2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Random Convex Programs for Distributed Multi-Agent Consensus

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Abstract— **We consider convex optimization problems with N randomly drawn convex constraints. Previous work has shown that the tails of the distribution of the probability that the optimal solution subject to these constraints will violate the next random constraint, can be bounded by a binomial distribution. In this paper we extend these results to the violation probability of convex combinations of optimal solutions of optimization problems with random constraints and different cost objectives. This extension has interesting applications to distributed multi-agent consensus algorithms in which the decision vectors of the agents are subject to random constraints and the agents' goal is to achieve consensus on a common value of the decision vector that satisfies the constraints. We give explicit bounds on the tails of the probability that the agents' decision vectors at an arbitrary iteration of the consensus protocol violate further constraint realizations. In a numerical experiment we apply these results to a model predictive control problem in which the agents aim to achieve consensus on a control sequence subject to random terminal constraints.**

I. Introduction

Whenever decision need to be made based on data collected from the real-world, e.g., in control or machine learning, uncertainty in the data will almost certainly occur. In order to make decisions robust against uncertainties and to assure unanticipated constraints are not violated, the uncertainties have to be accounted for *ex-ante* in the decision-making process. Several approaches have been studied to counteract these difficulties and make decisions robust to uncertainty. The *robust convex optimization* approach [1], [2] finds a solution to a convex optimization problem that is robust to all uncertainty realizations bound to lie in a given bounded uncertainty set. *Chance-constrained* approaches assume that there is a probability measure over the uncertainty set and try to find an optimal decision that satisfies the constraints with high probability [3]. *Random convex programs* (RCPs) find an optimal solution to an optimization problem subject to a finite number of randomly drawn constraints, which can be done fast and efficiently with modern convex solvers. Since the constraints are randomly drawn, the optimal solution of an RCP is a random variable. For RCPs [4] and [5] initially provided bounds on the tails of the probability that an optimal solution found with N random constraints will become infeasible for the next randomly drawn constraint. These bounds were refined in [6] and [7] and further extended to cases in which a violation of some random constraints is tolerated in [7].

In this paper we first extend theoretical results on RCPs to the following case: Optimal solutions of RCPs with different cost directions but with the same random constraints will in general be different vectors within the space of feasible decision vectors. If one considers an arbitrary convex combination of these optimal solutions, the convexity of the constraint function guarantees that the convex combination will be feasible for all N random constraints. However, there are no results on the violation probability of such a convex combination yet and the first contribution of this paper is to give explicit bounds on the tails of their violation probability. These bounds do not depend on the coefficients in the convex combination and, hence, they actually allow us to bound the violation probability of an entire set contained inside the feasible region of the initial RCPs. The set is the convex polytope of all convex combinations of the optimal solutions of the initial RCPs.

The second contribution of this paper is the application of the above theoretical results to distributed multi-agent consensus problems subject to random constraints. Here the task of the agents is to employ a consensus algorithm in order to reach agreement on a common decision vector that is subject to N random constraint realizations. We assume that the initial values for the decision vectors are computed by each agent as the optimal solution to RCPs and each agent can potentially have its own cost vector in this computation. This assumption is applicable for example when consensus algorithms are utilized to find a common estimate based on uncertain data and each agent uses a different estimation technique to determine its own initial estimate. Another application is the coordination of UAVs under exogenous disturbances when each UAV first determines a trajectory that is optimal with respect to its own priorities regarding e.g. fuel-efficiency or path-length. For such a consensus problem we show that with high probability the decision vector for each agent at an arbitrary iteration of the consensus algorithm remains feasible for further realizations of the random constraints. This result is a consequence of the previous result on convex combinations since the consensus rule with which the agents update their decision vector with the ones of the other agents is essentially a repeated convex combination.

These "averaging" multi-agent consensus algorithms have received great interest in the recent past starting with [8], [9] mostly due to their robustness against changes of the communication topology and their distributed nature. Current studies mostly deal with convergence of the agents' decision vectors under different assumptions on the communication structure and update rules, for details see [10]–[12]. Potential applications of consensus algorithms are data fusion in sensor networks [13], the coordination of multiple vehicles [14], distributed control [12] and dynamic load balancing in multiprocessor networks [15]. The work [16] proposes a distributed algorithm (based on constraint consensus strategies [17]) for solving one large RCP where the cost objective is shared among all agents but each agent only has knowledge about a small subset of the random constraints. In contrast, in this paper all agents have knowledge about all random

This work was funded by Italian Ministry of University and Research with PRIN grant n. 20087W5P2K "Large-scale and distributed systems: optimization, estimation and control, with applications" and by the Research Training Group 1194 "Self-organizing Sensor-Actuator-Networks" from the German Research Foundation.

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constraints but can have different cost objectives.

The paper is structured as follows. In Section II we formally introduce random convex programs and their most important properties. Further we briefly highlight the general class of distributed multi-agent consensus algorithms we consider in this paper. In Section III we establish the first contribution of this paper, the bounds on the tails of the violation probability of convex combinations of optimal solutions of different RCPs. In Section IV we apply this result to multi-agent consensus. In Section V we present a numerical example for multi-agent consensus for model predictive control with terminal constraints. Section VI concludes the paper.

II. Preliminaries

In this section we first introduce random convex programs and present the main result on their generalization properties. Then, we briefly highlight the main concepts of distributed multi-agent consensus algorithms.

A. Random Convex Programs

Let $\delta \in \Delta \subset \mathbb{R}^l$ be a random vector with probability distribution $\mathbb{P}$, and denote by $\omega := (\delta^{(1)}, \ldots, \delta^{(N)})$ N independent realizations drawn from the random vector δ. Let $f(x, \delta)$ be a function that is convex and lower-semicontinuous in the argument x for all δ.

Definition 1 (Random Convex Program (RCP)): An optimization problem of the form

$$\begin{aligned} \mathrm{P}[\omega]: \ & \min_{x} \ c^\top x \\ & \text{s.t.} \ f(x, \delta^{(j)}) \leq 0, \ j = 1, \ldots, N \\ & \quad\ x \in \Omega \ , \end{aligned} \tag{1}$$

with a linear objective $c \neq 0$ and decision variable x confined to lie within a compact domain $\Omega \subset \mathbb{R}^d$ is called a *random convex program* (RCP). ◇

Denote the optimal solution of an RCP depending on the (multi-)sample ω by $x^*(\omega)$ and the optimal objective by $\mathrm{Obj}(\mathrm{P}[\omega])$. We assume that the optimal solution $x^*(\omega)$ is unique, which is not a severe restriction, since tie-braking rules (like a lexicographic ordering) can be employed to always ensure uniqueness (for details cf. [5]).

Definition 2 (Support Constraints): A constraint $\delta^{(j)}$, $j \in \{1, \ldots, N\}$ is called *support constraint* of $\mathrm{P}[\omega]$ if the optimal objective of the RCP strictly improves when constraint $\delta^{(j)}$ is removed from the sample ω. Denote the set of support constraints of $\mathrm{P}[\omega]$ by $\mathrm{Sc}(\mathrm{P}[\omega]) \subset \{\delta^{(1)}, \ldots, \delta^{(N)}\}$. ◇

Definition 3 (Nondegenerate Problem): An RCP $\mathrm{P}[\omega]$ is called *nondegenerate* for an sample ω if for the optimal objective holds $\mathrm{Obj}(\mathrm{P}[\omega]) = \mathrm{Obj}(\mathrm{Sc}(\mathrm{P}[\omega]))$[1], i.e., when the optimal objective computed with all constraints ω equals the optimal objective computed when only the support constraints are considered. ◇

Definition 4 (Violation Probability): For an RCP $\mathrm{P}[\omega]$ the *violation probability* is defined as

$$V^*(\omega) := \mathbb{P}\{\delta \in \Delta \colon f(x^*(\omega), \delta) > 0\} \,, \tag{2}$$

that is, the probability that the optimal solution $x^*(\omega)$ of $\mathrm{P}[\omega]$ computed under the realization ω will become infeasible under the next realization $\delta \in \Delta$. ◇

[1] With slight abuse of notation we denote here the optimal objective computed on the support constraint set by $\mathrm{Obj}(\mathrm{Sc}(\mathrm{P}[\omega]))$

Since both the optimal solution and the optimal objective value of an RCP depend on the sample ω they are random variables, as is the violation probability $V^*(\omega)$. The main result on random convex programs is the following.

Theorem 1 ([6], [7]): Let $\epsilon \in (0, 1]$ and $N \geq d - 1$, where d is the dimension of the decision vector x. For an RCP $\mathrm{P}[\omega]$ that is feasible and nondegenerate with probability one, the following holds

$$\mathbb{P}^N\big\{\omega \in \Delta^N \colon V^*(\omega) > \epsilon\big\} \leq \Phi(\epsilon; d-1, N) \ , \tag{3}$$

where

$$\Phi(\epsilon; q, N) := \sum_{j=0}^{q} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \tag{4}$$

is the cumulative binomial distribution. ◇

This theorem presents an a-priori assessment of the probability that the optimal solution of an RCP that was found on a finite number of constraint realizations remains feasible and, hence, optimal for further yet "unseen" random constraints.

B. Distributed Multi-Agent Consensus

In this section we will briefly outline algorithms for distributed multi-agent consensus (for more details see e.g. [9]–[12]). In these algorithms the goal of the agents is to agree on a common value of a decision vector through distributed coordination techniques. Consider a set $\mathcal{A}$ of agents $\mathcal{A} = \{1, \ldots, n\}$. Each agent i starts with an initial value of the decision vector $x_i(0)$ only known to himself. For the consensus protocol we consider discrete time steps $t = 0, 1, \ldots$ and at each of these time steps each agent i receives values from other agents. Agent i then updates its own decision vector according to the update rule

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t) x_j(t) \ , \tag{5}$$

where the weight matrix $A(t) = (a_{ij}(t))_{i,j=1,\ldots,n}$ at iteration t is a possibly time-varying matrix representing the communication topology of the agents and the consensus rule. In particular, $a_{ij}(t) > 0$ only if agent i received a value from agent j at time step t. One important property of the matrices $A(t)$ that is generally required in the literature and will be also crucial for our derivations is that $A(t)$ is a stochastic matrix for all time steps.

Definition 5: A matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ is called *stochastic* when $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$, for all $i = 1, \ldots, n$, i.e., all row sums are equal to one. ◇

A consequence of the stochasticity of the update matrices is that the decision vector $x_i(t)$ arises through taking iterated convex combinations of the initial decision vectors $x_1(0), \ldots, x_n(0)$.

In current research on multi-agent consensus the main focus lies on establishing convergence of the agents' local decision vectors under distributed communication protocols for $t \to \infty$ with different restrictions and assumptions on the communication topology and the matrices $A(t)$ [9], [11], [12]. Also, explicit rates of convergence are studied in [10].

III. Convex Combinations of Optimal Solutions of RCPs

In this section we consider the setup were not only the solution to an RCP with a single cost direction is computed, but the solutions of several RCPs with different cost vectors with the *same* constraint realizations are determined and the violation probability of their convex combination is studied. More precisely, we consider the optimal solutions $x_i^* := x_i^*(\omega)$ for $i = 1, \dots, n$ to n RCPs

$$\begin{aligned} \mathrm{P}_i[\omega]: \quad & \min_{x_i} c_i^\top x_i \\ & \text{s.t. } f(x_i, \delta^{(j)}) \le 0,\ j = 1, \dots, N \\ & \quad x_i \in \Omega \end{aligned} \tag{6}$$

where each of the $\mathrm{P}_i[\omega]$ satisfies the assumptions of Theorem 1. In general, for different cost directions the optimal solutions will also be different vectors in $\mathbb{R}^d$. These different optimal solutions x_i^* have in common that they all are feasible for the constraints arising from the sample ω. Consider now a vector $x_\lambda(\omega)$ that is the convex combination of the vectors $x_i^*(\omega)$, i.e.,

$$x_\lambda(\omega) = \sum_{i=1}^{n} \lambda_i x_i^*(\omega) \ , \tag{7}$$

with coefficients $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$. Due to the convexity of the constraint function $f(x, \delta)$, the vector $x_\lambda(\omega)$ will be feasible for all constraints arising from the samples $\delta^{(1)}, \dots, \delta^{(N)}$. However, although we can employ Theorem 1 to derive tails on the violation probability of each of the optimal solutions of the $\mathrm{P}_i[\omega]$, it is not obvious how one should reason on the violation probability of the point $x_\lambda(\omega)$. More precisely, an open question is "what are the tails of the probability that the vector $x_\lambda(\omega)$ becomes infeasible under the next random constraint?". The following theorem establishes an answer to this question.

Theorem 2: Let $x_\lambda(\omega)$ be a convex combination of the optimal solutions $x_i^*(\omega)$ to n RCPs as in Eq. (6) and let $N \ge nd - 1$. Then it holds that

$$\mathbb{P}^N\big\{\omega \in \Delta^N : V_\lambda^*(\omega) > \epsilon\big\} \le \Phi(\epsilon; nd-1, N) \ , \tag{8}$$

where

$$V_\lambda^*(\omega) = \mathbb{P}\{\delta \in \Delta : f(x_\lambda(\omega), \delta) > 0\} \tag{9}$$

is the probability that $x_\lambda(\omega)$ becomes infeasible under the next realization of δ and $\Phi(\epsilon; q, N)$ is as in Eq. (4). ⋄

The proof of this theorem is lengthy and technical but since it is of general interest and of importance for the following results it is included in the Appendix.

In the statement of the theorem and its proof it becomes apparent that the theorem does not depend on the particular choice of the convex combination coefficients λ_i. Hence, it follows that the theorem holds for *any arbitrary* convex combination of optimal solutions of n RCPs. So actually the theorem gives a bound on the tails of the violation probability of a whole subset (a convex polytope with vertices $x_i^*(\omega)$) of the random feasible region. Therefore, Theorem 2 extends the results on RCPs from statements on the violation probability of a single point in the random feasible region, i.e., the optimal solution of an RCP to the violation probability of an entire set.

IV. Multi-Agent Consensus and RCPs

The bounds on the violation probability of convex combinations of different optimal solutions of RCPs from the previous section have interesting applications to multi-agent consensus algorithms when the agents aim to agree on the value of a decision vector subject to random constraints.

As in general multi-agent consensus we consider a set $\mathcal{A}$ of agents $\mathcal{A} = \{1, \dots, n\}$. We assume that each of the agents' initial decision vectors is the optimal solution of an RCP $\mathrm{P}_i[\omega]$ as in Eq. (6). After each agent solved $\mathrm{P}_i[\omega]$, the optimal solutions $x_i^*(\omega)$ are taken as the initial values for a consensus algorithm as described in Section II-B, i.e., the goal of the agents is to achieve consensus through coordination and the exchange of messages on a value of the decision vector that still satisfies the random constraints. These constraints can for example stem from a dataset that is known to all agents or could be drawn from a random variable where each agent initializes its pseudorandom generator with the same random seed. Hence, the problem we want to consider is the following:

Problem 1 (Consensus with Random Constraints): The goal of the agents is to employ the consensus averaging protocol to find a common value of x^* such that $f(x^*, \delta^{(j)}) \le 0$, $j = 1, \dots, N$ under the assumption that the $x_i(0)$ are solutions of RCPs $\mathrm{P}_i[\omega]$. ⋄

We are interested in the probability that the current value $x_i(t)$ of an agent i at an iteration t of the consensus algorithm becomes infeasible under the next realization of the random constraints which is a generalization of the violation probability for normal RCPs.

Definition 6 (Consensus Violation Probability): For a distributed multi-agent consensus algorithm in which the initial vectors $x_i(0)$ of each agent are the optimal solutions to RCPs $\mathrm{P}_i[\omega]$, define the consensus violation probability as

$$V_{t,i}^*(\omega) = \mathbb{P}\{\delta \in \Delta : f(x_i(t), \delta) > 0\} \ , \tag{10}$$

the probability that the decision vector of agent i at coordination iteration t becomes infeasible under the next realization of the random constraints. ⋄

Theorem 3 (Multi-agent Consensus Feasibility): Let x_i^*, $i = 1, \dots, n$ be optimal solutions to n RCPs $\mathrm{P}_i[\omega]$ for $N \ge nd - 1$ and assume that each $\mathrm{P}_i[\omega]$ satisfies the assumptions of Theorem 1. Consider a distributed multi-agent consensus algorithm with initial decision vectors $x_i(0) = x_i^*$ for all $i \in \mathcal{A}$ and update rule for the agents' decision vectors according to Eq. (5) with stochastic matrix $A(t)$ for all $t \ge 0$. Then for $\epsilon \in (0, 1]$ and each time step $t \ge 0$ it holds that

$$\mathbb{P}^N\big\{\omega \in \Delta^N : V_{t,i}^*(\omega) > \epsilon\big\} \le \Phi(\epsilon; nd-1, N) \tag{11}$$

where $V_{t,i}^*(\omega)$ is the consensus violation probability and $\Phi(\epsilon; q, N)$ as in Eq. (4). ⋄

This means that for each agent at each iteration of the consensus algorithm the tails of the consensus violation probability are bounded by a binomial distribution.

Proof: Let $t = 1$ and consider an arbitrary agent $i \in \mathcal{A}$. We have that

$$x_i(1) = \sum_{j=1}^{n} a_{ij}(0) x_j(0) \tag{12}$$

and because $a_{ij}(0) \ge 0$ and $\sum_{j=1}^n a_{ij}(0) = 1$, $x_i(1)$ is the convex combination of the x_j^* that are the optimal solutions of $\mathrm{P}_i[\omega]$. We apply Theorem 2 to obtain

$$\mathbb{P}^N\{\omega \in \Delta^N : V_{i,1}^*(\omega) > \epsilon\} \leq \Phi(\epsilon; nd-1, N) \ . \quad (13)$$

Let $t \geq 2$. We can write $x_i(t+1)$ as

$$x_i(t+1) = \sum_{j=1}^{n} \tilde{a}_{ij}(t) x_j(0) \quad (14)$$

where $A(t)A(t-1)\cdots A(0) = (\tilde{a}_{ij}(t))_{i,j=1,\ldots,n}$ and all matrices $A(s)$ for $s = 0, \ldots, t$ are stochastic matrices. Since the product of stochastic matrices is again a stochastic matrix we can deduce for the coefficients $\tilde{a}_{ij}(t)$ that $\tilde{a}_{ij}(t) \geq 0$ and

$$\sum_{j=1}^{n} \tilde{a}_{ij}(t) = 1 \ . \quad (15)$$

Hence, $x_i(t+1)$ is still a convex combination of the x_i^* and by Theorem 2 it holds that

$$\mathbb{P}^N\{\omega \in \Delta^N : V_{i,t}^*(\omega) > \epsilon\} \leq \Phi(\epsilon; nd-1, N) \quad (16)$$

and the statement of the theorem follows. ∎

This result is of relevance in applications in which the decision vectors of the agents need to satisfy certain random constraints. It guarantees that for an arbitrary distributed consensus algorithm, as long as the update rule is based on stochastic weighting matrices, the local decision vectors of the agents will with high probability remain feasible for future realizations of the random constraints at each iteration of the algorithm.

In practical applications the agents will not coordinate for an infinite amount of time but the coordination will halt at some finite instance t_0. While most studies on consensus algorithms are concerned with properties of the values for $t \to \infty$ we here guarantee bounds on the violation probability for every iteration of the consensus algorithm and, hence, also for the decision vectors at a stopping time t_0.

V. Numerical Example

We apply the results of the previous chapter to a model predictive control (MPC) example in which each agent has a different control objective and wants to optimally control a system with random terminal constraints over a horizon of length T. The terminal constraint set is random in the sense that it is given through linear inequalities that are perturbed by a random vector. We utilize the results of the previous section to construct a distributed consensus algorithm on the control inputs of the agents for that it is guaranteed that at each iteration the terminal states resulting from the current values of the controls of the agents will lie in the terminal constraint set for further random constraint realizations with high probability.

To be more precise, the linear, time-invariant, discrete time state equation of the system is given through

$$x_i(k) = Ax_i(k-1) + Bu_i(k) \quad (17)$$

for $k = 1, \ldots, T$ with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \ , \ B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad (18)$$

and initial state $x_i(0) = [7, \ 0]^\top$. The control inputs $u_i(k) \in \mathbb{R}$ are constrained by $\|u_i(k)\|_\infty \leq 2$ for all $k = 1, \ldots, T$.

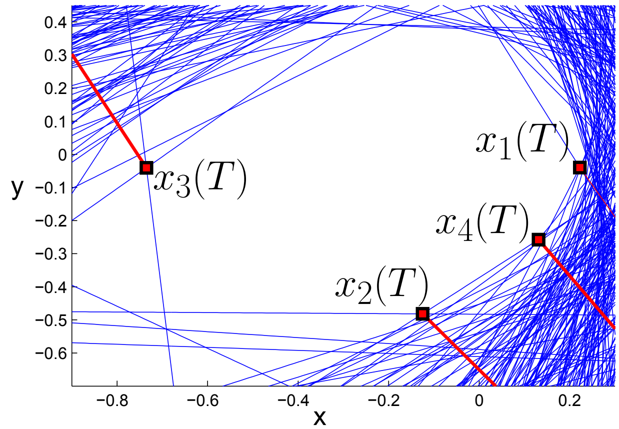

Fig. 1. Feasible set for the terminal states $x_i(T)$. The blue lines are the linear inequalities resulting from the realizations of the random constraints in Eq. (20) and the red squares are the terminal states $x_i(T)$ resulting from the optimal solutions of the agents' RCPs. The controls leading to these states are taken as the initial values for the consensus algorithm.

We consider $n = 4$ agents and the objective of agent i is

$$J_i(\mathbf{u}_i) = \sum_{k=1}^{T} (x_i(k) - z_i)^\top (x_i(k) - z_i) + ru_i(k), \quad (19)$$

for the control sequence $\mathbf{u}_i := (u_i(1), \ldots, u_i(T))$. Hence, we penalize deviations of the state from a point $z_i \in \mathbb{R}^2$ that is different for each agent, and the control effort is discounted by $r = 0.1$. Since the states $x_i(k)$ depend deterministically on the controls, only the controls $\mathbf{u}_i$ are decision variables. The values for the z_i we used are $z_1 = [7, \ 7]^\top$, $z_2 = [7, \ -7]^\top$, $z_3 = [10, \ 0]^\top$, $z_4 = [-7, \ 0]^\top$. The time horizon was set to $T = 7$.

Remark 1: Notice that the linear objective in the RCPs $\mathrm{P}_i[\omega]$ poses no restriction to the model and the quadratic objectives in Eq. (19) can be employed. This follows since a nonlinear convex objective function $g_i(x_i)$ can be transformed into a linear objective by adding a scalar slack variable t_i and considering the epigraphic reformulation of the problem with linear objective t_i and deterministic constraint $g_i(x_i) \leq t_i$. Theoretically the epigraphic reformulation adds another decision variable, the slack variable t_i, and, hence, the bound (3) in Thm. 1 would have to be formulated with d instead of $d-1$ in the binomial distribution. However, it is possible to show that in fact the slack variable t_i does not influence the probabilistic bounds and bound (3) holds for RCPs with general nonlinear convex objective without the need to increase the dimension of the decision variable. ◇

The terminal constraint set is given through the linear inequalities

$$a_l(\delta_l)^\top x_i(T) \leq b_l \ , l = 1, \ldots, 4 \quad (20)$$

with $a_1(\delta_1) = [1, \ 0]^\top + \delta_1$, $a_2(\delta_2) = [0, \ 1]^\top + \delta_2$, $a_3(\delta_3) = [1, \ 0]^\top + \delta_3$, $a_4(\delta_4) = [0, \ 1]^\top + \delta_4$ and the perturbations δ_l are zero-mean Gaussian random vectors with covariance matrices $\mathrm{diag}(0.5, 0.5)$ for $l = 1, 2, 3$ and $\mathrm{diag}(0.01, 0.01)$ for $l = 4$. The right hand sides in Eq. (20) are given through $b_1 = 2, \ b_2 = 1, \ b_3 = 0.5, \ b_4 = 0.5$.

Each agent's problem has T decision variables $(u_i(1), \ldots, u_i(T))$. We chose the confidence $\beta = 0.001$

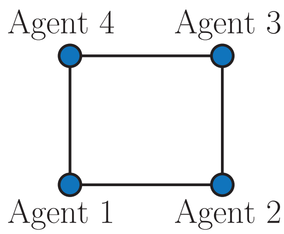

Fig. 2. Undirected cycle graph used for communication topology in the numerical example.

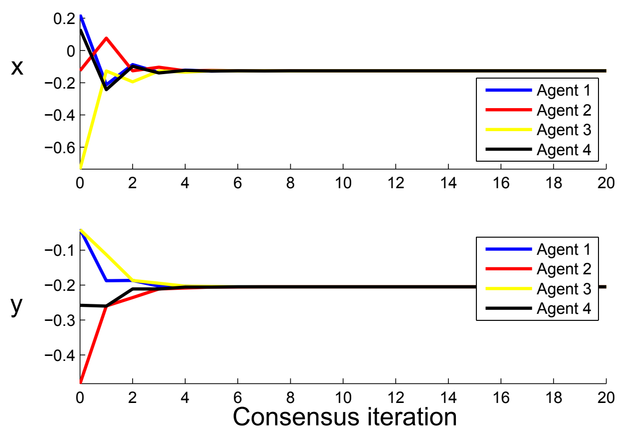

Fig. 3. Depicted are the x (top) and y (bottom) coordinate of the terminal states of each agent over 20 iterations of the consensus protocol.

and $\epsilon = 0.1$ and according to Thm. 3 we computed that we need to draw $N = 475$ samples of the terminal constraints to guarantee that $\mathbb{P}\{V_{t,i}^* > \epsilon\} \le \Phi(\epsilon; nd-1, N) \le \beta$. The different objectives will cause the agents to compute different optimal state trajectories and their goal is to reach consensus on a common state trajectory that still satisfies the terminal state constraints. The resulting MPC RCP problem for each agent i is

$$\begin{aligned} &\min_{\mathbf{u}_i} \ J_i(\mathbf{u}_i) \\ &\text{s.t.} \ \max_{l=1,\dots,4} \left\{ a_l(\delta_l^{(j)})^\top x_i(T) - b_l \right\} \le 0, \ j = 1,\dots,N \\ &\qquad \|u_i(k)\|_\infty \le 2, \ k = 1,\dots,T \\ &\qquad x_i(k) = Ax_i(k-1) + Bu_i(k), \ k = 1,\dots,T \\ &\qquad x_i(0) = [7, \ 0]^\top . \end{aligned} \tag{21}$$

and we solved these quadratic programs with solver software CPLEX [18]. The problem is feasible with probability one, since the terminal state $[0, \ 0]^\top$ is feasible with probability one and can be always reached given the input constraints. In Figure 1 we depict a realization of the random terminal constraint set where the blue lines are the realizations of the constraints in Eq. (20). The red squares are the terminal states $x_i(T)$ under the agents' controls and the red line connecting to the states is the end part of the state trajectory that lead to this particular terminal state.

The optimal controls $\mathbf{u}_i^* = (u_i^*(1), \dots, u_i^*(T))$ were then taken as initial values for multi-agent consensus. The communication topology is given by an undirected cycle graph (see Fig. 2 for an illustration). We assume a fixed coefficient, symmetric, equal neighbor model for the update rule, i.e., $a_{ij} = \frac{1}{3}$, if i communicates to j and $a_{ii} = \frac{1}{3}$. In Figure 3 we depict the impact of 20 iterations of the consensus algorithm on the x and y coordinate of the terminal states $x_i(T)$. Theorem 3 guarantees that with confidence of $1-\beta = 0.999$ when another sample of the random terminal constraints is drawn all terminal states $x_i(T)$ in Fig. 3 will still satisfy the terminal constraint with probability $1-\epsilon = 0.9$.

VI. Conclusions

We presented an extension of previous results on random convex programs by giving explicit bounds on the tails of the violation probability of convex combinations of optimal solutions of RCPs computed with different cost directions. This result allows us to reason on the violation probability of an entire set, the polytope with the aforementioned optimal solutions as vertices. As an application of this result we have studied a situation in which agents in a multi-agent system search for a common value of a decision vector subject to random constraints. We provided bounds on the tails of the consensus violation probability, i.e., the probability that the decision vector of an agent at an arbitrary iteration of the consensus algorithm becomes infeasible under the next random constraint realization. We applied this result to a multi-agent model predictive control problem with random terminal constraints.

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Appendix
Proof of Theorem 2

Proof: The proof proceeds in two steps. We first show that

$$\mathbb{P}^N\left\{\omega \in \Delta^N : V_\lambda^*(\omega) > \epsilon\right\} \le \mathbb{P}^N\left\{\omega \in \Delta^N : V_c^*(\omega) > \epsilon\right\} \tag{22}$$

where

$$\begin{aligned} V_c^*(\omega) &:= \mathbb{P}\{\delta \in \Delta\colon f(x_1^*, \delta) > 0 \textbf{ or } \ldots \textbf{or } f(x_n^*, \delta) > 0\} \\ &= \mathbb{P}\left\{\bigcup_{j=1}^n \{\delta \in \Delta\colon f(x_j^*, \delta) > 0\}\right\} . \end{aligned} \tag{23}$$

Then, we show that

$$\mathbb{P}^N\left\{\omega \in \Delta^N : V_c^*(\omega) > \epsilon\right\} \le \Phi(\epsilon; nd-1, N) . \tag{24}$$

a) Proof of (22): Because of the convexity of f we have for $x_\lambda := x_\lambda(\omega)$ that

$$f(x_\lambda, \delta) = f\left(\sum_{i=1}^n \lambda_i x_i^*, \delta\right) \le \sum_{i=1}^n \lambda_i f(x_i^*, \delta) . \tag{25}$$

It also holds that

$$\{\delta\colon f(x_\lambda, \delta) > 0\} = \Delta \setminus \{\delta\colon f(x_\lambda, \delta) \le 0\} . \tag{26}$$

From (25) it follows that if $\sum_{i=1}^n \lambda_i f(x_i^*, \delta) \le 0$ also $f(x_\lambda, \delta) \le 0$ and, hence,

$$\{\delta\colon \sum \lambda_i f(x_i^*, \delta) \le 0\} \subset \{\delta\colon f(x_\lambda, \delta) \le 0\} . \tag{27}$$

Then, it follows that

$$\begin{aligned} \{\delta\colon f(x_\lambda, \delta) > 0\} &\overset{(a)}{\subset} \{\delta\colon \sum \lambda_i f(x_i^*, \delta) > 0\} \\ &\overset{(b)}{\subset} \{\delta\colon f(x_1^*, \delta) > 0 \textbf{ or } \ldots \textbf{ or } f(x_n^*, \delta) > 0\} \\ &= \bigcup_{i=1}^n \{\delta\colon f(x_i^*, \delta) > 0\} , \end{aligned} \tag{28}$$

where (a) follows from equation (26) and inclusion (27) and (b) follows from the fact that all $\lambda_i \ge 0$ and, hence, for the sum to be strictly positive at least one of the summands has to be strictly positive. From the inclusions (28) it follows that $V_\lambda^*(\omega) = \mathbb{P}\{\delta\colon f(x_\lambda, \delta) > 0\} \le \mathbb{P}\{\delta\colon f(x_1^*, \delta) > 0 \textbf{ or } \ldots \textbf{ or } f(x_n^*, \delta) > 0\} = V_c^*(\omega)$ and if $V_\lambda^*(\omega) > \epsilon$ it follows that also $V_c^*(\omega) > \epsilon$ and so we obtain (22).

b) Proof of (24): Consider the RCP

$$\begin{aligned} \mathrm{P}_c[\omega] : \quad &\min_{x_1,\ldots,x_n} \sum_{i=1}^n c_i^\top x_i \\ &\text{s.t. } \max_{i=1,\ldots,n} \left(f(x_i, \delta^{(j)})\right) \le 0, \; j = 1, \ldots, N . \end{aligned}$$

$\mathrm{P}_c[\omega]$ is in fact convex since f is convex and taking the pointwise maximum of convex functions preserves convexity (see e.g. [19]). We want to apply Thm. 1 to $\mathrm{P}_c[\omega]$ and in order to do so we show that if each $\mathrm{P}_i[\omega]$ satisfies the assumptions of Thm. 1, then so will $\mathrm{P}_c[\omega]$.

We will show first that the component x_i^* of a joint feasible solution $(x_1^*, \ldots, x_n^*)$ of $\mathrm{P}_c[\omega]$ is feasible for $\mathrm{P}_i[\omega]$. Let $(x_1, \ldots, x_n)$ be feasible for $\mathrm{P}_c[\omega]$ which is equivalent to $\max_{i=1,\ldots,n}\left(f(x_i, \delta^{(j)})\right) \le 0$ for $j = 1, \ldots, N$, which is equivalent to $f(x_i, \delta^{(j)}) \le 0$ for all $j = 1, \ldots, N$ and for all $i = 1, \ldots, n$ and this is equivalent to the fact that every x_i is feasible for the corresponding $\mathrm{P}_i[\omega]$. Now we will show that each component of an optimal joint solution of $\mathrm{P}_c[\omega]$ is optimal for the corresponding $\mathrm{P}_i[\omega]$ and vice versa. Let $(x_1^*, \ldots, x_n^*)$ be optimal for $\mathrm{P}_c[\omega]$ and let $\hat{x}_1, \ldots, \hat{x}_n$ be optimal for the respective $\mathrm{P}_i[\omega]$. Since x_i^* is feasible for $\mathrm{P}_i[\omega]$ and $\hat{x}_i$ is optimal for $\mathrm{P}_i[\omega]$ we have that $c_i^\top \hat{x}_i \le c_i^\top x_i^*$. $(x_1^*, \ldots, x_n^*)$ is optimal for $\mathrm{P}_c[\omega]$ and $(x_1^*, \ldots, \hat{x}_i, \ldots, x_n^*)$ is feasible and we have $\sum_{j=1}^n c_j^\top x_j^* \le \sum_{j=1\ j\neq i}^n c_j^\top x_j^* + c_i^\top \hat{x}_i \Rightarrow c_i^\top x_i^* \le c_i^\top \hat{x}_i$ and it follows that $c_i^\top x_i^* = c_i^\top \hat{x}_i$ for all $i = 1, \ldots, n$. From the uniqueness of the optimal solution for each $\mathrm{P}_i[\omega]$ it follows that $x_i^* = \hat{x}_i$ for all i and it further follows that the optimal solution of $\mathrm{P}_c[\omega]$ is also unique.

Next, we prove that $\mathrm{P}_c[\omega]$ is nondegenerate with probability one. Let $\delta^{(j)}$ be a support constraint for $\mathrm{P}_c[\omega]$ i.e., $\delta^{(j)} \in \mathrm{Sc}(\mathrm{P}_c[\omega])$. If the objective of $\mathrm{P}_c[\omega]$ improves without constraint $\delta^{(j)}$, then $\sum_{i=1}^n c_i^\top x_i^*$ improves and hence some of the $\mathrm{P}_i[\omega]$ must have improved objectives. Hence, there are $i_1, \ldots, i_k$ with $\mathrm{P}_{i_l}(\omega)$ for $i_l \in \{i_1, \ldots, i_k\}$ has improved objective. So it follows that constraint $\delta^{(j)}$ is a support constraint for these $\mathrm{P}_{i_l}(\omega)$. Let $\delta^{(j)} \in \bigcup_{i=1}^n \mathrm{Sc}(\mathrm{P}_i[\omega])$ then there are $i_1, \ldots, i_k$ with $\mathrm{P}_{i_l}(\omega)$ for $i_l \in \{i_1, \ldots, i_k\}$ has improved objective and, hence, $\mathrm{P}_c[\omega]$ also has improved objective when constraint $\delta^{(j)}$ is omitted in $\mathrm{P}_c[\omega]$. So we obtain that $\mathrm{Sc}(\mathrm{P}_c[\omega]) = \bigcup_{i=1}^n \mathrm{Sc}(\mathrm{P}_i[\omega])$. Let all $\mathrm{P}_i[\omega]$ be nondegenerate, i.e., $\mathrm{Obj}(\mathrm{P}_i[\omega]) = \mathrm{Obj}(\mathrm{Sc}(\mathrm{P}_i[\omega]))$. We have that

$$\begin{aligned} &\mathrm{Obj}(\mathrm{P}_c[\omega]) \\ &= \sum_{i=1}^n \mathrm{Obj}(\mathrm{P}_i[\omega]) \overset{(1)}{=} \sum_{i=1}^n \mathrm{Obj}(\mathrm{Sc}(\mathrm{P}_i[\omega])) , \end{aligned} \tag{29}$$

where (1) follows because the $\mathrm{P}_i[\omega]$ are nondegenerate. Further, we have

$$\begin{aligned} \mathrm{Obj}(\mathrm{Sc}(\mathrm{P}_c[\omega])) &= \mathrm{Obj}\left(\bigcup_{i=1,\ldots,n} \mathrm{Sc}(\mathrm{P}_i[\omega])\right) \\ &= \sum_{i=1}^n \mathrm{Obj}\left(\bigcup_{i=1,\ldots,n} \mathrm{Sc}(\mathrm{P}_i[\omega])\right) \\ &= \sum_{i=1}^n \mathrm{Obj}(\mathrm{Sc}(\mathrm{P}_i[\omega]) \end{aligned} \tag{30}$$

and since the right hand sides of (29) and (30) are equal, so are the left hand sides and $\mathrm{P}_c[\omega]$ is nondegenerate. The samples $\delta^{(j)}$ are drawn independently with same distribution and the constraint sets in the $\mathrm{P}_i[\omega]$ do not depend on the order in which the samples are drawn. Hence, the samples for $\mathrm{P}_c[\omega]$ are also drawn independently and the constraints for $\mathrm{P}_c[\omega]$ also do not depend on the order in which the samples are drawn.

So we have proved that $\mathrm{P}_c[\omega]$ satisfies the assumptions of Thm. 1 and we apply it to obtain

$$\mathbb{P}^N\left\{\omega \in \Delta^N : V_c(\omega) > \epsilon\right\} \le \Phi(\epsilon; nd-1, N) \tag{31}$$

with

$$V_c(\omega) = \mathbb{P}\left\{\delta \in \Delta\colon \max_{i=1,\ldots,n} \left(f(x_i^*, \delta)\right) > 0\right\} \tag{32}$$

$$= \mathbb{P}\{\delta \in \Delta\colon \exists f(x_i^*, \delta) > 0\} \tag{33}$$

$$= \mathbb{P}\{\delta \in \Delta\colon f(x_1^*, \delta) > 0 \textbf{ or } \ldots \textbf{ or } f(x_n^*, \delta) > 0\}$$

because $\mathrm{P}_c[\omega]$ has $n \cdot d$ decision variables $(x_1, \ldots, x_n) \in \mathbb{R}^{nd}$. ∎