(\bar{p}_i(\lambda(\epsilon)) p_i'(\lambda(\epsilon)))}{w_i |p_i(\lambda(\epsilon))|} = 0, \\ & \Re(u(\epsilon)^* \left(\sum_{i=0}^m A_i p_i'(\lambda(\epsilon)) \right) v(\epsilon)) \\ & - \epsilon \sum_{i=0}^m \frac{\Re(\bar{p}_i(\lambda(\epsilon)) p_i'(\lambda(\epsilon)))}{w_i |p_i(\lambda(\epsilon))|} > 0, \end{aligned} \quad (22)$$

where

$$\delta A_i(\epsilon) = - \frac{\overline{p_i(\lambda(\epsilon))}}{w_i |p_i(\lambda(\epsilon))|} u(\epsilon) v(\epsilon)^* \epsilon, \quad 0 \leq i \leq m.$$

describe a smooth curve. For notational convenience we define

$$g_i(\epsilon) = \frac{|p_i(\lambda(\epsilon))|}{w_i}$$

for $i = 0, \dots, m$. We can rewrite the first two equations of (22) as

$$\left(\sum_{i=0}^m A_i p_i(\lambda(\epsilon)) \right) v(\epsilon) = \sum_{i=0}^m g_i(\epsilon) u(\epsilon) \epsilon \quad (23)$$

and

$$u(\epsilon)^* \left(\sum_{i=0}^m A_i p_i(\lambda(\epsilon)) \right) = \sum_{i=0}^m g_i(\epsilon) v(\epsilon)^* \epsilon. \quad (24)$$

Let us differentiate (23) w.r.t. ϵ , multiply it from the left by $u(\epsilon)^*$ and take the real part. Using

$$\begin{aligned} u'(\epsilon)^* u(\epsilon) + u(\epsilon)^* u'(\epsilon) = 0 & \implies u'(\epsilon)^* u(\epsilon) \in j\mathbb{R}, \\ v'(\epsilon)^* v(\epsilon) + v(\epsilon)^* v'(\epsilon) = 0 & \implies v'(\epsilon)^* v(\epsilon) \in j\mathbb{R}, \end{aligned}$$

²If the spectrum is symmetric w.r.t. the real axis, then only the eigenvalues in the upper half plane are considered.

we obtain:

$$\Re \left(u^* \left(\sum_{i=0}^m A_i p_i'(\lambda) \right) v \lambda' \right) = \epsilon \Re \left(\sum_{i=0}^m g_i'(\epsilon) \right) + \sum_{i=0}^m g_i(\epsilon). \quad (25)$$

The left hand side of (25) can be rewritten as

$$\begin{aligned} & \Re(u^* (\sum_{i=0}^m A_i p_i'(\lambda)) v) \Re(\lambda'(\epsilon)) \\ & - \Im(u^* (\sum_{i=0}^m A_i p_i'(\lambda)) v) \Im(\lambda'(\epsilon)) \\ & = \Re(u^* (\sum_{i=0}^m A_i p_i'(\lambda)) v) \Re(\lambda'(\epsilon)) \\ & - \epsilon \sum_{i=0}^m \Im \left(\frac{\bar{p}_i(\lambda) p_i'(\lambda)}{w_i |p_i(\lambda)|} \right) \Im(\lambda'(\epsilon)), \end{aligned}$$

where we used the optimality condition described by the last two conditions of (22). The right hand side of (25) can be computed as

$$\begin{aligned} & \epsilon \sum_{i=0}^m \Re \left(\frac{\bar{p}_i(\lambda) p_i'(\lambda)}{w_i |p_i(\lambda)|} \right) \Re(\lambda'(\epsilon)) \\ & - \epsilon \sum_{i=0}^m \Im \left(\frac{\bar{p}_i(\lambda) p_i'(\lambda)}{w_i |p_i(\lambda)|} \right) \Im(\lambda'(\epsilon)) + \sum_{i=0}^m g_i(\epsilon). \end{aligned}$$

Substituting the above expressions in (25) yields

$$\lambda'_{\epsilon} = \Re(\lambda'(\epsilon)) = \frac{\sum_{i=0}^m g_i(\epsilon)}{\Re \left(u(\epsilon)^* \left(\sum_{i=0}^m A_i p_i'(\lambda(\epsilon)) \right) v(\epsilon) \right) - \epsilon \sum_{i=0}^m \Re \left(\frac{\bar{p}_i(\lambda) p_i'(\lambda)}{w_i |p_i(\lambda)|} \right)}.$$

Evaluating this expression at $\hat{\epsilon}$, taking into account (22), we arrive at (20).

The second assertion follows from $\alpha_{\epsilon} = \Re(\lambda(\epsilon))$ whenever $\lambda(\epsilon)$ is the unique globally rightmost point of Λ_{ϵ} . ■

It follows from Theorem 3.2 that the pseudospectral abscissa function is differentiable whenever the globally rightmost point of the pseudospectrum is unique and the technical Assumption 2.5 is satisfied. This is the cases for almost all ϵ , that is, except for special choices where the pseudospectrum exhibits for instance more than one globally rightmost point in the upper half complex plane. Therefore, the pseudospectral abscissa is differentiable almost everywhere. The fact that the derivative must be strictly positive can be readily seen from expression (20) and optimality condition (15).

Finally, formulae (20)-(21) allow us to apply Newton's method to solve the equation

$$\alpha_{\epsilon}(F) = 0$$

for ϵ , in order to compute the stability radius. This results in Algorithm 2.

Algorithm 2: Initialize ϵ_0 and compute for $k = 1, 2, \dots$ (until convergence):

$$\epsilon_{k+1} = \epsilon_k - \frac{\alpha_{\epsilon_k}(F)}{\left. \frac{d\alpha_{\epsilon}(F)}{d\epsilon} \right|_{\epsilon=\epsilon_k}}.$$

IV. APPLICATION TO TIME-DELAY SYSTEMS

We apply the previous results to the delay eigenvalue problem [2], [5], [10],

$$F := -\lambda I + A_0 + \sum_{i=1}^m A_i e^{-\lambda \tau_i}, \quad (26)$$

where consider perturbations on $\Delta := (\delta A_0, \dots, \delta A_m)$ on (A_0, \dots, A_m) , measured by (5). Since the weights $\vec{w} = (w_0, \dots, w_m)$ play an important role in the subsequent analysis we will make the dependence of the pseudospectral abscissa and the stability radius on the weights explicit in the notation and write $\alpha_\epsilon(F; \vec{w})$ and $r(F; \vec{w})$.

The property

$$|e^{-j\omega \tau_i}| \equiv 1, \quad \forall \omega \in \mathbb{R}, \quad 0 \leq i \leq m,$$

lays at the basis of the following proposition, which directly follows from expression (10).

Proposition 4.1: Consider the function (26) Let $\vec{w}^{(1)} = (w_0^{(1)}, \dots, w_m^{(1)}) \in \overline{\mathbb{R}}_0^+{}^m$ and $\vec{w}^{(2)} = (w_0^{(2)}, \dots, w_m^{(2)}) \in \overline{\mathbb{R}}_0^+{}^m$ be such that

$$\sum_{i=0}^m \frac{1}{w_i^{(1)}} = \sum_{i=0}^m \frac{1}{w_i^{(2)}}. \quad (27)$$

Then we have

$$r(F; \vec{w}^{(1)}) = r(F; \vec{w}^{(2)}).$$

In other words the stability radius is independent of the distribution of the perturbations over the different coefficient matrices as long as (27) is satisfied. However, the shape of the pseudospectra and, in particular, the pseudospectral abscissa do depend on the distribution, as shown in the next proposition.

Proposition 4.2: Let $\vec{w} = (w_0, \dots, w_m) \in \overline{\mathbb{R}}_0^+{}^m$ and define

$$\vec{w}_0 = \left(\frac{1}{\sum_{i=0}^m w_i^{-1}}, \infty, \dots, \infty \right).$$

The corresponding pseudospectra satisfy

$$\begin{aligned} (\Lambda_\epsilon(F; \vec{w}) \cap \mathbb{C}_+) &\subseteq (\Lambda_\epsilon(F; \vec{w}_0) \cap \mathbb{C}_+), \\ (\Lambda_\epsilon(F; \vec{w}) \cap \mathbb{C}_-) &\supseteq (\Lambda_\epsilon(F; \vec{w}_0) \cap \mathbb{C}_-) \end{aligned}$$

and, consequently,

$$\begin{cases} \alpha_\epsilon(F; \vec{w}_0) \leq \alpha_\epsilon(F; \vec{w}), & \epsilon \leq r(F; \vec{w}_0), \\ \alpha_\epsilon(F; \vec{w}_0) \geq \alpha_\epsilon(F; \vec{w}), & \epsilon > r(F; \vec{w}_0). \end{cases}$$

Proof: For the delay eigenvalue problem (26) expression (8) becomes

$$\begin{aligned} \Lambda_\epsilon(F; \vec{w}) &= \left\{ \lambda \in \mathbb{C} : \sigma_n \left(-\lambda I + A_0 + \sum_{i=1}^m A_i e^{-\lambda \tau_i} \right) \right. \\ &\quad \left. \leq \epsilon \left(\frac{1}{w_0} + \sum_{i=1}^m \frac{e^{-\Re(\lambda) \tau_i}}{w_i} \right) \right\}, \quad (28) \end{aligned}$$

from which the assertions directly follow. ■

Proposition 4.1 allows some flexibility in (re)distributing the perturbations over the different coefficient matrices, via the choice of weights, in the application of Algorithm 2,

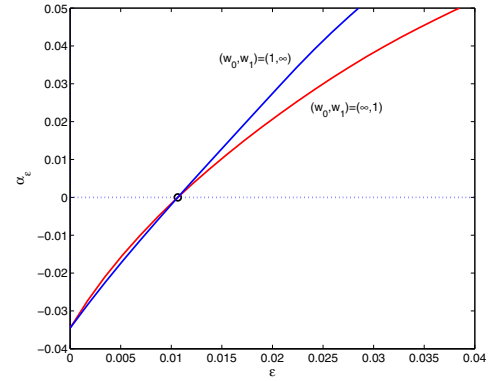


Fig. 1. Effect of distribution of weights over the coefficient matrices, for Example 1.

which is based on computing a zero of the pseudospectral abscissa function. Proposition 4.2 suggests to concentrate the perturbation on the non-delayed term, as this leads to the highest of slope the pseudospectral abscissa function around the intersection with the real axis, which is to be preferred in terms of condition of the zero. This is now illustrated with a numerical example.

Example 1: Consider (26) with $m = 1$,

$$A_0 = \begin{bmatrix} -0.080 & -0.030 & 0.200 \\ 0.200 & -0.040 & -0.005 \\ -0.060 & 0.200 & -0.070 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.0471 & -0.0504 & -0.0607 \\ -0.0942 & -0.1008 & -0.1214 \\ 0.0471 & 0.0504 & 0.0607 \end{bmatrix}$$

and $\tau_1 = 6$. In Figure 1 we plot the spectral abscissa function for two sets of weights: $(w_0, w_1) = (1, \infty)$, i.e., only the constant term is perturbed, and weights $(w_0, w_1) = (\infty, 1)$. The two sets of weights yield the same stability radius.

V. NUMERICAL EXPERIMENTS

First, we have applied Algorithm 2 to the benchmark collection³ used in [9]. This consists of delay eigenvalue problems of the form (26). The weights in the perturbation measure are taken as

$$w_i = \|A_i\|_2^{-1}, \quad 0 \leq i \leq m,$$

i.e. the value of ϵ refers to the maximal relative size of the perturbations. The results of Algorithm 2 are displayed in Table I: the second column describes the system dimension and the number of delays. The third and fourth column display the spectral abscissa and the stability radius. In the last column #it denotes the number of Newton iteration to reach 10 digits accuracy, starting from the initial value $\epsilon = 0$. In accordance with Section IV the perturbations were redistributed onto the non-delayed term via a change of the weights term before the computations took place.

³The collection is available at the webpage <http://twr.cs.kuleuven.be/research/software/delay-control/benchmarks/>.

Problem	($n, \#$ delays)	α	$r(F)$	#it
1	(3,1)	-2.866038425e-02	2.694529280e-2	5
2	(1,1)	-1.102659477+00	9.264754751e-1	6
3	(3,3)	-2.944578327e-01	4.347925858e-3	4
4	(4,9)	1.169539686+00	0	0
5	(8,20)	2.373606203+00	0	0
6	(10,7)	-3.775473572e-01	2.743086262e-2	4
7	(20,9)	-3.446892131e-02	2.566909280e-3	5
8	(40,3)	-1.044598769e-01	7.458287673e-3	5
9	(5,1)	-5.026086111e-01	3.524570232e-2	4
10	(4,3)	-9.858488139e-02	1.190747662e-2	9

TABLE I

STABILITY RADIUS COMPUTATION FOR DELAY EIGENVALUE PROBLEMS FROM THE SAME BENCHMARK COLLECTION AS USED IN [9].

k	ϵ_k	α_{ϵ_k}	$\frac{d\alpha_{\epsilon}}{d\epsilon}$ $\epsilon=\epsilon_k$
1	1.000000000e-05	-3.312014980e-01	1.183570721e+00
2	2.798424529e-01	4.488443869e-02	1.525601423e+00
3	2.504216370e-01	6.253282029e-04	1.483364073e+00
4	2.500000761e-01	1.250908710e-07	1.482770662e+00
5	2.499999918e-01	4.938617831e-15	1.482770544e+00

TABLE II

STABILITY RADIUS COMPUTATION FOR THE DELAYED PDE PROBLEM DESCRIBED IN [10].

To demonstrate the applicability to a large-scale problem we consider the PDE with delay from [10],

$$\frac{\partial v(x, t)}{\partial t} = \frac{\partial^2 v(x, t)}{\partial x^2} + a_0(x)v(x, t) + a_1(x)v(\pi - x, t - 1),$$

where $a_0(x) = -2 \sin(x)$, $a_1(x) = 2 \sin(x)$ and $v_x(0, t) = v_x(\pi, t) = 0$. The second derivatives in space in are approximated with central differences. This gives rise to a standard delay eigenvalue problem of the form (26), with one delay and sparse matrices A_0 and A_1 . The number of spatial discretization points is taken such that $n = 5000$. The weights are given by $(w_0, w_1) = (1/2, 1/2)$.

The spectral abscissa is equal to $-0.331213 < 0$. An application of Algorithm 2 yields the stability radius $r = 0.250000$. The result of successive Newton iterations is displayed in Table II.

VI. CONCLUDING REMARKS

We described an algorithm for computing stability radii of a large class of nonlinear eigenvalue problems of the form (1) and illustrated its effectiveness by means of the delay eigenvalue problem.

In many algorithms for solving nonlinear eigenvalue problems, the connection with a linearization of the eigenvalue problem plays an important role. In this sense the adopted approach for the computation of stability radii fully exploits the dual representation of the eigenvalue problem: on the one hand pseudospectra and stability radii are defined at the level of the nonlinear problem in such a way that the structure of the problem and the perturbations are fully respected; on the other hand the corresponding algorithm relies on the successive computation of selected eigenvalues, for which the connection with the linearization may be beneficial.

Future works consists of imposing additional structure on the perturbations of the individual coefficient matrices A_i in (1) as well as restricting them to be real valued.

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