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Convex optimization methods for computing the Lyapunov exponent of matrices.*

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Abstract

We introduce a new approach to evaluate the largest Lyapunov exponent of a family of matrices, which describes the stability with probability one of a randomly switching linear system. For positive systems, of particular importance in systems and control, the rate of convergence of our approximation is estimated and the efficiency of the algorithm is demonstrated on particular switching systems of different dimensions. This is done by introducing new upper and lower bounds for the largest Lyapunov exponent of nonnegative matrices. We generalize this approach to arbitrary systems (not necessarily positive), derive a new universal upper bound for the Lyapunov exponent, and show that a similar lower bound, in general, does not exist.

1. Introduction

A *switching discrete time linear dynamical system* in $\mathbb{R}^d$ is defined by a set of matrices $\mathscr{A} \subset \mathbb{R}^{d\times d}$. It is formally described by the equations

$$\begin{array}{rcl} x_{t+1} & = & A_{d_t} x_t \quad A_{d_t} \in \mathscr{A} \\ x_0 & \in & \mathbb{R}^d, \end{array} \tag{1}$$

where $\mathscr{A}$ is a given set of matrices that describes the system, and x_0 is the initial state. Switching systems, and in particular *positive* switching systems, have motivated a great deal of research efforts recently. They have applications in very diverse areas of engineering. See [2, 10, 11, 14, 18] for many examples.

In this paper, to each matrix A_j we associate a positive number p_j so that $\sum_{j=1}^m p_j = 1$. The number p_j represents the probability that $A_t = A_j$ in (1). So, the trajectory is not uniquely defined, as it depends on the sequence of matrices A_t, which is the result of a random process.

Consider a random product $X_k = A_{d_k} \cdots A_{d_1}$, where all indices $\{d_j\}$ are independent and identically distributed random variables; each d_j takes values $1, \dots, m$ with probabilities $p_1, \dots, p_m$ respectively. According to the Furstenberg-Kesten theorem [8] the value $\|X_k\|^{1/k}$ converges with probability 1 to a number ρ, which depends only on the family $\mathscr{A}$, and on the probabilities $\{p_j\}_{j=1}^m$. The number $\lambda = \log \rho$ is called the *largest Lyapunov exponent* of the family $\mathscr{A}$ (and the corresponding probabilities p_j). In this paper we do not deal with other Lyapunov exponents, and, for the sake of simplicity, we omit the word "largest". This number can be defined by the following limit formula

$$\lambda \quad = \quad \lim_{k\to\infty} \frac{1}{k} \mathbf{E} \log \left\| A_{d_k} \cdots A_{d_1} \right\|, \tag{2}$$

where $\mathbf{E}$ denotes the mathematical expectation. The results of this paper can be extended to more general matrix distributions, but we restrict ourselves to i.i.d. matrices taking values on a finite set $\mathscr{A}$.

As a direct corollary of the Furstenberg-Kesten theorem, we have the following result, of high importance for the stability of switching systems.

Theorem 1. *Suppose that the sequence A_t in System (1) is an i.i.d. sequence with corresponding probabilities p_j. Then, System (1) converges to zero with probability one if and only if*

$$\lambda \quad = \quad \lim_{t\to\infty} \frac{1}{t} \mathbf{E} \log \left\| A_{d_t} \cdots A_{d_1} \right\| < 1, \tag{3}$$

and moreover, with probability one, we have

$$\lim \log |x_t| / t = \lambda.$$

*This is a short version of our research paper [17]. See that reference for full developments and a literature review.

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The above theorem motivates the study of Lyapunov exponents of a set of matrices in systems and control. They are useful for many practical situations which can be modeled as switching systems, as for instance the design of multihop control networks [1, 12], the trackability of sensor networks [6], the capacity of codes avoiding forbidden differences [3][1].

We introduce a new approach for computing the Lyapunov exponent by deriving new lower and upper bound for it using special positive homogeneous functionals on $\mathbb{R}^d$. The idea is the following: Writing $\mathscr{A}^k$ the set of products of length k of matrices in $\mathscr{A}$, for any such a functional f the minimal and maximal expected value of $\frac{1}{k}\log\frac{f(Bx)}{f(x)}$, $B\in\mathscr{A}^k$ over all $x\in\mathbb{R}^d, x\neq 0$, give a lower and upper bound respectively for λ. For families of nonnegative matrices those bounds can be effectively computed and then optimized over certain families of functionals f, which leads to optimal bounds $\beta_k \leq \lambda \leq \alpha_k$ that, under some general assumptions, rapidly converge to λ as $k\to\infty$. For every k both α_k and β_k are found by solving unconstrained convex minimization problems. The rate of convergence is proved to be at least linear in k, but in most of practical examples it is much faster.

The problem of computing or estimating the Lyapunov exponent is known to be extremely hard. It is even algorithmically undecidable in general [4]. On the one hand, the most natural approach is to generate a long random sequence of indices $d_1, d_2, \ldots$ and compute the value $\frac{1}{k}\log\|A_{d_k}\cdots A_{d_1}\|$, which by the Furstenberg-Kesten theorem will be close to λ for large k with high probability. Many applications, however, require a theoretically proved upper or lower bound for λ, for which any sort of probabilistic or Monte-Carlo algorithms is not acceptable. A recent paper of Pollicott [15] presents an original approach for iterative approximation of the Lyapunov exponent that converges extremely fast to the real value. The error estimate for this approximation, however, has the same kind of disadvantage: being very good asymptotically as $n\to\infty$ (n is the number of iterations), it may be unsatisfactory in some practical cases, because the method does not come with a guarantee of accuracy so that one does not know when n is large enough so that the estimate is accurate enough.

A simple upper bound is well-known: this is the right hand side of (2) for an arbitrary k for the Euclidean norm. This naive approach appears to be quite good (sometimes, the best available) in several practical cases. However, just an upper bound, without a lower one, is useless, because it does not give an interval to localize the Lyapunov exponent. Finding a good lower bound is much more complicated. For nonnegative matrices the only known lower bound is due to Key [13]. One of the main results of our paper (Theorem 4) is a new lower bound that is better both theoretically and numerically. Then we improve the upper bound and show its efficiency in numerical examples.

The structure of the paper is as follows: In Section II we describe the new approach for nonnegative matrices, then in Section III we consider two special families of functionals and the corresponding bounds α_k and β_k. In Section IV we summarize our results, stating the simple convex problems that one has to solve numerically in order to obtain the bounds on the Lyapunov Exponent. Then, in Theorem 4 we prove that under some general assumptions on the matrices we have $\alpha_k-\beta_k\leq\frac{C}{k}$, where C is a constant. In Section V we apply our method to real switching systems (with randomly generated matrices) and show the good performance of our technique.

Summary of the method. Our method simply consists of computing an upper and a lower bound, as the solutions of two convex optimization problems. The method can be iterated by increasing the integer parameter k, which asymptotically leads to the exact value.
It has the following advantages: 1) it produces upper and lower bounds for the Lyapunov exponent λ that both converge to λ with a linear rate as the number of iterations grows (as we see in numerical examples, it converges much faster in practice). 2) The method works equally well for high dimensions. In examples with randomly generated matrices of dimension $d\leq 50$ it computes λ with a relative error less than 1% within a few iterations. 3) We relax the strict positivity assumption on the matrices imposed in the previous papers on the subject (as well as the invertibility assumption).

2. Matrices with a common invariant cone: Bounds for the Lyapunov exponent

Assume that all matrices $A_1,\ldots,A_m$ share a common invariant cone $K\subset\mathbb{R}^d$, which is supposed to be convex, closed, solid, pointed, and having its apex at the origin (i.e.: $\forall x\in K, A_i x\in K$). For any points $x,y\in\mathbb{R}^d$ we write $x\geq y$ if $x-y\in K$ and $x-y>0$ if $x-y\in\operatorname{int}K$. For a matrix A we write $A\geq 0$ if it leaves the cone K invariant. The same notation are used for the dual cone $K^*=\{v\in\mathbb{R}^d \mid \inf_{x\in K}(v,x)\geq 0\}$.

Consider a functional[2] $f:K\to\mathbb{R}_+$. In the sequel we impose the following assumptions on f:

[1] As the expert reader might know, the above mentioned applications were not introduced in the framework of random switching. However, random switching has a straightforward usefulness in these applications.

[2] By functional we mean a function from the state space $\mathbb{R}^d$ with values in $\mathbb{R}$, as customary in functional analysis.

1) f is positive, i.e., $f(x) > 0$, whenever $x \neq 0$;

2) f is homogeneous, i.e., $f(tx) = tf(x)$ for any $x \in K, t \in \mathbb{R}_+$.

Now we define two values $F_{\min}$ and $F_{\max}$ for any family $\mathscr{A}$ and for any functional f:

$$F_{\min}(f,\mathscr{A}) \quad = \quad \inf_{f(x)=1} \mathbf{E}\left\{ \log f(Ax) \;\middle|\; A \in \mathscr{A} \right\}$$

$$F_{\max}(f,\mathscr{A}) \quad = \quad \sup_{f(x)=1} \mathbf{E}\left\{ \log f(Ax) \;\middle|\; A \in \mathscr{A} \right\}.$$

We use the short notation $F_{\min}(f,\mathscr{A}) = F_{\min}$, and the same with $F_{\max}$, if the functional f and the family $\mathscr{A}$ are fixed. Denote also $F^{(k)}_{\min} = \frac{1}{k} F_{\min}(f,\mathscr{A}^k)$. Thus,

$$F^{(k)}_{\min} \quad = \quad \frac{1}{k} \inf_{f(x)=1} \mathbf{E}\left\{ \log f(Bx) \;\middle|\; B \in \mathscr{A}^k \right\},$$

and similarly with $F^{(k)}_{\max}$. Thus, $F^{(k)}_{\min}$ is the smallest expected value of the logarithm of the ratio $\frac{f(A_{d_k}\cdots A_{d_1}x)}{f(x)}$ over all $x \in K, x \neq 0$. Let us first make the following simple observation:

Lemma 1. *For every natural k and n we have*

$$(k+n)F^{(k+n)}_{\max} \;\leq\; kF^{(k)}_{\max} \;+\; nF^{(n)}_{\max}$$

and

$$(k+n)F^{(k+n)}_{\min} \;\geq\; kF^{(k)}_{\min} \;+\; nF^{(n)}_{\min}.$$

Corollary 1. *For an arbitrary functional f and for every k we have $F_{\min} \leq F^{(k)}_{\min}$ and $F_{\max} \geq F^{(k)}_{\max}$.*

Lemma 2. *For an arbitrary functional f and for any n we have $F^{(n)}_{\min} \leq \lambda \leq F^{(n)}_{\max}$. In particular, $F_{\min} \leq \lambda \leq F_{\max}$.*

By Fekete's lemma [7] for any sequence of nonnegative numbers $\{a_k\}_{k\in\mathbb{N}}$ such that $(k+n)a_{k+n} \leq ka_k + na_n$, $k,n \in \mathbb{N}$, the limit $\lim_{k\to\infty} a_k$ exists and equals to $\inf_{k\in\mathbb{N}} a_k$. Similarly, if $(k+n)a_{k+n} \geq ka_k + na_n$, $k,n \in \mathbb{N}$, then $\lim_{k\to\infty} a_k = \sup_{k\in\mathbb{N}} a_k$. Applying Lemma 1 we see that $\lim_{k\to\infty} F^{(k)}_{\min} = \sup_{k\in\mathbb{N}} F^{(k)}_{\min}$ (denote this limit by $F^{(\infty)}_{\min}$), and $\lim_{k\to\infty} F^{(k)}_{\max} = \inf_{k\in\mathbb{N}} F^{(k)}_{\max}$ (denote this limit by $F^{(\infty)}_{\max}$). Invoking now Lemma 2 we obtain the following

Proposition 1. *For every family $\mathscr{A}$ and for every functional f we have*

$$F_{\min} \;\leq\; F^{(\infty)}_{\min} \;\leq\; \lambda \;\leq\; F^{(\infty)}_{\max} \;\leq\; F_{\max}. \qquad (4)$$

Thus, to approximate the Lyapunov exponent one can take an arbitrary functional f and get the values $F_{\min}$ and $F_{\max}$ as a lower and upper bound respectively. Iterating, one obtains the bounds $F^{(k)}_{\min}$ and $F^{(k)}_{\max}$, which, by Corollary 1, are, at least, not worse. If the two inner inequalities in (4) become equalities, then $F^{(k)}_{\min}$ and $F^{(k)}_{\max}$ converge from different sides to λ, which allows us to compute λ with an arbitrary prescribed accuracy. Sufficient conditions for that are given in Theorem 2 below. Finally, in the ideal case, when $F_{\min} = F_{\max}$, all the inequalities in (4) become equalities. In this case we get a sharp value of λ immediately, just by evaluating $F_{\min}$. Such "ideal" functionals f are called *invariant*.

Definition 1. *A functional f is called invariant for a family $\mathscr{A}$ if $\forall\, x \in K \setminus \{0\}$,*

$$-f(x) \quad + \quad \mathbf{E}\left\{ \log f(Ax) \;\middle|\; A \in \mathscr{A} \right\} \;\equiv\; \text{const}.$$

Thus, f is invariant if and only if $F_{\min} = F_{\max}$. In view of Lemma 2 both these values equal to λ.

Corollary 2. *For any invariant functional f the constant in Definition 1 is equal to the Lyapunov exponent λ of $\mathscr{A}$.*

Certainly, invariant functionals do not necessarily exist for all families that have invariant cones. For nonnegative matrices sufficient conditions were obtained in [16], we shall cite that result in Theorem A (Section IV). However, even if an invariant functional exists, it may be very difficult to find or to approximate. Nevertheless, as the following theorem says, the very existence of an invariant functional guarantees that for an arbitrary functional f the values $F^{(k)}_{\min}$ and $F^{(k)}_{\max}$ both converge to λ at a linear rate.

Theorem 2. *For an arbitrary family $\mathscr{A}$ and for every functional f we have $F^{(\infty)}_{\max} = \lambda$.*

If, in addition, there is an invariant functional for this family, then for every functional f we have $F^{(\infty)}_{\min} = \lambda$. In this case

$$F^{(k)}_{\max} \;-\; F^{(k)}_{\min} \quad \leq \quad Ck^{-1}, \qquad k \in \mathbb{N},$$

where the constant C depends only on $\mathscr{A}$ and on f.

Thus, for every functional f we have $F^{(k)}_{\max} \to \lambda$. If the family $\mathscr{A}$ possesses an invariant functional on the cone K, then for every functional f the values $F^{(k)}_{\min}$ and $F^{(k)}_{\max}$ converge from two sides to the Lyapunov exponent λ, and the distance between them decays as Ck^{-1}. This provides a theoretical opportunity to compute the Lyapunov exponent with a given precision, using an arbitrary functional f. To realize this idea we need to compute the values $F^{(k)}_{\max}$ and $F^{(k)}_{\min}$ for large k. Each

computation actually requires the resolution of an optimization problem, for which one needs to find a global optimum of the function $\psi_k(x) = \frac{1}{k}\,\mathbf{E}\left\{\log\frac{f(Bx)}{f(x)} \;\middle|\; B \in \mathscr{A}^k\right\}$, on the cone K. Therefore, the functional f should be chosen in a special way, to obtain an objective function $\psi_k(x)$ that is convenient for global minimizing/maximizing. In the next section we define two families of functionals f (each depending on one d-dimensional parameter), and then we apply them for the cone $K = \mathbb{R}^d_+$ (i.e., for the case of nonnegative matrices).

3. Two special functionals f(x)

In this section we define two families of functionals f, which then will be applied to compute the Lyapunov exponent of nonnegative matrices. Let $\mathscr{A}$ be an arbitrary finite family of matrices sharing an invariant cone K. For every point $x > 0$ consider the functional $f(\cdot) = r_x(\cdot)$ defined on K as follows:

$$r_x(y) \quad = \quad \min\left\{r > 0 \,\middle|\, y \le rx\right\}. \tag{5}$$

Geometrically, this functional is a norm on K, whose unit ball is $K \cap (x - K)$. If $r_x(y) \le 1$, then $y \le x$, therefore $Ay \le Ax$ for any matrix $A \ge 0$, and hence $r_x(Ay) \le r_x(Ax)$. Consequently, for this functional we have

$$F_{\max} \quad = \quad \max_{y>0} \mathbf{E}\left\{\log\frac{r_x(Ay)}{r_x(y)} \;\middle|\; A \in \mathscr{A}\right\} \tag{6}$$

$$= \quad \mathbf{E}\left\{\log r_x(Ax) \,\middle|\, A \in \mathscr{A}\right\}. \tag{7}$$

Let us denote

$$\alpha(x) = \mathbf{E}\left\{\log r_x(Ax) \,|A \in \mathscr{A}\right\}; \alpha = \inf_{x>0}\alpha(x).$$

Similarly we define $\alpha_k(x) = F^{(k)}_{\max}$ and $\alpha_k = \inf_{x>0}\ \alpha_k(x)$. Applying now Lemma 2, we conclude that $\alpha_k(x) \ge \lambda$ for each $x > 0$ and $k \in \mathbb{N}$, and therefore $\alpha_k \ge \lambda$. To obtain a lower estimate for λ we take an arbitrary $v \in \operatorname{int} K^*$ and consider the linear functional $f(x) = (v,x)$. Again, this functional is a norm on K, whose unit ball is the intersection of K with a half-space. For this functional we have

$$F_{\min} = \inf_{x\in K} \mathbf{E}\left\{\log\frac{(v,Ax)}{(v,x)}\middle|\quad A \in \mathscr{A}\right\}.$$

Denote

$$\beta(v) = \inf_{x\in K}\mathbf{E}\left\{\log\frac{(v,Ax)}{(v,x)} \quad \middle| A \in \mathscr{A}\right\}; \beta = \sup_{v\in\operatorname{int} K^*}\beta(v).$$

Similarly we define $\beta_k(v) = F^{(k)}_{\min}$ and $\beta_k = \inf_{v>0}\ \beta_k$. Lemma 2 now implies that $\beta_k(v) \le \lambda$ for every $v \in \operatorname{int}(K^*)$ and $k \in \mathbb{N}$.

Thus, we have the following bounds for the Lyapunov exponent of a matrix possessing an invariant cone:

$$\beta_k(v) \quad \le \quad \lambda \quad \le \quad \alpha_k(x). \tag{8}$$

In general, they are not easy to compute. For instance, to evaluate $\beta(v)$ we need to find the global minimum over $x \in K$ of the function

$$\mathbf{E}\left\{\log\frac{(v,Ax)}{(v,x)} \;\middle|\; A \in \mathscr{A}\right\} \tag{9}$$

$$= -\log(v,x) + \sum_{j=1}^{m} p_j \log(v,A_jx). \tag{10}$$

This function is not convex in x, it is actually quasiconcave, and hence its minimization may be hard. Nevertheless, in case $K = \mathbb{R}^d_+$ the value $\beta_k(v)$ is not only computable, but can be efficiently optimized over all $v \in \operatorname{int} K^*$, and the same is for $\alpha_k(x)$.

Remark 1. *If we work with a general cone K, then it is more convenient to apply the linear functional $f(x) = (v,x)$ to get not a lower bound (as $\beta_k(v)$), but the upper one. Doing so, we write $F_{\max}$ for the functional $f(x) = (v,x)$ and obtain*

$$\gamma(v) \;=\; \max_{x\in K,\,(v,x)=1} \sum_{j=1}^{m} p_j \log(v,A_jx). \tag{11}$$

The objective function $\psi(x) = \sum_{j=1}^m p_j \log(v,A_jx)$ is concave, and hence its maximum on the convex set $\{x \in K \mid (v,x) = 1\}$ can be efficiently found. The same can be done for every k:

$$\gamma_k(v) \;=\; \max_{x\in K,\,(v,x)=1} \mathbf{E}\left\{\log(v,Bx) \;\middle|\; B \in \mathscr{A}^k\right\}. \tag{12}$$

The shortcoming of this estimate is that it is very hard to minimize over the set $v \in \operatorname{int} K^*$, even in the case $K = \mathbb{R}^d_+$. Nevertheless, choosing appropriate v one can obtain good upper bounds $\gamma_k(v)$ that converge fast to λ as $k \to \infty$. We use this bound in Theorem 5 for approximating Lyapunov exponents of general sets of matrices.

4. The bounds

In this section we gather the material presented above and present the theorems that allow to estimate the Lyapunov exponent. They make use of the developments above. The first two theorems are concerned with nonnegative matrices, while the last one is for general matrices, but does not allow for a guaranteed accuracy.

Theorem 3. *For any compact set of nonnegative matrices, and for any natural k, let us define α_k, β_k respectively as*

$$\inf_{u\in\mathbb{R}^d} \mathbf{E}\left\{ \max_{i=1,\dots,d} \log\Big(\sum_{j=1}^{d} a_{ij} e^{u_j-u_i}\Big) \;\Big|\; A\in\mathscr{A}^k \right\}, \tag{13}$$

$$\sup_{v\in\mathbb{R}^{+d}} \min_{1<j<d} \left(-\log v_j + \mathbf{E}\Big\{ \log\Big(\sum_{i=1}^{d} a_{ij} v_i\Big) \Big| A\in\mathscr{A}^k \Big\}\right). \tag{14}$$

Then, we have

$$\beta_k \;\le\; \lambda \;\le\; \alpha_k, \qquad k\in\mathbb{N}. \tag{15}$$

Theorem 4. *For every family of nonnegative matrices satisfying*
(a) *there is at least one strictly positive product of matrices from $\mathscr{A}$ (with repetitions permitted);*
(b) *matrices from $\mathscr{A}$ do not have zero rows nor zero columns,*

and for every vectors $x, v > 0$ we have $\beta_k(v) \le \lambda \le \alpha_k(x)$, $k\in\mathbb{N}$, and

$$\alpha_k(x) \;-\; \beta_k(v) \;\le\; Ck^{-1}, \qquad k\in\mathbb{N},$$

where the constant C depends on $\mathscr{A}, x$ and v.

Note that conditions **(a)** and **(b)** are assumed in most of papers studying random products of nonnegative matrices (see [9, 13, 16]). In the full version of our paper, we show that for any set of matrices (i.e. with positive and negative entries), our method also enables one to derive upper bounds on the Lyapunov Exponent. In the following theorem, $X \succeq 0$ means that the matrix X is positive semidefinite.

Theorem 5. *For every positive definite matrix V, the value $\Gamma_k(V)$*

$$\Gamma_k(V) \;=\; \frac{1}{2k} \sup_{X\succeq 0,\ \mathrm{tr}(VX)=1} \mathbf{E}\left\{ \log\ \mathrm{tr}\,(VBXB^*) \;\Big|\; B\in\mathscr{A}^k \right\},$$

is an upper bound for $\lambda(\mathscr{A})$, and it converges to it as $k\to\infty$.

To compute $\Gamma_k(V)$ one needs to find the maximum of a smooth concave function on the intersection of the cone $\mathscr{K}_d$ of semidefinite positive matrices with a hyperplane $\{X\in\mathscr{A}_d \mid \mathrm{tr}\,(VX)=1\}$. This problem can be efficiently solved by standard tools of semidefinite programming (see for instance [5]). In many cases the most natural choice is to take $V = I$ (the identity matrix), which yields the following upper bound:

$$\Gamma_k(I) = \frac{1}{2k} \sup_{X\succeq 0,\ \mathrm{tr}(X)=1} \mathbf{E}\left\{ \log\ \mathrm{tr}(BXB^*) \;\Big|\; B\in\mathscr{A}^k \right\}. \tag{16}$$

One can show that this upper bound is better than the standard upper bound with the Euclidean norm:

Proposition 2. *For every matrix family we have $\Gamma_k(I) \le \frac{1}{k}\mathbf{E}\left\{\log\|B\|_2 \;\big|\; B\in\mathscr{A}^k\right\}$, $k\in\mathbb{N}$.*

5. Numerical example

In this section, we show the efficiency of our method on a randomly switching system. As mentioned above, all optimization problems that we need to solve for computing these estimates are convex unconstrained problems. We apply standard tools of convex optimization, namely, the matlab function *fminunc*, which uses a quasi-Newton procedure (the so-called *BFGS*-scheme), together with a cubic line search procedure. All computations took a few minutes on a standard desktop pc.

As a general observation, our upper bound generally converges faster than the Euclidean bound (that is, the bound obtained from (2) with the Euclidean norm) towards the real value in the first steps of the algorithm. This makes it possible to have a fair upper estimate without having to compute too large products of matrices. On the other hand, for our lower bound, not only it also converges faster than the previously available lower estimates (see [13]), but in addition, generally these other lower bounds do not allow to reach a satisfactory accuracy at all. Even though they converge asymptotically towards the exact value, they appear to be often relatively far from it for the largest values of k that a classical computer can handle (say, $k = 14$ or so). We graphically present our results in terms of the Lyapunov radius $\rho = \exp(\lambda)$, because in most applications, it is the "physical" quantity that one wants to compute. More precisely, ρ is the minimal r such that $\mathscr{A}/r$ is stable almost surely (writing "$\mathscr{A}/r$" for $\{A/r : A\in\mathscr{A}\}$).

We ran our algorithms on randomly selected nonnegative matrices. We report here the results of computation of ρ on two different sizes: reasonably small matrices of dimension 5, and large matrices of dimension 60. For both these sizes, we generated dense and sparse matrices. For the dense matrices, the entries are drawn uniformly and independently between zero and one, while for the second case, we sparsify the matrices by putting each entries to zero with a probability $p = 5/7$. For matrices of size 60, computing long products rapidly becomes prohibitive. As one can see on Fig. 5 (c) and (d), already with products of length 1

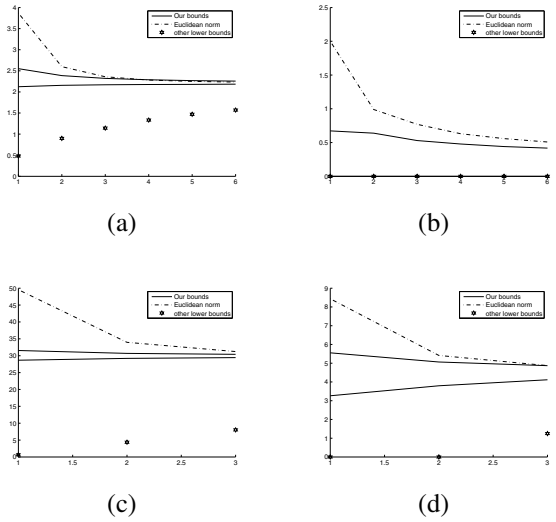

Figure 1. **Evolution (with the length of the computed products) of the different bounds for random matrices of dimension** 5 **((a) and (b)) and** 60 **((c) and (d)). In ((b) and (d)), we sparsify the matrices by independently putting each entry to zero with a probability** $5/7$**; the other entries are i.i.d homogeneously between zero and one. (In (b) all the lower bounds give the trivial zero bound.)**

size	k	lower bound	upper bound	accuracy
10	12	5.175	5.205	0.6%
20	12	10.16	10.22	0.6%
30	11	15.28	15.35	0.4%
40	11	19.74	19.78	0.2%
50	11	24.67	24.83	0.7%

Table 1. **Results of our algorithm on random pairs of matrices. For all matrices, each entry was drawn i.i.d. at random homogeneously between zero and one. In the second column,** k **indicates the length of the products computed in order to derive the bounds in columns** 3 **and** 4.

we have a fairly good approximation of the Lyapunov exponent, while the estimate with the Euclidean norm is still far from convergence. As one can see in figure (b), if the matrices become so sparse that zero rows can appear, then our lower bound β_k may vanish. In this case we apply the modified bound $\tilde{\beta}_k$. Table 1 summarizes the scalability of our method when the size of the matrices increases. For the same effort of computation, the accuracy seems to be somewhat independent of the dimension.

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