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Fault-Tolerant Pole-Placement in Single-Integrator Networks*

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Abstract— **The paper faces a fault-tolerant control problem for a network of agents with single-integrator internal dynamics, which share information on their states according to an arbitrary topology. Specifically, we aim at designing a decentralized regulator able to place the closed-loop dominant poles near prespecified locations. Furthermore, this goal has to be achieved even in the presence of faults of the transmitting and receiving apparatuses of the single agents. We prove a necessary and sufficient solvability condition for our pole-placement problem, and show that it can be solved if and only if a simpler fault-tolerant stabilization problem admits a solution. Then, we give an explicit formula for a class of possible regulators.**

I. INTRODUCTION

The decentralized control of networks composed of identical subsystems, usually called agents, has been attracting the interest of the researchers for many years. The first papers on this topic are probably [1], [2]. After their publication, the literature got rich of a huge number of contributions, which often make reference to agents having single- or multiple-integrator dynamics. Many papers deal with consensus, synchronization, flocking and related problems. See, for instance, [3]–[15]. Other articles focus on stabilization and pole-placement [16]–[21].

Following the lead of the just mentioned papers, we consider here a decentralized dominant-pole-placement problem for a network of single integrators, which share pieces of information on their states according to an arbitrary topology. Moreover, we add a fault-tolerance constraint. In particular, we consider the possibility that: (i) one or more agents become unable to transmit information on their states to the other agents; (ii) one or more agents become unable to receive information on the states of the other agents. In all these situations, the dominant poles of the controlled network must remain ”near” prespecified locations.

We believe that this problem deserves attention. Indeed, it is preliminary to the more challenging, yet unsolved, consensus problem with the additional requirement that the agreement law and the dominant dynamics are arbitrarily imposed and not affected by faults.

Problems of the kind outlined above have scarcely been faced in the classical literature on fault-tolerance [22], [23]. However, some exceptions exist. With reference to networks of continuous-time double integrators, the authors of [18] mainly concentrate on dominant-pole-placement in nominal conditions, and incidentally give sufficient conditions for fault-tolerant stabilization. Starting from the results of that paper, we solved a fault-tolerant dominant-pole-placement problem for continuous-time double-integrator networks [21] and a fault-tolerant stabilization problem for continuous-time and discrete-time multiple-integrator networks in [20] and [19], respectively.

We supply a constructive necessary and sufficient solvability condition for the present problem. Quite unexpectedly, fault-tolerant dominant-pole-placement can be achieved if and only if the network is stabilizable in a fault-tolerant way. However, pole-placement generally requires dynamical regulators, whereas simpler nondynamical ones are always sufficient for stabilization [20]. Here, we give explicit formulas for possible regulators of order equal to the number of the agents. We did not try to extend classical general-purpose techniques for decentralized control design [24]–[26], since they usually lead to high-order regolators. On the contrary, our results are based on the already cited papers [19]–[21], which have their roots on some previous works of ours concerning what is called regulator problem in the presence of actuator and sensor faults [27]–[30].

The paper is organized as follows. Section II is devoted to formally describe the overall control system: network, regulator and faults. It also contains an analysis of the closed-loop characteristic equation. Then, Section III briefly summarizes some definitions and results on the cited fault-tolerant stabilization problem. In a sense, it is preliminary to the fault-tolerant dominant-pole-placement problem, which is stated and solved in Section IV. The paper ends with a tutorial example (Section V) and some concluding remarks (Section VI). The proofs are omitted because of space limitations.

Notation. Let $\emptyset$ denote the empty set, and, for any set ϑ, let κ_ϑ be its cardinality. For any $\sigma \times \sigma$ matrix K, let $k_{i,j}$ be its (i,j) element. For any two sets of integers

$$\vartheta_1 := \{\vartheta_{1i} \,|0 < \vartheta_{1i} \leq \sigma \,\text{and}\, \vartheta_{1i} \neq \vartheta_{1j}, \forall i \neq j\} \ ,$$
$$\vartheta_2 := \{\vartheta_{2i} \,|0 < \vartheta_{2i} \leq \sigma \,\text{and}\, \vartheta_{2i} \neq \vartheta_{2j}, \forall i \neq j\} \ ,$$

let

$$K_{[\vartheta_1,\vartheta_2]} := \left\{k_{[\vartheta_1,\vartheta_2]i,j}\right\} \ ,$$

$$k_{[\vartheta_1,\vartheta_2]i,j} := \begin{cases} 0 & , \ j \in \vartheta_1 \quad , \ i \neq j \\ 0 & , \ i \in \vartheta_2 \quad , \ j \neq i \\ k_{i,j} & , \ \text{otherwise} \end{cases} \ .$$

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Matrix $K_{[\vartheta_1,\vartheta_2]}$ is $\sigma \times \sigma$. Of course, $K_{[\emptyset,\emptyset]} = K$. For any set of integers ϑ, of the same form as ϑ_1 and ϑ_2 above, let $K_{(\vartheta)}$ denote the $(\sigma - \kappa_\vartheta) \times (\sigma - \kappa_\vartheta)$ matrix obtained from K after removing its rows and columns with indices in the set ϑ. Conventionally, if $\vartheta = \emptyset$ no row or column is removed. Analogously, for any scalar variable z, let adopt the notations $K_{[\vartheta_1,\vartheta_2]}(z)$ and $K_{(\vartheta)}(z)$ for a matrix function $K(z)$. As usual, I_σ denotes the $\sigma \times \sigma$ identity matrix.

II. CONTROL SYSTEM DESCRIPTION

This section introduces the control system we consider in the rest of the paper. Specifically, we introduce the model of the network, the class of admissible regulators and the model of the faults, which can occur on the communication apparatuses. We also add an analysis of the characteristic equation of the overall control system, in both nominal and faulty conditions.

A. Network

The network is constituted of m identical agents with single-integrator internal dynamics. As a whole, it is described in the time-domain by

$$\dot{x}(t) = u(t) \ , \tag{1}$$

where $u(t)$ and $x(t)$ are m-dimensional vectors which collect the inputs and the states of all the agents, respectively. The information available to each agent is a scalar linear combination of (some of) the states. Let $y(t)$ be the m-dimensional observation vector and G an $m \times m$ real matrix, called topology matrix. Then, the network description is completed by the equation

$$y(t) = Gx(t) \ . \tag{2}$$

Of course, the ith components of $u(t)$, $x(t)$ and $y(t)$ are the input, state and output of the ith agent, $i \in \mathcal{M} := \{1, 2, \dots, m\}$. The network model specified by (1), (2) has frequently been adopted in the literature, for instance in [7] and [12]. In the complex variable domain it can be given the form

$$Y(s) = \frac{1}{s} G U(s) \ , \tag{3}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$.

In order to avoid trivialities, we assume that the topology matrix G is nonsingular. Otherwise, the network (1), (2) is not observable, and no stability/pole-placement problem can be solved, not even by means of a centralized regulator and in the absence of faults.

B. Regulator

As already mentioned, we are interested in decentralized regulators. Therefore, our regulators are composed of m local elements, the ith of which is applied to the ith agent. Thus, they are described by

$$U(s) = R(s)Y(s) \ , \tag{4a}$$

$$R(s) := N(s)D(s)^{-1} \ , \tag{4b}$$

$$N(s) := \operatorname{diag}\{n_1(s), n_2(s), \dots, n_m(s)\} \ , \tag{4c}$$

$$D(s) := \operatorname{diag}\{d_1(s), d_2(s), \dots, d_m(s)\} \ , \tag{4d}$$

where $R(s)$ is proper, $d_i(s)$ and $n_i(s)$ are coprime, and the degree of $d_i(s)$ is ρ_i, $i \in \mathcal{M}$.

C. Faults

The network is subject to faults, which can occur in the transmitting and/or receiving apparatuses of one or more agents. Specifically, we assume that some agents may become unable to transmit information on their states to the other agents. We call $f_T \subset \mathcal{M}$ the set of the indices of such agents. Similarly, we assume that some agents may become unable to receive information on the states of the other agents. We call $f_R \subset \mathcal{M}$ the set of the indices of such agents.

Equations (1), (2) or (3) represent the network in nominal conditions, whereas we are interested also in describing the network in faulty conditions. A model able to represent the network in all the situations of interest can immediately be obtained by substituting (3) with

$$Y(s) = \frac{1}{s} G_{[f_T,f_R]} U(s) \ , \ f_T \subset \mathcal{M} \ , \ f_R \subset \mathcal{M} \ , \tag{5}$$

where $G_{[f_T,f_R]}$ replaces the topology matrix G.

D. Characteristic Equation

Of course, stability and locations of the poles of the controlled network depend on its characteristic equation. For any $f_T \subset \mathcal{M}$ and for any $f_R \subset \mathcal{M}$, the characteristic equation of the positive feedback system (4a), (4b), (5) is

$$\det\left(P_{[f_T,f_R]}(s)\right) = 0 \ , \tag{6}$$

where

$$P(s) := sD(s) - GN(s) \ . \tag{7}$$

A moment reflection shows that, for any couple $(f_T, f_R) \subset \mathcal{M} \times \mathcal{M}$,

$$\det\left(P_{[f_T,f_R]}(s)\right) = \chi\left(s, f_T, f_R\right) \prod_{i \in f_T \cup f_R} \pi_i(s) \ , \tag{8}$$

where

$$\chi\left(s, f_T, f_R\right) := \det\left(P_{(f_T \cup f_R)}(s)\right) \ , \tag{9}$$

$$\pi_i(s) := P_{(\mathcal{M}\setminus\{i\})}(s) = s d_i(s) - g_{i,i} n_i(s) \ , \ i \in f_T \cup f_R \ . \tag{10}$$

Hence, the characteristic equation (6) of system (4a), (4b), (5) is equivalent to the set of equations

$$\chi\left(s, f_T, f_R\right) = 0 \ , \tag{11}$$

$$\pi_i(s) = 0 \ , \ i \in f_T \cup f_R \ , \tag{12}$$

which have a nice interpretation. Equation (11) is the characteristic equation of the controlled subnetwork composed of the agents not involved in faults. On the other hand, (12) are the characteristic equations of the single controlled agents whose transmitting and/or receiving apparatuses are faulty.

III. FAULT-TOLERANT STABILIZATION

This section is devoted to briefly review some previous results of ours on fault-tolerant stabilization. For the proofs and a more detailed presentation of the topic the reader is referred to [19]. In a certain sense, pole-placement can be considered as a reinforcement of stabilization. For this reason, comparing solvability conditions and regulator complexity for the two problems is of obvious interest.

A. Problem Statement

The problem of designing a regulator which stabilizes the network in nominal and faulty conditions can be stated as follows.

Problem 3.1: Find a regulator $R(s)$, of the form (4b)-(4d), such that, for all $f_T \subset \mathcal{M}$ and for all $f_R \subset \mathcal{M}$, system (4a), (5) is stable, that is, all the roots of the characteristic equation (6) have negative real parts. □

We point out that we require that the network, when controlled by the decentralized regulator (4), is stable whichever the number of agents unable to transmit and/or to receive information on their position states may be. Hence, a regulator solving Problem 3.1 actually supplies the control system (4a), (5) with a real fault-tolerance property.

B. Problem Solution

A careful analysis of (6)-(10) allows us to realize that the set of the characteristic equations involved in Problem 3.1 coincides with the set of the characteristic equations

$$\det\left(P_{(f)}(s)\right) = 0 \ , \ \ \forall f \subset \mathcal{M} \ . \tag{13}$$

Moreover, the set of regulators which place the roots of (6) in the left-half of the complex plane, for all couples $(f_T, f_R) \subset \mathcal{M} \times \mathcal{M}$, coincides with the set of regulators which place the roots of (13) in the left-half of the complex plane.

The reformulated problem corresponding to the above remark is more suitable to face than the original one. Actually, it was expedient to solve Problem 3.1.

A necessary and sufficient solvability condition for it is reported in the next theorem, where

$$\tilde{G} := G_{[\mathcal{M},\mathcal{M}]} = \mathrm{diag}\left\{g_{1,1}, g_{2,2}, \ldots, g_{m,m}\right\} \ . \tag{14}$$

Theorem 3.1: Problem 3.1 admits a solution if and only if condition

$$\det\left(G_{(f)}\right)\det\left(\tilde{G}_{(f)}\right) > 0 \ , \ \forall f \subset \mathcal{M} \ , \tag{15}$$

holds. □

Furthermore, it is possible to identify a class of regulators solving Problem 3.1. We present it in the forthcoming theorem, where

$$R(s) = C(s)V\mu \ , \tag{16a}$$

$$V := -\tilde{G}\mathrm{Z}(\zeta) \ , \quad \mathrm{Z}(\zeta) := \mathrm{diag}\left\{\zeta, \zeta^2, \ldots, \zeta^m\right\} \ , \tag{16b}$$

$$C(s) := \mathrm{diag}\left\{C_1(s), C_2(s), \ldots, C_m(s)\right\} \ , \ C(0) = I_m \ , \tag{16c}$$

with μ scalar, ζ scalar, and the $C_i(s)$s, $i \in \mathcal{M}$, rational transfer functions. The order of $C_i(s)$ is ρ_i, $i \in \mathcal{M}$.

Theorem 3.2: Assume that (15) holds, and let $C(s)$ be any stable proper transfer function. Then, there exist $\bar{\zeta} > 0$ and $\bar{\mu}_{C(s)}(\cdot) > 0$ such that, for all $\zeta \in (0, \bar{\zeta})$, and for all $\mu \in (0, \bar{\mu}_{C(s)}(\zeta))$, the regulator (14), (16) solves Problem 3.1. □

According to Theorems 3.1 and 3.2, a regulator solving Problem 3.1 can be designed by means of a three-step procedure.

1) The first step consists in finding a sufficiently small positive ζ such that the eigenvalues of the matrices $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}(\zeta)$, $f \subset \mathcal{M}$, are real negative. The existence of such a ζ is guaranteed.
   As a matter of fact, what is really needed for the subsequent steps is finding a matrix V such that matrices $G_{(f)}V_{(f)}$, $f \subset \mathcal{M}$, are Hurwitz. The matrix V given by (16b) enjoys this property, since it actually makes negative the eigenvalues of the matrices $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}(\zeta)$, $f \subset \mathcal{M}$. See [31].
2) The second step is choosing the stable proper transfer function $C(s)$, with $C(0) = I_m$.
   Of course, this can be done in an infinity of ways, and the dynamical behavior of the controlled network will depend on the choice. However, observe that the order of the regulator (14), (16) coincides with the order of $C(s)$. Hence, in the absence of other specifications, the most reasonable choice is $C(s) = I_m$. In so doing, the regulator turns out to be nondynamical.
3) The third step calls for assigning a sufficiently small positive value to μ. This value has to be such that all the roots of the characteristic equations (13) have negative real parts. Its existence is guaranteed.
   The rationale is as follows. For any fault $f \subset \mathcal{M}$: (i) when $\mu = 0$, (13) has $m - \kappa_f$ roots at $s = 0$ (the poles of the subnetwork not involved in faults), together with $\sum_{i \in f} \rho_i$ roots with negative real parts (the poles of $C_{(f)}(s)$); (ii) if μ is taken positive small, (13) has $\sum_{i \in f} \rho_i$ roots "near" the poles of $C_{(f)}(s)$ together with $m - \kappa_f$ roots "near" the origin of the complex plane; (iii) the latter roots turn out to lie in the left-half of the complex plane, because of the particular choice of matrix V.
   Notice that, in the special case where $C(s) = I_m$, the scalar μ can be taken large at will. Indeed, (13) becomes
   $$\det\left(sI_{m-\kappa_f} + G_{(f)}\tilde{G}_{(f)}\mathrm{Z}_{(f)}(\zeta)\mu\right) = 0 \ , \ \forall f \subset \mathcal{M} \ , \tag{17}$$
   and the roots of these equations are μ times the eigenvalues of the matrices $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}(\zeta)$, $f \subset \mathcal{M}$, which are real negative.

IV. FAULT-TOLERANT POLE-PLACEMENT

Besides stabilizing the network, we might be willing to end up with a fault-tolerant closed-loop system with well damped and sufficiently fast transients. On this aspect, the design procedure outlined in the preceding section supplies a first partial answer. Indeed, for any $f \subset \mathcal{M}$, the choice

$C(s) = I_m$ leads to real negative roots of (13). Hence, the damping factors of the controlled network eigenvalues are anyway unitary. Moreover, an appropriate choice of μ allows us to give all the roots of (17) real parts less than any prespecified value. Thus, we are able to make the speed of response arbitrarily large. However, for any fixed $f \subset \mathcal{M}$, the eigenvalues of $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}(\zeta)\mu$ have different order of magnitude with respect to ζ [31], which can be an unpleasant fact. Moreover, the actual speed of response does depend on the particular fault $f \subset \mathcal{M}$.

In this section, we state and solve a problem which overcomes these drawbacks, by means of a dominant-pole-placement approach. It still leads to a regulator of the form (16), but the criteria for choosing $C(s)$ and μ are rather different.

A. Problem Statement

Problem 4.1: Let a set of m negative numbers $\widehat{\mathcal{L}} := \left\{\hat{\lambda}_i | i \in \mathcal{M}\right\}$ and two positive scalars $\alpha \leq 1$ and β be given. Find a regulator $R(s)$, of the form (4b)-(4d), such that, for all $f_T \subset \mathcal{M}$ and for all $f_R \subset \mathcal{M}$, the following conditions are satisfied.

1) Equation (11) possesses $m - \kappa_{f_T \cup f_R}$ roots λ_i with

$$\left|\frac{\lambda_i - \hat{\lambda}_i}{\hat{\lambda}_i}\right| < \alpha \;,\; i \in \mathcal{M} \setminus \{f_T \cup f_R\} \;, \tag{18}$$

and $\sum_{i \in \mathcal{M} \setminus \{f_T \cup f_R\}} \rho_i$ roots ν_{i,l_i} with

$$\mathrm{Re}\{\nu_{i,l_i}\} < -\beta \;, \tag{19a}$$

$$l_i = 1, 2, \ldots, \rho_i \;,\; i \in \mathcal{M} \setminus \{f_T \cup f_R\} \;. \tag{19b}$$

2) Each one of (12) possesses one root λ_i with

$$\left|\frac{\lambda_i - \hat{\lambda}_i}{\hat{\lambda}_i}\right| < \alpha \;,\; i \in f_T \cup f_R \;, \tag{20}$$

and ρ_i roots ν_{i,l_i} with

$$\mathrm{Re}\{\nu_{i,l_i}\} < -\beta \;, \tag{21a}$$

$$l_i = 1, 2, \ldots, \rho_i \;,\; i \in f_T \cup f_R \;. \tag{21b}$$

□

The rationale behind Problem 4.1 can be explained as follows. The elements $\hat{\lambda}_i$s of $\widehat{\mathcal{L}}$ are the desired negative dominant poles, whereas α and β are design parameters. Then, the problem requires that the controlled network possesses m eigenvalues λ_i located "near" the $\hat{\lambda}_i$s. The parameter α specifies how "near" they have to be (see (18), (20)). Of course, it is reasonable that α be sufficiently small. Moreover, the locations of the $\sum_{i=1}^{m} \rho_i$ remaining eigenvalues is constrained by (19), (21). If $\beta \gg \max_{i \in \mathcal{M}} |\hat{\lambda}_i|$, then the λ_is can legitimately be interpreted as the dominant poles of the controlled network. The above must be true not only in nominal, but also in all possible faulty conditions. Hence, the problem can be interpreted as the formal statement of a fault-tolerant dominant-pole-placement issue.

Since the $\hat{\lambda}_i$s are arbitrary, the speed of response of the overall control system can be made as large as desired. Besides that, it turns out to be the same for all the couples of faults $(f_T, f_R) \subset \mathcal{M} \times \mathcal{M}$.

B. Problem Reformulation and Solution

The solution of the above fault-tolerant dominant-pole-placement problem can be obtained if reference is made to an appropriate reformulation of it. The idea underneath is similar to the one that was expedient in solving the fault-tolerant stabilization Problem 3.1. However, in the present case we prefer stating the reformulated problem explicitly.

Problem 4.2: Let a set of m negative numbers $\widehat{\mathcal{L}} := \left\{\hat{\lambda}_i | i \in \mathcal{M}\right\}$ and two positive scalars $\alpha \leq 1$ and β be given. Find a regulator $R(s)$, of the form (4b)-(4d), such that each one of the characteristic equations (13) possesses $m - \kappa_f$ roots λ_i with

$$\left|\frac{\lambda_i - \hat{\lambda}_i}{\hat{\lambda}_i}\right| < \alpha \;,\; i \in \mathcal{M} \setminus \{f\} \;,$$

and $\sum_{i \in \mathcal{M} \setminus \{f\}} \rho_i$ roots ν_{i,l_i} with

$$\mathrm{Re}\{\nu_{i,l_i}\} < -\beta \;,\; l_i = 1, 2, \ldots, \rho_i \;,\; i \in \mathcal{M} \setminus \{f\} \;.$$

□

The relationship between Problems 4.1 and 4.2 is very tight, as specified in the following lemma.

Lemma 4.1: Problem 4.1 admits a solution if and only if Problem 4.2 admits a solution.

Furthermore, a regulator $R(s)$, of the form (4b)-(4d), solves Problem 4.1 if and only if it solves Problem 4.2. □

A necessary solvability condition for Problem 4.1 immediately follows if we observe that pole-placement can be achieved only if the network is stabilizable. Thus, Problem 4.1 is solvable only if Problem 3.1 is solvable as well. Hence, the next result holds, in view of the preceding lemma and Theorem 3.1.

Lemma 4.2: Problem 4.2 admits a solution only if condition (15) holds. □

As far as sufficiency is concerned, we state a condition not expressed in terms of the problem data in the forthcoming lemma.

Lemma 4.3: If there exists a diagonal matrix V such that, for all $f \subset \mathcal{M}$, the matrices $G_{(f)}V_{(f)}$ are Hurwitz, then Problem 4.2 admits a solution.

Furthermore, there exist $\bar{\mu} > 0$ and $\bar{\varepsilon}(\cdot) > 0$ such that, for all $\mu > \bar{\mu}$ and for all $\varepsilon \in (0, \bar{\varepsilon}(\mu))$, the regulator

$$R(s) = \frac{\mu}{1 + \varepsilon s} V \left(sI_m - \widehat{\Lambda}\right) \;, \tag{22}$$

where

$$\widehat{\Lambda} := \mathrm{diag}\left\{\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_m\right\} \;, \tag{23}$$

solves Problem 4.2. □

This lemma gives a formula for the regulator, but neither explicitly supplies V, neither guarantees that it exists. These subproblems are the same as those called forth in the first step of the design procedure of the previous section. Here, we prefer to formally state their solutions in the next lemma, which directly follows from a result in [31].

Lemma 4.4: There exists a diagonal matrix V such that, for all $f \subset \mathcal{M}$, the matrices $G_{(f)}V_{(f)}$ are Hurwitz if and only if condition (15) holds.

Furthermore, under this condition, there exists $\bar{\zeta} > 0$ such that, for all $\zeta \in (0, \bar{\zeta})$, the eigenvalues of matrices $G_{(f)}V_{(f)}$, where V is given by (14), (16b), are all negative. □

In conclusion, we are able to state the main achievement of this paper, which results by combining the preceding lemmas.

Theorem 4.1: Problem 4.1 admits a solution if and only if condition (15) holds.

Furthermore, there exist $\bar{\zeta} > 0$, $\bar{\mu}(\cdot) > 0$ and $\bar{\varepsilon}(\cdot) > 0$ such that, for all $\zeta \in (0, \bar{\zeta})$, for all $\mu > \bar{\mu}(\zeta)$ and for all $\varepsilon \in (0, \bar{\varepsilon}(\mu))$, the regulator (14), (16a), (16b), (22), (23) solves the problem.

□

According to this theorem, a regulator solving Problem 4.1 can be designed by resorting to the following three-step procedure.

1) The first step consists in choosing a sufficiently small positive ζ such that the eigenvalues of the matrices $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}(\zeta)$, $f \subset \mathcal{M}$, are real negative. The existence of such a ζ is guaranteed.
   The comments on the first step of the stabilizing design procedure of Section II apply also here.
2) The second step calls for setting $\varepsilon = 0$ and finding a sufficiently large μ such that (18)-(21) are satisfied. Its existence is guaranteed.
   The idea underlying this point can easily be explained. Setting $\varepsilon = 0$ means that a proportional-derivative regulator is considered at the moment. Then, when $\mu \to +\infty$, the closed-loop eigenvalues tend to the zeros of the regulator, which are the desired dominant eigenvalues $\hat{\lambda}_i \in \widehat{\mathcal{L}}$.
3) The third step calls for assigning a sufficiently small positive value to ε such that (18)-(21) still hold. Such value exists.
   In so doing, negative eigenvalues with highly negative real parts are added to the regulator, which becomes proper. Standard singular perturbation theory [32] shows that the eigenvalues resulting when using the proportional-derivative regulator do not significantly move, whereas the new ones remain well far from the origin in the left-half of the complex plane.

We want to notice that condition (15) is necessary and sufficient for both Problems 3.1 and 4.1. Put in a different way, fault-tolerant dominant-pole-placement does not require anything more than simple fault-tolerant stabilization.

However, stabilization can be achieved by means of nondynamical regulators. On the contrary, our solution to dominant-pole-placement calls for the adoption of mth order regulators (see (22)).

Observe that the same regulator can also solve a dominant-pole-placement problem for networks of m double integrators, as we have shown in [21].

TABLE I

STABILIZATION PROBLEM: LOCATIONS OF THE POLES.

	λ_1/μ	λ_2/μ	λ_3/μ	λ_4/μ
$f_T = \emptyset$ $f_R = \emptyset$	-0.49	-6.50	-20.56	-56.45
$f_T = \{1\}$ $f_R = \{1\}$	-1.00	-5.96	-21.08	-55.96
$f_T = \{1\}$ $f_R = \{2\}$	-1.00	-9.00	-21.22	-52.78
$f_T = \{1\}$ $f_R = \{3\}$	-1.00	-5.61	-25.00	-52.39
$f_T = \{2\}$ $f_R = \{4\}$	-0.19	-9.00	-25.81	-49.00

V. TUTORIAL EXAMPLE

The theory presented in the preceding sections is now illustrated by means of a simple example.

A. Network

With reference to a network composed of 4 agents, we assume that the topology matrix is

$$G = \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 3 & 2 & 7 \\ 4 & 0 & 5 & 3 \\ 0 & 1 & 1 & 7 \end{bmatrix},$$

which fulfils (15). Then, simple computations show that matrices $-G_{(f)}\tilde{G}_{(f)}\mathrm{Z}_{(f)}(\zeta)$, $\zeta \in \mathcal{M}$, are Hurwitz for all positive ζ. Then, we set $\zeta = 1$ in (16b), so that $V = -\tilde{G}$.

B. Stabilization

The simplest regulator stabilizing the network corresponds to choosing $C(s) = I_4$ in (16c). Thus, from (16) it follows $R(s) = -\mu \mathrm{diag}\{1, 3, 5, 7\}$.

The values of the closed-loop eigenvalues divided by μ are collected in Table I. They refer to the nominal and to four faulty conditions. The table shows that stability is obtained in all cases. However, notice that the locations of the eigenvalues undergo significant changes.

C. Pole-Placement

In order to get the locations of the dominant poles independent of the particular fault, we resort to the pole-placement technique of Section IV.

We choose $\widehat{\Lambda} = -I_4$ in (23), and $\alpha = 0.05$, $\beta = 1000$ in Problem 4.1. With these parameters, the performances of the regulator (14), (16a), (16b), (22), (23), with $\mu = 300$, $\varepsilon = 0.01$, can be evaluated by looking at Table II. This table collects the relative errors in the locations of the dominant eigenvalues in the nominal and in the four faulty conditions considered before. It is evident that (18) and (20) are satisfied. Moreover, all the remaining eigenvalues are negative with minimum absolute values equal to 1.47×10^4, 3.01×10^4, 3.01×10^4, 3.01×10^4, 5.91×10^4 in the five considered cases. Thus, also (19) and (21) are fulfilled.

TABLE II

POLE-PLACEMENT PROBLEM: RELATIVE ERRORS IN THE LOCATIONS OF THE DOMINANT POLES.

	$\left\lvert\frac{\hat{\lambda}_1-\lambda_1}{\hat{\lambda}_1}\right\rvert$ $\times 10^3$	$\left\lvert\frac{\hat{\lambda}_2-\lambda_2}{\hat{\lambda}_2}\right\rvert$ $\times 10^3$	$\left\lvert\frac{\hat{\lambda}_3-\lambda_3}{\hat{\lambda}_3}\right\rvert$ $\times 10^3$	$\left\lvert\frac{\hat{\lambda}_4-\lambda_4}{\hat{\lambda}_4}\right\rvert$ $\times 10^3$
$f_T = \emptyset$ $f_R = \emptyset$	6.73	0.51	0.16	0.06
$f_T = \{1\}$ $f_R = \{1\}$	3.29	0.55	0.16	0.06
$f_T = \{1\}$ $f_R = \{2\}$	3.29	0.37	0.15	0.06
$f_T = \{1\}$ $f_R = \{3\}$	3.28	0.59	0.13	0.06
$f_T = \{2\}$ $f_R = \{4\}$	16.75	0.37	0.13	0.07

VI. CONCLUDING REMARKS

This paper faced a fault-tolerant dominant-pole-placement problem for a network of agents with single-integrator dynamics. We proved a necessary and sufficient solvability condition for it. Rather surprisingly, this problem and the less demanding one of fault-tolerant stabilization are solvable under the same condition. We also presented an explicit formula for the decentralized regulator. The local regulators to be applied to each agents are of the first order.

We may wonder whether this order is the least possible one. Within the general framework adopted in this paper, the answer to this question is negative. Indeed, it can easily be checked that when the topology matrix is diagonal a nondynamical regulator can solve the problem. However, this is plainly a trivial case. We conjecture that the order of our regulator is minimum, provided that the topology is such that each agent transmits and receives information in nominal conditions. This topic will be matter of future research.

Among other possible issues, we mention: (i) extending the present results to cases where the agents have internal dynamics different from that considered here; (ii) adapting the theory to deal with different (more general) models of faults; (iii) applying the here approach to consensus problems.

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