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Economic model predictive control with self-tuning terminal weight

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***Abstract*— In this paper, we propose an economic model predictive control (MPC) framework with a self-tuning terminal weight, which builds on a recently proposed MPC algorithm with a generalized terminal state constraint. First, given a general time-varying terminal weight, we derive an upper bound on the closed-loop average performance which depends on the limit value of the predicted terminal state. After that, we derive conditions for a self-tuning terminal weight such that bounds for this limit value can be obtained. Finally, we propose several update rules for the self-tuning terminal weight and analyze their respective properties. We illustrate our findings with several examples.**

I. INTRODUCTION

Model predictive control has become a very popular and successful control strategy in recent years, which is based on the repeated solution of a finite horizon optimal control problem. So far, the most commonly used MPC formulation is that of a tracking problem, meaning that the involved cost function is assumed to be positive definite with respect to a certain setpoint or trajectory to be tracked (see, e.g., [1]). However, this basic assumption need not be satisfied in general, which in particular is the case when optimizing the economics of a process (for a recent example, see, e.g., [2]). In order to overcome this limitation, an *economic* MPC formulation has recently been proposed [3], where this assumption is not made but a general cost function can be used. Within such an economic MPC framework, different properties of the resulting closed-loop system have been studied such as average performance or convergence to the optimal steady-state. To this end, different assumptions and variants of economic MPC have been used, including terminal constraints [3–6], Lyapunov-like constraints [7] and certain controllability conditions [8].

In this paper we consider an economic MPC setup involving a *generalized* terminal state constraint, meaning that the endpoint of the predicted state sequence has to be equal to some arbitrary steady-state in contrast to the optimal one as in [3, 4]. Such a setup has been proposed in the context of tracking MPC (see, e.g., [9–12]) and recently also within an economic MPC framework [11–14]. The big advantage of such a generalized terminal constraint is that a possibly much larger region of attraction is obtained, and a loss of feasibility can be prevented which otherwise might occur if the cost function (and hence the optimal steady-state) changes online. In particular, in [13, 14], the authors propose to use a slightly modified economic cost function and in addition an offset cost function which penalizes the distance of the predicted terminal steady-state $x(N|t)$ to the optimal steady-state x^*. On the other hand, in [11, 12] the authors use the original (economic) cost function and in addition a terminal cost term which penalizes the economic cost of the predicted terminal steady-state $x(N|t)$. It is shown that if the terminal weight is large enough, then the cost of the predicted terminal steady-state will be arbitrarily close to the cost of the best reachable steady-state. Furthermore, under additional assumptions and by further modifying the MPC algorithm, i.e., if necessary, following the previously optimal solution, the cost of the predicted terminal steady-state, and hence also the average performance of the closed-loop system, will eventually be arbitrarily close to the cost of the best overall steady-state [12].

In this work, we further study an economic MPC algorithm with generalized terminal constraint, using a setup similar to [11]. In particular, we develop an economic MPC algorithm with a self-tuning terminal weight. The advantage of such a self-tuning terminal weight in comparison to a fixed terminal weight is that we do not necessarily need to make the terminal weight large in order to guarantee certain performance properties. This is desirable both from a numerical point of view as well as in order not to modify the original economic cost function too much. In fact, we illustrate with a simple example that in some cases, a smaller terminal weight leads to a better closed-loop average performance than a large terminal weight. Furthermore, we want to give (average) performance guarantees for the closed-loop system without further modifying the MPC algorithm as in [12]. The remainder of this paper is organized as follows. In Section II, some preliminaries and the precise problem statement are introduced; furthermore, we illustrate by means of an example that larger values of the terminal weight do not necessarily lead to a better closed-loop average performance. Section III then gives an upper bound for the closed-loop average performance when using a general time-varying terminal weight. In Section IV, we consider certain self-tuning update rules for the terminal weight and show that the cost of the predicted terminal steady-state (and hence also the closed-loop average performance) can be upper bounded by the best "robustly" achievable steady-state cost of the ω-limit set of the associated closed-loop system. We illustrate

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our findings and the different update rules with various examples. Section V concludes the paper.

II. Preliminaries and problem setup

We consider discrete-time nonlinear systems of the form

$$x(t+1) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{1}$$

with[1] $t \in \mathbb{I}_{\geq 0}$, where $x \in \mathbb{X} \subseteq \mathbb{R}^n$, $u \in \mathbb{U} \subseteq \mathbb{R}^m$, and f is assumed to be continuous. The system is subject to (possibly coupled) state and input constraints $(x, u) \in \mathbb{Z}$ for some compact set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. Denote by $\mathbb{Z}_\mathbb{X}$ the projection of $\mathbb{Z}$ on $\mathbb{X}$. Define the set of steady-states which are reachable in $N > 0$ steps from a point $y \in \mathbb{Z}_\mathbb{X}$ as

$$\begin{aligned}&\mathcal{X}_N(y) := \{x \in \mathbb{Z}_\mathbb{X} : \ \exists \mathbf{u} \in \mathbb{U}^{(N+1)} \ s.t. \ x(0) = y,\\ &x(j+1) = f(x(j), u(j)) \ \forall j \in \mathbb{I}_{[0,N-1]}, x(N) = x,\\ &x = f(x, u(N)), \ (x(j), u(j)) \in \mathbb{Z} \ \forall j \in \mathbb{I}_{[0,N]}\}.\end{aligned} \tag{2}$$

Note that for each $y \in \mathbb{Z}_\mathbb{X}$, the set $\mathcal{X}_N(y)$ is compact as $\mathbb{Z}$ is compact and f is continuous. System (1) is equipped with a stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, which is assumed to be continuous. Denote the best achievable steady-state cost from a point $y \in \mathbb{Z}_\mathbb{X}$ by[2]

$$\begin{aligned}\ell_{\min}(y) := \min_{x,u} \ & \ell(x, u)\\ \text{s.t. } & x \in \mathcal{X}_N(y), \ (x, u) \in \mathbb{Z},\\ & x = f(x, u)\end{aligned} \tag{3}$$

Furthermore, in the following we need the notion of the best *robustly* achievable steady-state cost from a point $y \in \mathbb{Z}_\mathbb{X}$, which we define as follows. For each $\varepsilon \geq 0$, denote by

$$\ell_{\min}(y, \varepsilon) := \sup_{z \in B_\varepsilon(y) \cap \mathbb{Z}_\mathbb{X}} \ell_{\min}(z) \tag{4}$$

the supremum of the best achievable steady-state cost on the set[3] $B_\varepsilon(y) \cap \mathbb{Z}_\mathbb{X}$. With this, we define the best robustly achievable steady-state cost from a point $y \subseteq \mathbb{Z}_\mathbb{X}$ as

$$\bar{\ell}_{\min}(y) := \lim_{\varepsilon \searrow 0} \ell_{\min}(y, \varepsilon). \tag{5}$$

Note that the limit in (5) exists as $\ell_{\min}(y, \varepsilon)$ is monotonically decreasing when ε decreases to zero. From the definitions in (3) and (5), it immediately follows that for each $y \in \mathbb{Z}_\mathbb{X}$ we have $\ell_{\min}(y) \leq \bar{\ell}_{\min}(y)$; however, equality does in general not hold as $\ell_{min}(y, \varepsilon)$ is not necessarily continuous in ε at $\varepsilon = 0$. Finally, let (x_s, u_s) denote an overall optimal steady-state, i.e., (x_s, u_s) satisfies

$$\ell(x_s, u_s) = \min_{(x,u) \in \mathbb{Z}, x = f(x,u)} \ell(x, u); \tag{6}$$

note that without loss of generality, we can assume that $\ell(x_s, u_s) = 0$.

[1] $\mathbb{I}_{\geq 0}$ denotes the set of nonnegative integers, and $\mathbb{I}_{[a,b]}$ the set of all integers in the interval $[a, b] \subseteq \mathbb{R}$.

[2] In the following, if a minimum is taken over the empty set, then by convention the minimum is $+\infty$.

[3] $B_\varepsilon(y)$ denotes the ball of radius ε around the point y, i.e., $B_\varepsilon(y) := \{x \in \mathbb{R}^n : |x - y| \leq \varepsilon\}$.

Now consider the following economic MPC algorithm, which is a variation of the one introduced in [11]. Namely, at each time t with $x := x(t)$, the following optimization problem is solved:

$$\begin{aligned}\min_{\mathbf{u}} J(x, \mathbf{u}, \beta) = &\sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))\\ &+ \beta(t)\ell(x(N|t), u(N|t))\end{aligned} \tag{7}$$

subject to

$$x(0|t) = x \tag{8a}$$

$$x(k+1|t) = f(x(k|t), u(k|t)) \quad k \in \mathbb{I}_{[0,N-1]} \tag{8b}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{[0,N]} \tag{8c}$$

$$x(N|t) = f(x(N|t), u(N|t)), \tag{8d}$$

$$\ell(x(N|t), u(N|t)) \leq \lambda(t). \tag{8e}$$

for some possibly time-varying terminal weight β and λ specified in the following. As pointed out in the Introduction, the special feature of the above optimization problem is the generalized terminal constraint in (8d), meaning that the predicted terminal state $x(N|t)$ has to be equal to an arbitrary steady-state and not to a specific one. Denote the optimal solution to problem (7)–(8) by $\mathbf{u}^* := [u^*(0|t)^T, \ldots, u^*(N|t)^T]^T$ and the corresponding state sequence by $\mathbf{x}^* := [x^*(0|t)^T, \ldots, x^*(N|t)^T]^T$. As usual in MPC, the first part of the optimal input sequence, $u^*(0|t)$, is applied to the system at time t. The optimal value function is denoted by $V(x, \beta) := J(x, \mathbf{u}^*, \beta)$, which depends on the terminal weight β. The parameter λ is updated according to the cost of the previous terminal state, i.e., the following closed-loop system is obtained:

$$\begin{aligned}x(t+1) &= f(x(t), u^*(0|t)),\\ \lambda(t+1) &= \ell(x^*(N|t), u^*(N|t)).\end{aligned} \tag{9}$$

From (8d)–(9), it follows that the sequence $\lambda(t)$ is non-increasing and bounded from below (by $0 = \ell(x_s, u_s)$), hence it converges. Denote the limit by $\lambda_\infty := \lim_{t \to \infty} \lambda(t) \geq 0$. Note that the sequence $\lambda(t)$ is convergent irrespective of the evolution of the terminal weight β, however, the limit λ_∞ does in general depend on β. As described in the Introduction, in the following we want to analyze the behavior of the closed-loop system (9) when using a self-tuning, time-varying terminal weight β which is not unnecessarily large, and without further modifying the MPC algorithm as in [12, Algorithm 3]. Before doing so, we shortly motivate our research with a simple example showing that larger values of β do not necessarily lead to smaller values of λ_∞ and to a better average performance of the closed-loop system.

Example 1: Consider the system $x(t+1) = x(t)u(t)$ with state and input constraint set $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$ with $\mathbb{U} = [-1.2, 1.2]$ and $\mathbb{X} = [-5, 5]$, cost function

$$\ell(x, u) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 + \frac{9}{4} + (u-1)^2$$

and prediction horizon $N = 1$. Figure 1 shows closed-loop state sequences for four different constant values of β. As can be seen, x converges to 0 and hence $\lambda_\infty = \ell(0, 1) = 9/4$ for both $\beta \equiv 1.5$ and $\beta \equiv 5$, whereas x converges to 3 and hence $\lambda_\infty = \ell(3, 1) = 0$ for both $\beta \equiv 0.1$ and $\beta \equiv 1$. □

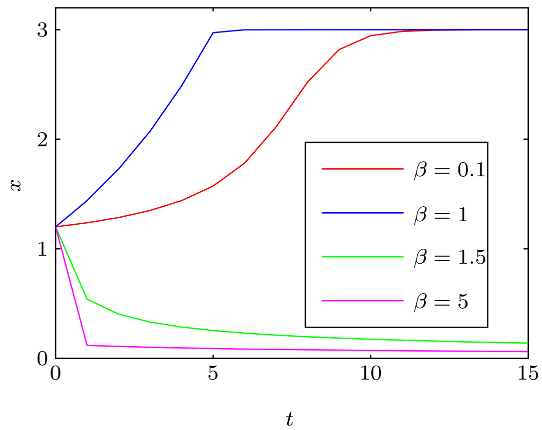


Fig. 1: Closed-loop state sequences for different values of β in Example 1.

III. Average performance with time-varying terminal weight

In this section, we study the average performance of the closed-loop system (9) when the terminal weight β in (7) is time-varying. Namely, we assume that β evolves according to some update rule B, i.e.,

$$\beta(t+1) = B(\beta(t), x(t), \lambda(t)), \quad \beta(0) = \beta_0 \geq 0. \tag{10}$$

In order to prove certain bounds on the closed-loop average performance, we consider the following two assumptions on β. Later, in Section IV-B, we propose several update rules B such that these assumptions are satisfied. Let $\gamma(t) := \beta(t+1) - \beta(t)$.

Assumption 1: The update rule B in (10) is such that the sequence $\beta(\cdot)$ satisfies $\gamma(t) \leq c$ and $\beta(t) \geq \underline{\beta}$ for all $t \in \mathbb{I}_{\geq 0}$ and some constants $c, \underline{\beta} \in \mathbb{R}$, and $\limsup_{t\to\infty} \gamma(t) \leq 0$.

Assumption 2: The update rule B in (10) is such that the sequence $\beta(\cdot)$ satisfies $\gamma(t) \leq c$ and $\beta(t) \geq \underline{\beta}$ for all $t \in \mathbb{I}_{\geq 0}$ and some constants $c, \underline{\beta} \in \mathbb{R}$, and $\liminf_{t\to\infty} \beta(t) < \infty$.

Theorem 1: Assume that the optimization problem (7)–(8) is feasible at $t = 0$. Consider the closed-loop system (9) and (10). If Assumption 1 is satisfied, then

$$\limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T} \leq \lambda_\infty. \tag{11}$$

If Assumption 2 is satisfied, then

$$\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T} \leq \lambda_\infty. \tag{12}$$

Proof: As usual in MPC, a feasible solution to problem (7)–(8) at time $t+1$ is given by the endpiece of the previously optimal solution appended by the steady-state input, i.e., $\hat{\mathbf{u}}(t+1) := [u^*(1|t)^T, \ldots, u^*(N|t)^T, u^*(N|t)^T]^T$. This means that the optimization problem (7)–(8) is recursively feasible. With this, we obtain

$$\begin{aligned} &V(x(t+1), \beta(t+1)) - V(x(t), \beta(t)) \\ &\leq J(x(t+1), \hat{\mathbf{u}}(t+1), \beta(t+1)) - J(x(t), \mathbf{u}^*(t), \beta(t)) \\ &= (1+\gamma(t))\ell(x^*(N|t), u^*(N|t)) - \ell(x(t), u(t)) \end{aligned} \tag{13}$$

As discussed above, the sequence $\ell(x^*(N|t), u^*(N|t))$ is non-increasing in t and converges to λ_∞ for $t \to \infty$. This means that $\varepsilon(t) := \ell(x^*(N|t), u^*(N|t)) - \lambda_\infty$ converges to zero for $t \to \infty$. Furthermore, $0 \leq \varepsilon(t) \leq \varepsilon(0) < \infty$ for all $t \in \mathbb{I}_{\geq 0}$, where the last inequality follows from continuity of ℓ and compactness of $\mathbb{Z}$. Summing the inequality in (13), for each $T \geq 1$ we obtain

$$\begin{aligned} &V(x(T), \beta(T)) - V(x_0, \beta_0) \\ &\leq \sum_{t=0}^{T-1} \big[(1+\gamma(t))(\lambda_\infty + \varepsilon(t)) - \ell(x(t), u(t))\big] \end{aligned} \tag{14}$$

Now first consider the case where Assumption 1 is satisfied. Taking averages on both sides of (14), we obtain

$$\begin{aligned} &\liminf_{T\to\infty} (1/T)\big(V(x(T), \beta(T)) - V(x_0, \beta_0)\big) \\ &\leq \liminf_{T\to\infty} (1/T)\Big(T\lambda_\infty + \\ &\qquad + \sum_{t=0}^{T-1} \big[\gamma(t)\lambda_\infty + (1+\gamma(t))\varepsilon(t) - \ell(x(t), u(t))\big]\Big) \\ &\leq \lambda_\infty + \liminf_{T\to\infty} (1/T)\Big(\sum_{t=0}^{T-1} -\ell(x(t), u(t))\Big) \\ &\qquad + \limsup_{T\to\infty} (1/T)\Big(\sum_{t=0}^{T-1} \big[\gamma(t)\lambda_\infty + (1+\gamma(t))\varepsilon(t)\big]\Big) \\ &\leq \lambda_\infty - \limsup_{T\to\infty} (1/T)\Big(\sum_{t=0}^{T-1} \ell(x(t), u(t))\Big), \end{aligned} \tag{15}$$

where the last inequality follows from the fact that $\lim_{t\to\infty} \varepsilon(t) = 0$, $\limsup_{t\to\infty} \gamma(t) \leq 0$, $\lambda_\infty \geq 0$, and $0 \leq \varepsilon(t) \leq \varepsilon(0) < \infty$ and $\gamma(t) \leq c < \infty$ for all $t \in \mathbb{I}_{\geq 0}$. On the other hand, as $\beta(t) \geq \underline{\beta}$ for all $t \in \mathbb{I}_{\geq 0}$ and furthermore ℓ is continuous and $\mathbb{Z}$ compact, by definition of the cost function J in (7) there exists a constant $\underline{V} \in \mathbb{R}$ such that $V(x(t), \beta(t)) \geq \underline{V}$ for all $t \in \mathbb{I}_{\geq 0}$. Hence we obtain

$$\begin{aligned} &\liminf_{T\to\infty} \frac{V(x(T), \beta(T)) - V(x_0, \beta_0)}{T} \\ &\geq \liminf_{T\to\infty} \frac{V(x(T), \beta(T))}{T} + \liminf_{T\to\infty} \frac{-V(x_0, \beta_0)}{T} \\ &\geq \liminf_{T\to\infty} (\underline{V}/T) + \liminf_{T\to\infty} \frac{-V(x(0), \beta_0)}{T} = 0. \end{aligned} \tag{16}$$

Combining (15) and (16) yields that (11) is satisfied, which concludes the proof of the first statement of Theorem 1.

Second, consider the case where Assumption 2 is satisfied. Then, there exists an infinite sequence of time instants $\{t_i\} \subseteq \mathbb{I}_{\geq 0}$ such that $\beta(t_i) \leq \beta_\infty$ for all t_i and some $\beta_\infty < \infty$. Using the fact that $\sum_{t=0}^{t_i-1} \gamma(t) = \beta(t_i) - \beta_0$ by definition of γ, and $\gamma(t) \leq c$ for all $t \in \mathbb{I}_{\geq 0}$ by assumption, from (14) we obtain that

$$\begin{aligned} &V(x(t_i), \beta(t_i)) - V(x_0, \beta_0) \\ &\leq (t_i + \beta_\infty - \beta_0)\lambda_\infty + \sum_{t=0}^{t_i-1} \big[(1+c)\varepsilon(t) - \ell(x(t), u(t))\big]. \end{aligned}$$

Taking averages on both sides, we have

$$\begin{aligned}
&\liminf_{T\to\infty}(1/T)\big(V(x(T),\beta(T))-V(x_0,\beta_0)\big)\\
&\le \liminf_{i\to\infty}(1/t_i)\big(V(x(t_i),\beta(t_i))-V(x_0,\beta_0)\big)\\
&\le \liminf_{i\to\infty}(1/t_i)\Big((t_i+\beta_\infty-\beta_0)\lambda_\infty\\
&\qquad\qquad +\sum_{t=0}^{t_i-1}\big[(1+c)\varepsilon(t)-\ell(x(t),u(t))\big]\Big)\\
&\le \lambda_\infty+\liminf_{i\to\infty}(1/t_i)\sum_{t=0}^{t_i-1}\big[(1+c)\varepsilon(t)-\ell(x(t),u(t))\big]\\
&\le \lambda_\infty-\limsup_{i\to\infty}(1/t_i)\sum_{t=0}^{t_i-1}\ell(x(t),u(t))\\
&\le \lambda_\infty-\liminf_{T\to\infty}(1/T)\sum_{t=0}^{T-1}\ell(x(t),u(t)), \qquad (17)
\end{aligned}$$

where the fourth inequality is again due to the fact that $\lim_{t\to\infty}\varepsilon(t)=0$ and $\varepsilon(t)\le\varepsilon(0)<\infty$ for all $t\in\mathbb{I}_{\ge 0}$, and the last inequality follows from the fact that $\{t_i\}$ is an infinite subsequence of $\mathbb{I}_{\ge 0}$. Combining (17) with (16) (which is valid independent of whether Assumption 1 or 2 is satisfied) results in (12), which concludes the proof of the second statement of Theorem 1. □

Remark 1: Assumption 2 is weaker than requiring β to be bounded. In fact, if Assumption 2 is strengthened to β being bounded, then again the stronger conclusion (11) instead of (12) follows. Namely, if $\beta(t)\le\beta_\infty$ for all $t\in\mathbb{I}_{\ge 0}$, then the calculations in (17) hold for all $t\in\mathbb{I}_{\ge 0}$ and not only for the subsequence $\{t_i\}\subseteq\mathbb{I}_{\ge 0}$. Then, the second-to-last line of (17) together with (16) imply that (11) is satisfied. □

IV. Self-tuning terminal weights

Having established that λ_∞ is an upper bound for the average performance of the closed-loop system, we would like to determine update rules B such that β satisfies Assumption 1 or Assumption 2 and which lead to a small value of λ_∞. Clearly, $\lambda_\infty\ge\ell(x_s,u_s)=0$ for each update rule B, with $\ell(x_s,u_s)$ defined in (6). As demonstrated in Example 1, it is not necessarily true that larger values of β lead to smaller values of λ_∞. In general, and without further assumptions on the system and the cost function, the best one can hope for is that λ_∞ will be equal to the best achievable steady-state cost of the ω-limit set corresponding to the specific update rule. In the following, we will formalize this notion and give a sufficient condition for update rules in order that this property holds true.

A. Specifications for update rules B

Let $\omega_B(x_0)$ be the ω-limit set of the closed-loop state sequence (9) starting at x_0 and using the update rule B (10). We then have the following result:

Theorem 2: (i) For each update rule B which is such that β satisfies

$$\lambda_\infty-\liminf_{t\to\infty}\ell_{\min}(x(t))>0 \;\Rightarrow\; \liminf_{t\to\infty}\beta(t)=\infty, \quad (18)$$

it holds that $\lim_{t\to\infty}\ell_{\min}(x(t))$ exists and

$$\lambda_\infty=\lim_{t\to\infty}\ell_{\min}(x(t))\le\inf_{y\in\omega_B(x_0)}\bar{\ell}_{\min}(y). \quad (19)$$

(ii) For each update rule B which is such that β satisfies

$$\lambda_\infty-\limsup_{t\to\infty}\ell_{\min}(x(t))>0 \;\Rightarrow\; \limsup_{t\to\infty}\beta(t)=\infty, \quad (20)$$

it holds that

$$\lambda_\infty=\limsup_{t\to\infty}\ell_{\min}(x(t))\le\sup_{y\in\omega_B(x_0)}\bar{\ell}_{\min}(y). \quad (21)$$

Proof: (i) First, we show that $\lambda_\infty-\liminf_{t\to\infty}\ell_{\min}(x(t))\le 0$ if β satisfies (18). Namely, assume for contradiction that $\lambda_\infty-\liminf_{t\to\infty}\ell_{\min}(x(t))>0$. According to (18), this implies that $\liminf_{t\to\infty}\beta(t)=\infty$. However, in [12, Proposition 2] it was shown that for each $\varepsilon>0$, if $\beta>a/\varepsilon$ for some finite constant $a>0$, then $\ell(x^*(N|t),u^*(N|t))\le\ell_{\min}(x(t))+\varepsilon$ for all $x(t)$. But then, as $\liminf_{t\to\infty}\beta(t)=\infty$, we obtain that $\lambda_\infty=\liminf_{t\to\infty}\ell_{\min}(x(t))$, which contradicts our assumption. Thus it holds that $\lambda_\infty-\liminf_{t\to\infty}\ell_{\min}(x(t))\le 0$, which by the definition of λ_∞ and $\ell_{\min}$ implies that $\lim_{t\to\infty}\ell_{\min}(x(t))$ exists and $\lambda_\infty-\lim_{t\to\infty}\ell_{\min}(x(t))=0$. This establishes the equality in (19). Second, according to the definition of the ω-limit set $\omega_B(x_0)$, for each $y\in\omega_B(x_0)$ and each $\varepsilon>0$ there exists an infinite sequence of time instants $\{t_i^y\}$ such that $x(t_i^y)\in B_\varepsilon(y)\cap\mathbb{Z}_{\mathbb{X}}$. But this implies that

$$\lambda_\infty=\lim_{t\to\infty}\ell_{\min}(x(t))=\lim_{i\to\infty}\ell_{\min}(x(t_i^y))\le\ell_{\min}(y,\varepsilon).$$

As this holds for each $\varepsilon>0$ and each $y\in\omega_B(x_0)$, we obtain

$$\lambda_\infty=\lim_{t\to\infty}\ell_{\min}(x(t))\le\inf_{y\in\omega_B(x_0)}\bar{\ell}_{\min}(y),$$

which establishes statement (i) of the Theorem.

(ii) The proof of this statement is similar the proof of statement (i). We first show that $\lambda_\infty-\limsup_{t\to\infty}\ell_{\min}(x(t))\le 0$ if β satisfies (20). Namely, assume for contradiction that $\lambda_\infty-\limsup_{t\to\infty}\ell_{\min}(x(t))>0$. According to (20), this implies that $\limsup_{t\to\infty}\beta(t)=\infty$. However, in [12, Proposition 2] it was shown that for each $\varepsilon>0$, if $\beta>a/\varepsilon$ for some finite constant $a>0$, then $\ell(x^*(N|t),u^*(N|t))\le\ell_{\min}(x(t))+\varepsilon$ for all $x(t)$. But then, as $\limsup_{t\to\infty}\beta(t)=\infty$, we obtain that $\lambda_\infty\le\limsup_{t\to\infty}\ell_{\min}(x(t))$, which contradicts our assumption. By definition of λ_∞ and $\ell_{\min}$, the above inequality has to hold with equality, i.e., $\lambda_\infty=\limsup_{t\to\infty}\ell_{\min}(x(t))$, which establishes the equality in (21). Now let $\{t_i\}$ be an infinite sequence of time instants such that $\lim_{i\to\infty}\ell_{\min}(x(t_i))$ exists and satisfies $\lim_{i\to\infty}\ell_{\min}(x(t_i))=\limsup_{t\to\infty}\ell_{\min}(x(t))$. According to the definition of the ω-limit set $\omega_B(x_0)$, there exists a point $y^*\in\omega_B(x_0)$ and for each $\varepsilon>0$ an infinite subsequence $\{t_r\}$ of the sequence $\{t_i\}$ such that $x(t_r)\in B_\varepsilon(y^*)\cap\mathbb{Z}_{\mathbb{X}}$. But this implies that $\limsup_{t\to\infty}\ell_{\min}(x(t))=\lim_{r\to\infty}\ell_{\min}(x(t_r))\le\ell_{\min}(y^*,\varepsilon)$. As this holds for each $\varepsilon>0$, we obtain

$$\lambda_\infty=\limsup_{t\to\infty}\ell_{\min}(x(t))\le\bar{\ell}_{\min}(y^*)\le\sup_{y\in\omega_B(x_0)}\bar{\ell}_{\min}(y),$$

which establishes statement (ii) of the Theorem. □

In Theorem 2, condition (20) is weaker than (18) and hence more update rules B will be likely to fulfill it. However, then also the resulting conclusion (21) which can be made is weaker than (19). Namely, in case (i), we can ensure that the best achievable steady-state cost along the closed-loop system converges, i.e., $\lim_{t\to\infty} \ell_{\min}(x(t))$ exists. Furthermore, λ_∞ is equal to this limit, which is at least as good as the minimum of the best robustly achievable steady-state cost on the ω-limit set of the closed-loop system. On the other hand, in case (ii), we cannot necessarily ensure that $\lim_{t\to\infty} \ell_{\min}(x(t))$ exists, but only that λ_∞ is equal to $\limsup_{t\to\infty} \ell_{\min}(x(t))$, which in turn is at least as good as the supremum of the best robustly achievable steady-state cost on the ω-limit set of the closed-loop system. Furthermore, we remark that if $\omega_B(x_0)$ is just a singleton, or, more general, if $\bar{\ell}_{\min}(y_1) = \bar{\ell}_{\min}(y_2)$ for all $y_1, y_2 \in \omega_B(x_0)$, then the right hand sides of (21) and (19), i.e., the upper bounds for λ_∞, are the same.

B. Several update rules B and their properties

In the following, we propose and discuss several different update rules for β which ensure that the conditions of Theorems 1 and 2 are satisfied. We start with rather simple update rules which lead to monotonically increasing β. To this end, let $\delta(t) := \ell(x^*(N|t), u^*(N|t)) - \ell_{\min}(x(t))$.

- Update rule 1: $B_1(\beta(t), x(t), \lambda(t)) := \beta(t) + d$ for some $d > 0$.
- Update rule 2: $B_2(\beta(t), x(t), \lambda(t)) := \beta(t) + \alpha(\delta(t))$ for some $\alpha \in \mathcal{K}$.

The appeal of update rules 1 and 2 is clearly their simplicity. A further advantage of update rule 1 is the fact that in contrast to update rule 2, $\ell_{\min}(x(t))$ does not have to be known at each time t. However, with update rule 1, in any case $\beta \to \infty$, which might not be desirable.

Lemma 1: The update rules 1 and 2 are such that β satisfies (18); furthermore, for update rule 2 Assumption 1 is satisfied.

Proof: With update rule 1, β trivially satisfies (18) as $\liminf_{t\to\infty} \beta(t) = \infty$ independent of the behavior of $x(t)$. For update rule 2, consider the following. If $\lambda_\infty - \liminf_{t\to\infty} \ell_{\min}(x(t)) > 0$, then $\delta(t_i) \geq c > 0$ for an infinite sequence of time instants t_i and some constant $c > 0$. But this immediately implies that $\liminf_{t\to\infty} \beta(t) = \infty$ as $\alpha \in \mathcal{K}$. Hence with both update rules 1 and 2, β satisfies (18). In order to establish the second claim of the lemma, consider the following. For update rule 2, we have $\gamma(t) = \beta(t+1) - \beta(t) = \alpha(\delta(t))$. But $\delta(t)$ and hence also $\gamma(t)$ is bounded above by some finite constant due to continuity of ℓ and compactness of $\mathbb{Z}$, i.e., $\gamma(t) \leq c < \infty$ for all $t \in \mathbb{I}_{\geq 0}$. Furthermore, $\beta(t) \geq \beta_0 =: \underline{\beta}$ for all $t \geq 0$. Finally, as (18) is satisfied, by Theorem 2(i) we conclude that $\lim_{t\to\infty} \delta(t) = 0$, and hence also $\lim_{t\to\infty} \gamma(t) = 0$ as $\alpha \in \mathcal{K}$. Hence for update rule 2 Assumption 1 is satisfied. □

We now present a slightly more complex update rule which is nonmonotonic, but allows for a reset of β. The benefit of such an update rules is that β might stay smaller, which, as mentioned above, might be good for performance reasons. Moreover, one might also have a greater robustness for β to stay bounded in case of disturbances (see, e.g., Example 2 below).

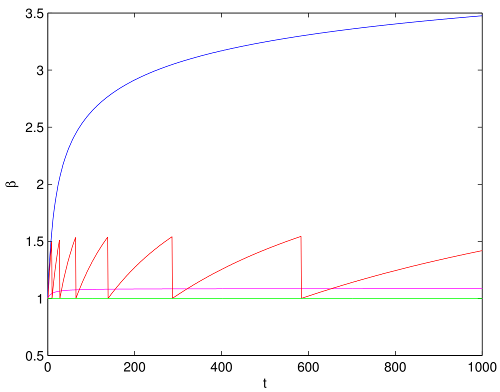
- Update rule 3: Let $\alpha_1, \alpha_2 \in \mathcal{K}$.
  $$B_3(\beta(t), x(t), \lambda(t)) := \begin{cases} 1 & \text{if } C_3(t) \leq 0, \\ \beta(t) + \alpha_2(\delta(t)) & \text{else,} \end{cases}$$
  where $C_3(0) = 0$ and for each $t \in \mathbb{I}_{\geq 1}$, $C_3(t) := \ell(x^*(N|t), u^*(N|t)) - \ell(x^*(N|t_{last}), u^*(N|t_{last})) + \alpha_1(\delta(t))$ with $t_{last} := \max_{s \leq t, \beta(s)=1} s - 1$.

Lemma 2: Update rule 3 is such that β satisfies (20); moreover, at least one of Assumptions 1 and 2 holds.

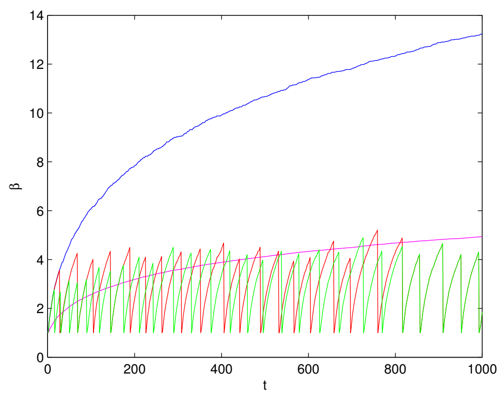
Proof: We start by proving the first claim of the lemma. If $\lambda_\infty - \limsup_{t\to\infty} \ell_{\min}(x(t)) > 0$, then there exists $\hat{t} > 0$ such that $\delta(t) \geq c > 0$ for all $t \geq \hat{t}$ and some $c > 0$. We will show that in this case the number of resets of β to 1 in update rule 3 is finite. Namely, assume it was not. Then, both $\ell(x^*(N|t), u^*(N|t))$ as well as $\ell(x^*(N|t_{last}), u^*(N|t_{last}))$ converge to λ_∞. But this implies that there exists $\bar{t} \geq \hat{t}$ such that $C_3(t) > 0$ for all $t \geq \bar{t}$ (as $\delta(t) \geq c > 0$), i.e., the reset condition is not satisfied anymore, which gives a contradiction. Hence the number of resets of β is finite. But then according to the definition of B_3, we immediately obtain that $\liminf_{t\to\infty} \beta(t) = \limsup_{t\to\infty} \beta(t) = \infty$, i.e., (20) is satisfied. To prove the second claim of the lemma, consider the following. First, the definition of B_3 implies that $\beta(t) \geq \min\{\beta_0, 1\} \geq 0 =: \underline{\beta}$ for all $t \in \mathbb{I}_{\geq 0}$. Second, whenever $\beta(t+1) = 1$, we have $\gamma(t) \leq 1$ (as $\beta(t) \geq 0$ for all $t \in \mathbb{I}_{\geq 0}$); if $\beta(t+1) > 1$, we again obtain $\gamma(t) = \alpha(\delta(t))$, which is bounded above by some finite constant as established in the proof of Lemma 1. Hence $\gamma(t) \leq c < \infty$ for all $t \in \mathbb{I}_{\geq 0}$. Thus, in order to show that at least one of Assumptions 1 and 2 is satisfied, it remains to show that either $\limsup_{t\to\infty} \gamma(t) \leq 0$ or $\liminf_{t\to\infty} \beta(t) < \infty$. The latter is immediately satisfied if the number of resets of β is infinite, i.e., $C_3(t_i) \leq 0$ for an infinite sequence $\{t_i\} \subseteq \mathbb{I}_{\geq 0}$. Now assume for contradiction that both $\limsup_{t\to\infty} \gamma(t) > 0$ and the number of resets of β is finite, i.e., there exists $\check{t} \in \mathbb{I}_{\geq 0}$ such that $C_3(t) > 0$ for all $t \in \mathbb{I}_{\geq \check{t}}$. Then, the definition of B_3 immediately implies that β is monotonically nondecreasing for $t \geq \check{t}$ with $\liminf_{t\to\infty} \beta(t) = \infty$. As shown in the proof of Theorem 2(i), this would result in (19). But the first equality in (19) implies that $\lim_{t\to\infty} \delta(t) = 0$ and hence $\limsup_{t\to\infty} \gamma(t) \leq 0$, which contradicts our assumption. Hence at least one of Assumptions 1 and 2 is satisfied, which concludes the proof of Lemma 2. □

Remark 2: With update rule 3, the number of resets of β can very well be infinite. In the proof of Lemma 2, we only showed that if $\lambda_\infty - \limsup_{t\to\infty} \ell_{\min}(x(t)) > 0$, then the number of resets is bounded. However, this situation does not occur according to Theorem 2(ii). □

Example 2: Consider the system $x(t+1) = (1 - u(t))x(t)$ with state and input constraint set $\mathbb{Z} = \mathbb{X} \times$



(a) Without noise.



(b) With additive noise.

Fig. 2: Closed-loop evolution of β in Example 2 using different update rules: blue – update rule 2 with $\alpha(r) = r$; magenta – update rule 2 with $\alpha(r) = r^2$; red – update rule 3 with $\alpha_1(r) = \alpha_2(r) = r$; green – update rule 3 with $\alpha_1(r) = r^2$ and $\alpha_2(r) = r$.

$\mathbb{U} = [0,1]^2$, cost function $\ell(x,u) = x + (1/2)u^2$ and prediction horizon $N = 1$. The optimal steady-state is given by $(x_s, u_s) = (0,0)$, and for all $x \in \mathbb{X}$, we obtain $\ell_{\min}(x) = \ell(x_s, u_s) = 0$. It is straightforward to calculate that the optimal solution to problem (7)–(8) with $N = 1$ is given by $u^*(0|t) = \min\{\beta(t)x(t), 1\}, u^*(1|t) = 0$. The resulting closed-loop system is then given by $x(t+1) = (1 - \min\{\beta(t)x(t), 1\})x(t)$, with $\beta(t)$ according to the specific used update rule. For each of the above discussed update rules and each initial condition $x_0 \in \mathbb{X}$ and $\beta_0 > 0$, the ω-limit set is given by $\omega_B(x_0) = \{0\}$, and $\max_{y \in \omega_B(x_0)} \bar{\ell}_{\min}(y) = \min_{y \in \omega_B(x_0)} \bar{\ell}_{\min}(y) = 0$. Hence for each of the above update rules, by Theorem 2 we conclude that $\lambda_\infty = 0$. However, the closed-loop evolution of β is quite different when using different update rules. As discussed above, update rule 1 leads to an unbounded β in any case. For update rule 2, one can show that when using $\alpha(r) = r$, β also does not stay bounded, as x converges to zero very slowly (blue curve in Figure 2(a)). On the other hand, when using $\alpha(r) = r^2$, one observes that β stays bounded and converges (magenta curve in Figure 2(a)). Finally, using update rule 3 with $\alpha_1(r) = r^2$ and $\alpha_2(r) = r$, one can show that $\beta \equiv 1$, as the reset condition is always fulfilled (green curve in Figure 2(a)). For $\alpha_1(r) = \alpha_2(r) = r$, on the other hand, one observes kind of a sawtooth behavior of β (red curve in Figure 2(a)).

As discussed earlier, an advantage of nonmonotonic update rules is that they might be more robust to disturbances. Namely, consider again the same update rules as used in Figure 2(a), but now an additive random disturbance (uniformly distributed over the interval $[0, 0.2]$) acts on the system. As one can see from the simulations (see Figure 2(b)), the monotonic update rule 2 leads now to an unbounded β, for both the choices $\alpha(r) = r$ and $\alpha(r) = r^2$. On the other hand, update rule 3 leads to a bounded β. $\square$

V. Conclusions

In this paper, we considered an economic MPC algorithm with self-tuning terminal weight. We proposed several update rules for the terminal weight β which resulted in an average performance which is at least as good as the best robustly achievable steady-state cost of the ω-limit set of the corresponding closed-loop system. Future research could include analyzing the interplay between the terminal weight β and the prediction horizon length N such that this obtained upper bound is as small as possible.

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