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Escape time formulation of state estimation and stabilization with intermittent communication

Chun-Chia Huang, *Student Member, IEEE*, Robert R. Bitmead, *Fellow, IEEE*

***Abstract*— The problems of state estimation and feedback stabilization of an unstable system including a communications channel are quantified as escape or survival times, which yield stochastic processes describing the time of first exit of the state estimate error or of the state itself from a specific domain. The system complications introduced by communications – intermittency, channel noise, etc – are evaluated using a Markov hitting time formulation. This is compared to and contrasted with earlier analyses which considered: the behavior of Kalman filters with intermittent data, the evaluation of the minimal number of bits required for mean square stabilization, and the Large Deviations analysis of probability theory. The main result shows the the escape time is characterized by a Markov chain which is amenable to explicit analysis in the linear gaussian case.**

I. Introduction

We study the problems of state estimation and output feedback stabilization of an unstable linear time-invariant system including a single communications link.

$$x_{k+1} = Ax_k + Bu_k + w_k, \quad (1)$$
$$y_k = Cx_k + Du_k + v_k, \quad (2)$$

Here, as usual, x_k, u_k, y_k, w_k, v_k are the system state, input, output, process noise and measurement noise signals of dimensions n, p, m, n, m respectively and $[A, B, C, D]$ are the system matrices of conformable dimensions. As in [1], [2], the intermittency of the communication channel is modeled by the random 0-1 $\{\gamma_k\}$ sequence.

Definition 1 (Escape time): Given a closed domain $\mathcal{D} \subset \mathbb{R}^d$ and a stochastic process $\{\xi_k : k = 1, \dots\}$ on $\mathbb{R}^d$ with initial condition ξ_0, the escape time is defined to be

$$\tau_e = \begin{cases} \arg\min_k \xi_k \notin \mathcal{D}, & \\ \infty, & \text{if } \xi_k \in \mathcal{D} \ \forall k. \end{cases}$$

Sometimes the escape time is called the 'first exit time', or 'hitting time' We shall be concerned with the escape time for the state process, x_k, or the output process, y_k, of (1-2) when the control input is causally computed.

Clearly the escape time is a random variable provided the infinite value has zero probability. We have the following simple result, applicable in the linear gaussian case and independent of the system matrices $[A, B, C, D]$.

Lemma 1: If the noise process $\{w_k\}$ in (1) is ergodic and possesses a density function of unbounded support in all directions then almost surely the escape time of x_k is finite if $\mathcal{D}$ has any part of its boundary, $\partial\mathcal{D}$, finite. The equivalent statement holds for the pair $\{v_k\}$ and y_k.

The import of Lemma 1 is that it ensures that, in the linear gaussian case or equivalent problems able to be transformed to linear gaussian, using say Girsanov's Theorem, the finite escape of the state and/or output from any compact domain is ensured. The analysis of these processes then needs concentration on the description of the escape time rather than any attempt to establish confinement to a compact set or simply to characterize moment properties. This hearkens back to the escape time or residence time analysis of, say, [3], [4], [5], [6], [7]. These earlier treatments focus on stable continuous-time systems with small stochastic perturbations and use the Theory of Large Deviations to develop escape time characterizations as the noise power tends to zero. Our approach will maintain discrete time and deal with unstable systems with non-infinitesimal perturbations. This will not draw on Large Deviations Theory other than for comparison.

Our treatment of (1-2) endeavors to blend two distinct trains of research. The first is associated with the behavior of state estimators for such systems as treated in, say, [1], [2], [8], [9] with or without control being applied. Since the system is linear and if the applied control is known, the controlled state behavior is derivable from the estimator. The second class of problems, characterized by results such as [10], [11], [12] concentrates on the stabilization aspects of the feedback control. The distinction between the two sets of problems in the literature rests with the description of the communications channel. In the estimator problems, the communication is taken to be intermittent — that is the stochastic process $\{\gamma_k\}$ operates in a persistent fashion to cause arbitrarily long outages of communications — but the communications is not limited in bitrate and full state reconstruction occurs with any successful communication packet. While in the stabilization problem, the emphasis is on the quantizer and its associated bitrate limit and the channel is assumed not intermittent, i.e. $\gamma_k = 1$ for all k, with a deterministic maximal delay and possible additive channel noise. We pose a different set of questions dealing with escape time, which we regard as being more apropos for these problems. These focus not on limiting behaviors or mean-square stabilization but on characterizing the probability distribution function of the escape time of the system state, output or state estimate error, since in general there is no almost sure bound on either, as stated in Lemma 1.

Before launching into the analysis, it is pertinent to examine some practical sources of estimation and control

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problems associated with systems described by (1-2), since the presence of a single communications link rules out teleoperation-styled feedback control problems. Utility management of a geographically distributed system, such as a power grid or radar network, where the sensors, but not the actuators, are remotely placed and linked back to base by communications networks, is the clearest application of state estimation operating with communications limits. The study of sensor fusion and its sibling area of sensor scheduling [13], [14] has a long history in these arenas. Schweppe [15] was a pioneer in the application of such methods in power system state estimation using data of variable reliability. More generally, the study of missing data has been longstanding in statistics [16] and in array beamforming [17] with studies of estimation in high noise going back to Wiener [18], whose work was connected with the origins of radar.

Closing the loop on a system to achieve stabilization using communicated data would appear to be a more recent problem. Interestingly, Wong and Brockett study first the state estimation problem [19] and then the feedback stabilization problem [20] for systems with reliable but bandlimited communications. This extends earlier results due to Williamson [21][22] and Delchamps [23] on finite-wordlength effects on estimation and control in deterministic contexts. Sensor scheduling is also a feedback control or decision problem with a solution achievable via dynamic programming. In these earlier analyses of the estimation and the control problems the covariance function of the state estimate error or of the state itself is the target of the analysis. Thus, mean-square stabilization is the objective in [12].

For communications problems with intermittency, the conditional covariance, $\Sigma_{k+1|k}$, of the state prediction error is a random process adapted to the $\{\gamma_k\}$ sequence. Recent works have been concerned with the distributional and moment properties of this $\{\Sigma_{k+1|k}\}$ process. Thus: [1] considers the convergence of the expectation $\mathrm{E}[\Sigma_{k+1|k}]$ as $k \to \infty$; [2] quantifies the probability $\Pr\left(\Sigma_{k+1|k} > G\right)$ for given matrix G; [8] analyzes conditions for the weak (in distribution) convergence of $\Sigma_{k+1|k}$; and, [9] treats the *tail* distribution properties of this covariance. (In this latter reference 'tail' refers to the distribution on the tail σ-algebra as $k \to \infty$ and not the tail probabilities in the sense of $\Sigma_{k+1|k}$ taking on large values for any specific value of k.) The approach to be taken in this paper, as hinted at in the preceding parenthetical remark, will be to study the finite-time, sample-path behavior of state estimate errors in the situation of intermittency. Rather than to seek conditions under which asymptotic distributional or moment properties are established, our aim will be to study the escape time of the sample paths of the state prediction error, $\tilde{x}_{k+1|k}$, in the estimation case.

The paper is structured as follows. In Section II, we formulate precisely the problem statement and clarify the specific conditions that we assume to aid simplicity. Escape times and survival times are defined and refined for the estimation and stabilization problems for linear gaussian system. Section III develops the Markov chain description of escape time in a general context, which is our main result, and then goes on to make this explicit for linear gaussian systems. Section IV applies the analytic results from Section III to the specific cases of Kalman filter survival and escape. Numerical examples are given to draw the comparison with earlier results. The paper concludes in Section V.

II. Problem formulation

A. Assumptions

We commence with the communications-linked control system described by (1-2) and make the following assumptions.

Assumption 1: 1.1 A has at least one eigenvalue strictly outside the unit circle, so the system is unstable, and $[A, B]$ is stabilizable.

1.2 $[A, C]$ is observable and we define r to be the least value for which $\begin{bmatrix} C^T & A^T C^T & \dots & A^{(r-1)^T} C^T \end{bmatrix}$ is full rank.

1.3 $u_k = -K\hat{x}_{k|k-1}$ with the closed-loop matrix, $A - BK$, having all eigenvalues strictly inside the unit circle, so that constant linear state prediction feedback is applied.

1.4 w_k and v_k are independent, white, zero-mean gaussian processes with covariance matrices Q and R, which we take to be positive definite.

1.5 $\{\gamma_k\}$ is a random process taking values 0 or 1 to describe the non-arrival or arrival of a data packet. Process $\{\gamma_k\}, k = 1 \dots$ may be taken Bernoulli, as in [1], or Markov as in [8], in the analysis and simulations.

B. Kalman state estimation

The state estimate used in Assumption 1.3 is derived from the Kalman filtering equations commencing from initial values $\hat{x}_{1|0}$ and $\Sigma_{1|0}$, with $\hat{x}_{1|0}$ independent from $\{w_k\}$ and $\{v_k\}$. The Kalman filtering equations for an estimator with intermittent observations described by $\{\gamma_k\}$ are as follows. Note the γ_k appears explicitly solely in the Kalman gain L_k.

measurement update

$$\begin{aligned} L_k &= \gamma_k \left[\Sigma_{k|k-1} C^T (C \Sigma_{k|k-1} C^T + R)^{-1}\right], \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k (y_k - C\hat{x}_{k|k-1}), \\ \Sigma_{k|k} &= \Sigma_{k|k-1} - L_k C \Sigma_{k|k-1}. \end{aligned}$$

time update

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\ \Sigma_{k+1|k} &= A\Sigma_{k|k} A^T + Q, \end{aligned}$$

filtered error

$$\tilde{x}_{k|k} = x_k - \hat{x}_{k|k},$$

prediction error

$$\begin{aligned} \tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k}, \\ \tilde{x}_{k+1|k} &= (A - AL_k C)\tilde{x}_{k|k-1} + w_k - AL_k v_k. \end{aligned} \tag{3}$$

The Kalman predictor error equation (3) will be central to both the estimator and the output escape time formulations below. We note that, when $\gamma_k = 0$ then $L_k = 0$ and (3) is unstable and driven by w_k. We have the following result using the Information Filter form of the Kalman filter equations above [24], [25].

Lemma 2: For any initial prediction covariance $\Sigma_{1|0}$ and $\gamma_k = 1$ for $k = 1, \ldots, r-1$, where r is defined in Assumption 1.2, we have a finite non-negative definite $\bar{M}$ such that $\Sigma_{r+1|r} \le \bar{M}$.

C. Escape times and survival times

We consider the following two escape time problems.

Definition 2 (Estimator escape time τ_e): For given positive scalar bound G, we define the estimator escape time to be the escape time for the random sequence $\tilde{x}_{k+1|k}$ starting from the initial condition $\tilde{x}_{1|0} \sim N(0, \bar{M})$ and $\mathcal{D} = \{\|\tilde{x}_{k+1|k}\|_2 \le G\}$.
The estimator escape time is a random variable depending on the bound G, the initial covariance $\bar{M}$ and the realization.

Definition 3 (Survival time): For positive scalar bound G and stochastic process $\{\xi_k\}$ adapted to the sequence $\{\gamma_k\}$, we define the survival time for $\{\xi_k\}$ as the escape time from the domain $\mathcal{D} = \{\|\xi_k\|_2 \le G\}$ from an initial condition under the condition that $\{\gamma_k = 0, k = 1, 2, \ldots\}$.
That is, the survival time is the escape time when all communication is unsuccessful from time 1 onwards. This is a worst-case scenario but helps to elucidate the escape time properties, because the analysis is simpler.

III. Markov chain analysis

A. General Markov Chain model

Following Definition 1, given a closed domain $\mathcal{D} \subset \mathbb{R}^d$ and a stochastic process $\{\xi_k : k = 1, \ldots\}$ on $\mathbb{R}^d$, the escape time of ξ_k from $\mathcal{D}$ is described by a Markov chain. This is a general result concerning adapted processes and is not limited to linear gaussian systems nor to hyperspherical domains.

Theorem 1: For the stochastic process $\{\xi_k : k = 1, \ldots\}$ the random variable

$$J_k = \begin{cases} 1, & \text{if } \xi_k \in \mathcal{D} \text{ and } J_{k-1} = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

is a Markov process and, denoting

$$\Pi_k = \begin{bmatrix} \Pr(J_k = 1) \\ \Pr(J_k = 0) \end{bmatrix},$$

is described by the Markov chain.

$$\Pi_{k+1} = \begin{bmatrix} \alpha_k & 0 \\ 1-\alpha_k & 1 \end{bmatrix} \Pi_k,$$

$$\alpha_k = \frac{\Pr(J_{k+1} = 1, J_k = 1 \mid J_{k-1} = 1)}{\Pr(J_k = 1 \mid J_{k-1} = 1)} \tag{5}$$

The escape time of the stochastic process $\{\xi_k\}$ is the value of k when $J_k = 0$ for the first time.

Proof: From the definition (4), J_k satisfies the Markov property.

$$\Pr(J_{k+1} \mid J_k, \ldots J_1) = \Pr(J_{k+1} \mid J_k),$$

and so $\{J_k\}$ is described by a Markov chain. The transition matrix of this chain is given by

$$\begin{bmatrix} \Pr(J_{k+1} = 1 | J_k = 1) & \Pr(J_{k+1} = 1 | J_k = 0) \\ \Pr(J_{k+1} = 0 | J_k = 1) & \Pr(J_{k+1} = 0 | J_k = 0) \end{bmatrix}$$

The (1,2)-element of this matrix is zero, because there is no possibility of moving from $J_k = 0$ to $J_{k+1} = 1$. Denoting the (1,1)-element as α_k we have, using the Markov property and Bayes theorem,

$$\begin{aligned} \alpha_k &= \Pr(J_{k+1} = 1 \mid J_k = 1), \\ &= \Pr(J_{k+1} = 1 \mid J_k = 1, J_{k-1} = 1), \\ &= \frac{\Pr(J_{k+1} = 1, J_k = 1 \mid J_{k-1} = 1)}{\Pr(J_k = 1 \mid J_{k-1} = 1)}. \end{aligned}$$

■

B. Markov analysis of escape times for linear gaussian systems

Consider the following time-varying linear gaussian system,

$$\psi_{k+1} = A_k \psi_k + \omega_k, \quad \psi_0, \tag{6}$$

$$\eta_k = C_k \psi_k + \epsilon_k, \tag{7}$$

assuming independence between $\psi_0 \sim N(0, \Psi_0)$ and the white sequence

$$\begin{bmatrix} \omega_k \\ \epsilon_k \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega_k & S_k \\ S_k^T & E_k \end{bmatrix} \right).$$

As above, denote the domain to be exited as $\mathcal{D}$. As developed in Theorem 1, the escape time of gaussian process $\{\eta_k\}$ is governed by a Markov chain and the probability transition matrix is determined by a sole value α_k.

Using the linear gaussian property of the system (6-7),

$$\begin{bmatrix} \eta_{k+1} \\ \eta_k \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma(\eta_{k+1}, \eta_k) \right), \tag{8}$$

where the state and joint output covariances are:

$$\Sigma(\eta_{k+1}, \eta_k) = \begin{bmatrix} C_{k+1}\Psi_{k+1}C_{k+1}^T + E_{k+1} & C_{k+1}(A_k\Psi_k C_k^T + S_k) \\ (C_k\Psi_k A_k^T + S_k^T)C_{k+1}^T & C_k\Psi_k C_k^T + E_k \end{bmatrix}, \tag{9}$$

and $\Psi_{k+1} = A_k \Psi_k A_k^T + \Omega_k$. These expressions describe the joint density of successive η_k and the covariance update.

Theorem 2: For the time-varying linear gaussian system (6-7), the escape time of the η_k process is described by a Markov chain with α_k parameter given by

$$\alpha_k = \frac{\Theta\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma(\eta_{k+1}, \eta_k), \mathcal{D} \times \mathcal{D} \right)}{\Theta(0, C_k\Psi_k C_k^T + E_k, \mathcal{D})}, \tag{10}$$

where the function $\Theta(\mu, \Sigma, \mathcal{S})$ is the integral of the multivariate gaussian density function $\mathrm{N}(\mu, \Sigma)$ over the domain $\mathcal{S}$. $\Sigma(\eta_{k+1}, \eta_k)$ is the joint covariance of η_{k+1}, η_k in (9).

We note that the multivariate normal integral in (10) is a standard function in packages such as `matlab` (where it is called `mvncdf`). Depending on the complexity of the domain $\mathcal{D}$, (10) can be simply computed. Thereom 2 will be applied to the estimator and output escape problems shortly. The important feature of Theorem 2 is that the conditioning on $J_{k-1} = 1$ in (5) is captured by the gaussian property and the covariance-related calculation of α_k in (10).

C. Markov analysis of survival times for linear time-invariant gaussian systems

Following Definition 3, survival time describes the escape time in the case where the communication persistently fails from time 1, i.e. $\gamma_k = 0, k = 1, 2, \ldots$. As will be shown shortly, the survival time analysis for both the estimator escape and for the output escape with linear gaussian systems is a time-invariant analysis, since the Kalman gain $L_k = 0$. Accordingly, the study of survival time can be explicitly conducted, which provides insight into the processes underlying escape times, which usually require simulation for evaluation. We derive our results for the more general output escape problem and then specialize to the estimator escape problem later by taking $C = I$ and state update (3).

Consider the time-invariant version of (6-7),

$$\psi_{k+1} = A\psi_k + \omega_k, \quad \psi_0, \tag{11}$$

$$\eta_k = C\psi_k + \epsilon_k, \tag{12}$$

with $\omega_k \sim N(0, \Omega)$, $\epsilon_k \sim N(0, E)$, $\psi_0 \sim N(0, \Psi_0)$ and hence $\eta_0 \sim N(0, C\Psi_0 C^T + E)$. The escape domain is $\mathcal{D} = \{\|\eta_k\|_2 \leq Z\}$. The state and joint output covariances, Ψ_{k+1} and $H(\eta_{k+1}, \eta_k)$ satisfy

$$\Psi_{k+1} = A\Psi_k A^T + \Omega, \tag{13}$$

$$H(\eta_{k+1}, \eta_k) = \begin{bmatrix} C\Psi_{k+1}C^T + E & CA\Psi_k C^T \\ C\Psi_k A^T C^T & C\Psi_k C^T + E \end{bmatrix} \tag{14}$$

and thus α_k is given by the corresponding version of (10) with time-invariant A and C.

D. Properties of survival times for linear time-invariant gaussian systems

Denote by Φ_s the probability distribution function of the survival time for the linear time-invariant system (11-12). By Lemma 1 if $\mathcal{D}$ is bounded, $\Phi_s(t) \to 1$ as $t \to \infty$. Also from the covariance equations (13-14), it is apparent that, for fixed A and C, Φ_s is a function of t, Ψ_0, Ω, E and Z. From the covariance recursion (13) we further have the following properties holding for all $t > 0$.

$$\begin{aligned}
\Psi_0^1 \geq \Psi_0^2 &\implies \Phi_s(t, \Psi_0^1, \Omega, E, Z) \geq \Phi_s(t, \Psi_0^2, \Omega, E, Z),\\
\Omega^1 \geq \Omega^2 &\implies \Phi_s(t, \Psi_0, \Omega^1, E, Z) \geq \Phi_s(t, \Psi_0, \Omega^2, E, Z),\\
E^1 \geq E^2 &\implies \Phi_s(t, \Psi_0, \Omega, E^1, Z) \geq \Phi_s(t, \Psi_0, \Omega, E^2, Z),\\
Z^1 \leq Z^2 &\implies \Phi_s(t, \Psi_0, \Omega, E, Z^1) \geq \Phi_s(t, \Psi_0, \Omega, E, Z^2).
\end{aligned}$$

However, the time-invariance of the covariance recursion for survival time admits the following more informative result concerned with the quantified extension of survival time.

Theorem 3: For the linear time-invariant gaussian system (11-12), the distribution function of the survival time, Φ_s, satisfies

$$\Psi_0^1 \geq A^n \Psi_0^2 A^{n^T} + A^{n-1}\Omega A^{n-1^T} + \cdots + A\Omega A^T + \Omega \tag{15}$$

$$\implies \Phi_s(t, \Psi_0^1, \Omega, E, Z) \geq \Phi_s(t + n, \Psi_0^2, \Omega, E, Z). \tag{16}$$

Theorem 3 will be used when studying the effect of quantization errors on the survival time. In general, while the results of this section pertain exactly to the study of survival times, the implication is that the underlying principles describing the nature of and dependencies of escape time will remain. We shall make this implication via simulations later in the paper.

IV. Estimator escape time analysis

A. Kalman predictor escape time with intermittent data

The estimator error dynamics are given by (3) and do not depend on the feedback control sequence, u_k. The complete behavior is given by the following.

$$\begin{aligned}
L_k &= \gamma_k[\Sigma_{k|k-1}C^T(C\Sigma_{k|k-1}C^T + R)^{-1}]\\
\tilde{x}_{k+1|k} &= (A - AL_kC)\tilde{x}_{k|k-1} + w_k - AL_k v_k\\
\Sigma_{k+1|k} &= (A - AL_kC)\Sigma_{k|k-1}A^T + Q
\end{aligned}$$

Adopting the Markov Chain method of Section III, we compute $\Pr(\|\tilde{x}_{k-j+1|k-j}\|_2 \leq G, j = 0\ldots k)$ from the initial condition $\tilde{x}_{1|0} \sim N(0, \bar{M})$, where $\bar{M}$ is the starting covariance, which might be taken to be that in Lemma 2 or some other suitable value. The starting probability vector is given by

$$J_1 = \begin{bmatrix} \Pr(\|\tilde{x}_{1|0}\|_2 \leq G) \\ \Pr(\|\tilde{x}_{1|0}\|_2 > G) \end{bmatrix}.$$

The joint covariance of $\tilde{x}_{k+1|k}$ and $\tilde{x}_{k|k-1}$ is given by

$$\Sigma(\tilde{x}_{k+1|k}, \tilde{x}_{k|k-1}) \triangleq \begin{bmatrix} \Sigma_{k+1|k} & (A - AL_kC)\Sigma_{k|k-1} \\ \Sigma_{k|k-1}(A - AL_kC)^T & \Sigma_{k|k-1} \end{bmatrix}.$$

This yields the following Markov chain parameter.

$$\begin{aligned}
\alpha_k &= \frac{\Pr(\|\tilde{x}_{k+1|k}\|_2 \leq G, \|\tilde{x}_{k|k-1}\|_2 \leq G)}{\Pr(\|\tilde{x}_{k|k-1}\|_2 \leq G)},\\
&= \frac{\Theta\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma(\tilde{x}_{k+1|k}, \tilde{x}_{k|k-1}), \mathcal{D} \times \mathcal{D}\right)}{\Theta(0, \Sigma_{k|k-1}, \mathcal{D})},
\end{aligned} \tag{17}$$

where domain $\mathcal{D} = \{\tilde{x} : \|\tilde{x}\|_2 \leq G\}$ and $\Theta(\mu, \Sigma, \mathcal{F})$ is the multivariate normal distribution function for an $N(\mu, \Sigma)$ process on the set $\mathcal{F}$.

B. Example 1: Estimator survival time

Because (17) provides and explicit formula for α_k using the `matlab` function `mvncdf`, we study first the survival time of the estimator. This example, while simple, permits revealing comparisons between the escape/survival time analysis and the moment-based methods of [1], [2]. We take the following conditions:

– a scalar state example with the following parameters: $A = 1.5, B = 1, C = 1, D = 0, Q = 0.005, R = 0.001, G = 10$.
– the initial condition of the estimator, $\tilde{x}_{1|0} \sim N(0, \bar{M})$, where $\bar{M}$ is the maximal value of the state-estimate covariance from Lemma 2. $\bar{M} = ARA^T + Q = 0.00725$. This corresponds to the largest possible covariance immediately after receiving a data sample from, say, having $\gamma_0 = 1$.
– in line with Definition 3 for survival time, we take $\{\gamma_k = 0 : k = 1, 2, \dots\}$.

Using Theorems 1 and 2 with (13-14) yielding the covariance updates, the explicit probability density function for the estimator survival time is computed and plotted in Figure 1. The calculation of the density function of τ_s was verified,

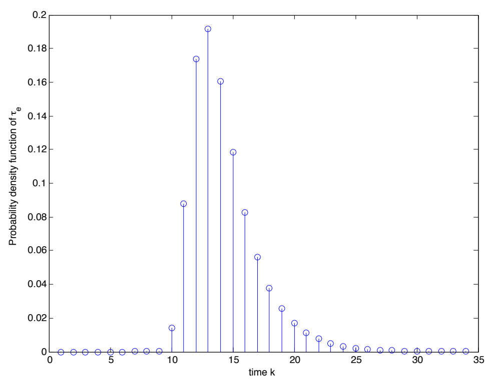

Fig. 1. Probability density function of estimator survival time for Example 1

needlessly, by simulation. Figure 2 displays three sample $\tilde{x}_{k+1|k}$ trajectories of differing survival times: 8, 12, and 19, from a Monte Carlo simulation with initial covariance $\Sigma_{1|0} = 0.00725$. We have selected three trajectories which escaped with positive state-estimate error values. From the

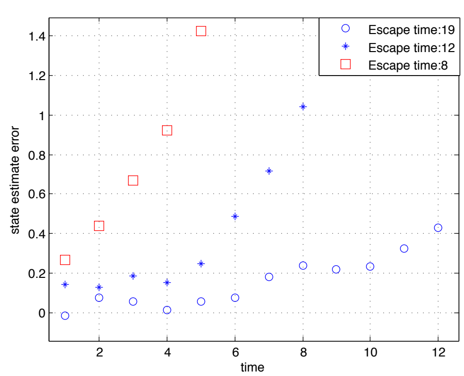

Fig. 2. Three different estimator survival time trajectory samples for Example 1 with initial error covariance $\bar{M} = 0.00725$.

problem description, the state-estimate error satisfies,

$$\tilde{x}_{k+1|k} = A^k \tilde{x}_{1|0} + \sum_{j=0}^{k-1} A^{k-j} w_j,$$

which displays the relative importance of the random terms. Since $w_k \sim N(0, 0.005)$, once the state exceeds roughly 0.2 (three σ) in magnitude it becomes most unlikely that w_k will arise to bring the error back to a small magnitude. Once the three trajectories exceed this value, we see that they all escape with roughly similar behavior, i.e. at a rate of 1.5^k. The difference between the trajectories lies in their residence time in the neighborhood $|\tilde{x}| < 0.2$. Here the disturbance process is as likely to drag the error back towards small magnitudes as it is to increase the magnitude, resulting in some trajectories remaining close to zero for an extended time. This is captured by the Markov analysis.

C. Markov versus covariance comparison

Comparing the Markov results to the covariance calculations of [1], [2], we see that, in both survival time ($\gamma_k = 0, k = 1, 2, \dots$) analyses, the covariance is given by

$$\Sigma_{k+1|k} = A^k \Sigma_{1|0} A^{k^T} + \sum_{j=0}^{k-1} A^j Q A^{j^T}.$$

The covariance $\Sigma_{k+1|k}$ may be used to compute directly the probabilities $\Pr(\|\tilde{x}_{k+1|k}\|_2 \le G)$ and $\Pr(\|\tilde{x}_{k+1|k}\|_2 > G)$ using the normal cumulative distribution function. Whence, the escape probability may be calculated via

$$\Pr(\left\|\tilde{x}_{k+1|k}\right\|_2 > G) \times \prod_{j=1}^{k} \Pr(\left\|\tilde{x}_{k+1-j|k-j}\right\|_2 < G). \quad (18)$$

Figure 3 shows the survival time probability density function computed using this method based on covariances versus the exact calculation using the Markov approach. This is done with the same parameters as Example 1 save for $A = 1.01$ to amplify the distinction. This figure illustrates that the analysis based on covariance alone, which presumes independence of the errors, is significantly more pessimistic than that using the Markov model. This is tied to the covariance being dominated by errors which escape early and to the independence assumed for successive times.

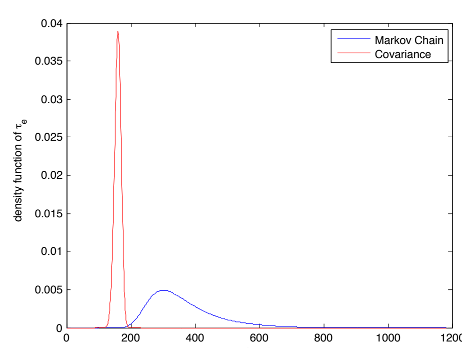

Fig. 3. Comparison between probability density function calculations using the Markov method and using (18) based on covariances alone; Example 1 with A=1.01

D. Escape time of Sinopoli's example

In [1] and subsequent papers, Sinopoli et al. define a critical value of the probability, P_γ, of packet arrival above which the expected covariance matrix, $\mathrm{E}[\Sigma_{k+1|k}]$, is finite for all k. For the example:

$$A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix}, \quad Q = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

$$\bar{M} = \begin{bmatrix} 67.36 & -.052 \\ -.052 & 22.73 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad R = 2.5, \quad G = 100,$$

$P_\gamma > 0.15$ suffices for finiteness of $\Sigma_{k+1|k}$ in the linear, time-invariant estimator case. Figure 4 displays the probability distribution functions of survival time and of escape time for the case where P_γ is chosen as 0.2, and therefore yields a finite covariance. This example indicates that, despite

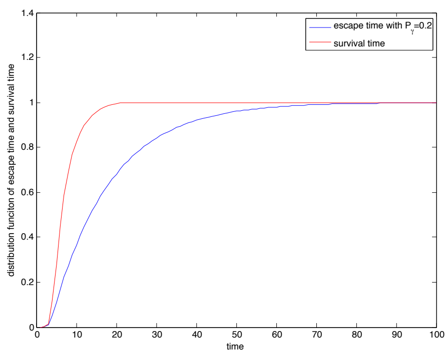

Fig. 4. Probability distribution function of survival time (red, leftmost) and escape time (blue, rightmost) for bound $G = 100$ for Sinopoli's state estimator example with $P_\gamma = 0.2$, above the critical lower value for finite covariance.

the finiteness of $\Sigma_{k+1|k}$, the state estimate error exceeds the bound 100. Indeed, according to Lemma 1, eventually it exceeds any given bound in this gaussian case.

V. Comments and Conclusions

We have presented an approach to the study of state estimation with intermittent measurements. This is based on escape time analysis and computation, which is compared to earlier works in estimation and stabilization with differing descriptions of the properties – finite expected covariance and mean-square stabilization – and of the communication system – intermittent but exact and quantized but certain. The central result is that the escape time can be described by a Markov chain, which in the linear gaussian case is easily computed. This yields much less conservative evaluation of behavior than covariance calculations. In the full version of this paper [26], these issues will be used to examine the bitrate assignment questions associated with retransmission schemes for dropped data.

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