

# Robust Prescribed Performance Tracking Control for Unknown Underactuated Torpedo-like AUVs

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**Abstract**—This paper addresses the tracking control problem of 3D trajectories for underactuated torpedo-like Autonomous Underwater Vehicles (AUVs). A smooth control scheme is designed, without any a priori knowledge of the AUV model parameters, guaranteeing prescribed performance tracking despite the presence of external disturbances representing ocean currents and waves. This work is the first approach on underactuated systems employing the Prescribed Performance notion, originally proposed in [1]–[3]. Moreover, only the desired trajectory and none of its higher order derivatives is employed in the control scheme. Furthermore, the stability of the unactuated degrees of freedom is secured without incorporating the corresponding measurements in the control signal, thus simplifying implementation and increasing robustness against measurement noises. The proposed control scheme: i) is of low complexity, ii) avoids singularities that arise owing to the error definition in the kinematic model and iii) guarantees, through the appropriate selection of certain performance functions, configuration constraints that may arise owing to the limited field of view of cameras in case a vision based sensor system with on board cameras is adopted. Finally, simulation results clarify and verify the approach.

## I. INTRODUCTION

The motion control problem of AUVs continues to pose considerable challenges to system designers, especially when the vehicles are underactuated and exhibit large model uncertainty. A typical motion control problem is trajectory tracking which is concerned with the design of control laws that force a vehicle to reach and follow a reference trajectory. Classical approaches such as local linearization and input-output decoupling have been used in the past to design tracking controllers for underactuated vehicles. See [4] for a relevant survey on trajectory tracking control schemes. Nevertheless, the aforementioned methods yielded poor closed loop performance and the results were local, around only certain selected operating points. An alternative approach involved output feedback linearization [5]–[7], which, however, was not always possible. Finally, based on a combined approach involving Lyapunov theory and the backstepping technique various nonlinear trajectory tracking controllers have been reported during the last two decades [8]–[17].

Despite the recent progress in the tracking control for underactuated AUVs, certain issues still remain open. First,

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notice that even in case the actual AUV model is considered accurately known, external disturbances affect severely the tracking performance thus making the problem of guaranteeing prescribed performance difficult or impossible in certain situations. By prescribed performance we mean that the tracking error should converge to a predefined arbitrarily small residual set with convergence rate no less than a prespecified value. The tracking performance deterioration becomes more intense when uncertainty in the AUV model is also present. Moreover, all aforementioned works require measurement of the velocity and in some cases of the acceleration of the desired target, which is difficult or impossible in practice, reducing thus their applicability.

Contrary to the current state of the art in the AUV control literature, the controller proposed here achieves tracking with prescribed performance despite the presence of external disturbances representing ocean currents and waves and without requiring prior information of the AUV model parameters. This work is also the first approach on underactuated systems employing the Prescribed Performance notion, originally proposed in [1]–[3]. Moreover, only the desired trajectory and none of its higher order derivatives is incorporated in the control scheme. Furthermore, no measurement of the unactuated degrees of freedom is needed increasing thus its robustness against noises that corrupt the corresponding measurements. The proposed control scheme is of low complexity, avoids any singularities that arise owing to the error definition in the kinematic model and guarantees, through the appropriate selection of certain performance functions, configuration constraints that may arise owing to the limited field of view of cameras in case a vision based sensor system with on board cameras is adopted. Finally, simulation results clarify and verify the approach. Future research will address the extension of these results in case of low cost torpedo-like vehicles equipped with only one stern thruster and a couple of fins, where the pitch and yaw moments satisfy  $M \propto u^2 \delta_q$  and  $N \propto u^2 \delta_r$  with  $\delta_q, \delta_r$  denoting the angles of the corresponding fins.

## II. DEFINITIONS AND PRELIMINARIES

At this point, we recall some definitions and preliminary results that are necessary in the subsequent analysis.

### A. Prescribed Performance

It will be clearly demonstrated in the Main Results Section, that the control design is heavily connected to the pre-

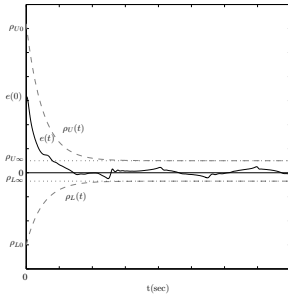


Fig. 1. Graphical illustration of the prescribed performance definition.

scribed performance notion that was originally employed to design neuro-adaptive controllers, for various classes of non-linear systems, namely feedback linearizable [1], strict feedback [2] and general MIMO affine in the control [3], capable of guaranteeing output tracking with prescribed performance. In this work, by prescribed performance, it is meant that the output tracking error converges to a predefined arbitrarily small residual set with convergence rate no less than a certain prespecified value. For completeness and compactness of presentation, this subsection summarizes preliminary knowledge on prescribed performance. In that respect, consider a generic scalar tracking error  $e(t)$ . Prescribed performance is achieved if  $e(t)$  evolves strictly within a predefined region that is bounded by certain functions of time. The mathematical expression of prescribed performance is given,  $\forall t \geq 0$ , by the following inequalities:

$$\rho_L(t) < e(t) < \rho_U(t) \quad (1)$$

where  $\rho_L(t)$ ,  $\rho_U(t)$  are smooth and bounded functions of time satisfying  $\rho_U(t) > \rho_L(t)$ ,  $\forall t \geq 0$  and  $\lim_{t \rightarrow \infty} \rho_U(t) > \lim_{t \rightarrow \infty} \rho_L(t)$ , called performance functions. The aforementioned statements are clearly illustrated in Fig. 1 for exponential performance functions  $\rho_i(t) = (\rho_{i0} - \rho_{i\infty})e^{-l_i t} + \rho_{i\infty}$  with  $\rho_{i0}$ ,  $\rho_{i\infty}$ ,  $l_i$ ,  $i \in \{L, U\}$  appropriately chosen constants. The constants  $\rho_{L0} = \rho_L(0)$ ,  $\rho_{U0} = \rho_U(0)$  are selected such that  $\rho_{U0} > e(0) > \rho_{L0}$ . The constants  $\rho_{L\infty} = \lim_{t \rightarrow \infty} \rho_L(t)$ ,  $\rho_{U\infty} = \lim_{t \rightarrow \infty} \rho_U(t)$  represent the maximum allowable size of the tracking error  $e(t)$  at the steady state, which may be set arbitrarily small to a value reflecting the resolution of the measurement device, thus achieving practical convergence of  $e(t)$  to zero. Moreover, the decreasing rate of  $\rho_L(t)$ ,  $\rho_U(t)$  which is affected by the constants  $l_L$ ,  $l_U$  in this case, introduces a lower bound on the required speed of convergence of  $e(t)$ .

### B. Dynamical Systems

Consider the initial value problem:

$$\dot{\xi} = h(t, \xi), \quad \xi(0) = \xi^0 \in \Omega_\xi \quad (2)$$

with  $h : \mathbb{R}_+ \times \Omega_\xi \rightarrow \mathbb{R}^n$  where  $\Omega_\xi \subset \mathbb{R}^n$  is a non-empty open set.

**Definition 1:** [18] A solution  $\xi(t)$  of the initial value problem (2) is maximal if it has no proper right extension that is also a solution of (2).

**Theorem 1:** [18] Consider the initial value problem (2). Assume that  $h(t, \xi)$  is: a) locally Lipschitz on  $\xi$  for almost all  $t \in \mathbb{R}_+$ , b) piecewise continuous on  $t$  for each fixed  $\xi \in \Omega_\xi$  and c) locally integrable on  $t$  for each fixed  $\xi \in \Omega_\xi$ . Then, there exists a maximal solution  $\xi(t)$  of (2) on the time interval  $[0, \tau_{\max})$  with  $\tau_{\max} > 0$  such that  $\xi(t) \in \Omega_\xi$ ,  $\forall t \in [0, \tau_{\max})$ .

**Proposition 1:** [18] Assume that the hypotheses of Theorem 1 hold. For a maximal solution  $\xi(t)$  on the time interval  $[0, \tau_{\max})$  with  $\tau_{\max} < \infty$  and for any compact set  $\Omega'_\xi \subset \Omega_\xi$  there exists a time instant  $t' \in [0, \tau_{\max})$  such that  $\xi(t') \notin \Omega'_\xi$ .

## III. PROBLEM FORMULATION

A six DoFs dynamic model of a torpedo-like AUV is presented in this section and a rigorous formulation of the problem of tracking a desired trajectory is introduced. The vehicle considered in this work is actuated by a force  $X$  along its longitudinal axis (surge), by a torque  $M$  around its transverse axis (pitch) as well as by a torque  $N$  around its vertical axis (yaw). The aforementioned force  $X$  and torques  $M$ ,  $N$  define the input control variables of the dynamic system. In this respect, the vehicle is underactuated since only three of its six DoFs are actuated (sway, heave and roll are unactuated).

In that respect, consider a neutrally buoyant AUV modeled as a rigid body subject to external forces and torques. Let  $\{I\}$  be an inertial coordinate frame and  $\{B\}$  a body-fixed coordinate frame with orthonormal axes  $\bar{n} = [n_x, n_y, n_z]^T$ ,  $\bar{o} = [o_x, o_y, o_z]^T$ ,  $\bar{t} = [t_x, t_y, t_z]^T$  relatively to  $\{I\}$ , whose origin  $O_B$  is located at the center of mass of the vehicle that coexists with the center of buoyancy. Further let  $\mathbf{p} = [x, y, z]^T$  be the position of  $O_B$  in  $\{I\}$  and  $R = [\bar{n}, \bar{o}, \bar{t}]$  a rotation matrix that describes the orientation of the vehicle. Let  $\mathbf{v} = [u, v, w]^T$  be the linear velocity ( $u$ ,  $v$ ,  $w$  are the longitudinal-surge, transverse-sway and vertical-heave velocities respectively) and  $\mathbf{w} = [p, q, r]^T$  be the angular velocity ( $p$ ,  $q$ ,  $r$  are the angular velocities around the longitudinal-roll, the transverse-pitch and vertical-yaw axis respectively) of  $O_B$  with respect to  $\{I\}$  expressed in  $\{B\}$ . Hence, the kinematic equations of motion for the considered AUV can be written as:

$$\dot{\mathbf{p}} = R\mathbf{v} + \delta_{\mathbf{v}}(t) \quad (3)$$

$$\dot{R} = RS(\mathbf{w}) \quad (4)$$

where  $\delta_{\mathbf{v}}(t) = [\delta_x(t), \delta_y(t), \delta_z(t)]^T$  denotes bounded ocean currents and  $S(\mathbf{w}) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$ .

The dynamic equations of motion for the underactuated AUV considered in this work can be written as:

$$m_u \dot{u} = m_v v r - m_w w q + X_u u + X + \delta_u(t) \quad (5)$$

$$m_v \dot{v} = m_w w p - m_u u r + Y_v v + \delta_v(t) \quad (6)$$

$$m_w \dot{w} = m_u u q - m_v v p + Z_w w + \delta_w(t) \quad (7)$$

$$m_p \dot{p} = m_{vv} v w + m_{qr} q r + K_p p + \delta_p(t) \quad (8)$$

$$m_q \dot{q} = m_{uu} u w + m_{rp} r p + M_q q + M + \delta_q(t) \quad (9)$$

$$m_r \dot{r} = m_{uv} u v + m_{pq} p q + N_r r + N + \delta_r(t) \quad (10)$$

where  $m_u, m_v, m_w, m_p, m_q, m_r$  denote the AUV's mass/moment of inertia and added mass/moment of inertia terms with  $m_{vv} = m_v - m_w, m_{wu} = m_w - m_u, m_{uv} = m_u - m_v, m_{qr} = m_q - m_r, m_{rp} = m_r - m_p, m_{pq} = m_p - m_q$  and  $X_u, Y_v, Z_w, K_p, M_q, N_r$  are negative hydrodynamic damping coefficients. Moreover,  $\delta_u(t), \delta_v(t), \delta_w(t), \delta_p(t), \delta_q(t), \delta_r(t)$  denote bounded exogenous forces and torques acting on surge, sway, heave, roll, pitch, yaw owing to ocean waves and  $X, M, N$  denote the control input force and torques respectively that are applied in order to produce the desired motion of the body fixed frame. Furthermore, notice that there is no control input in the sway, heave and roll equation of motion. Hence, Eqs. (6)-(8) are uncontrolled making thus (3)-(10) an underactuated dynamical system. Finally, it should be noted that contrary to the majority of the works in the current AUV literature, where the Euler angles parametrization is mainly employed owing to its physical meaning, our approach, which is based on the rotation matrix parametrization, does not suffer from geometric singularities when the pitch angle approaches  $\pm 90^\circ$ .

To proceed, let  $x_d(t), y_d(t), z_d(t)$  denote a smooth, bounded desired trajectory with bounded time derivatives. The objective of this paper is to design a robust controller, without incorporating any information regarding the AUV model, such that it tracks the desired trajectory with bounded closed loop signals and prescribed performance (concerning the steady state error and the speed of convergence) despite the presence of exogenous disturbances representing ocean currents and waves. Let us now define the position error:

$$e_d = \sqrt{e_x^2 + e_y^2 + e_z^2} \quad (11)$$

where

$$e_x = x_d(t) - x, e_y = y_d(t) - y, e_z = z_d(t) - z \quad (12)$$

as well as the orientation errors:

$$e_o = \frac{e_x}{e_d} o_x + \frac{e_y}{e_d} o_y + \frac{e_z}{e_d} o_z = \cos(\theta_o) \quad (13)$$

$$e_t = \frac{e_x}{e_d} t_x + \frac{e_y}{e_d} t_y + \frac{e_z}{e_d} t_z = \cos(\theta_t) \quad (14)$$

where  $\theta_o, \theta_t$  are the angles measured from the vector  $\bar{e}_d = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}^T$  to the transverse  $\bar{o}$  and vertical  $\bar{t}$  axis of the AUV respectively. Eqs. (13) and (14) can be easily verified if we consider the inner product of the unite vector  $\bar{e}_d$  with the unite vectors of the transverse  $\bar{o}$  and vertical  $\bar{t}$  axis respectively, which equals to the cosine of the angle defined by the aforementioned vectors. Fig. 2 illustrates the error coordinates at a time instant  $t_0$ . Hence, the tracking problem stated above is solved if the position error  $e_d$  and the orientation errors  $e_o, e_t$  reduce to zero (i.e., the AUV is heading to the desired trajectory since in such case the unite vector  $\bar{e}_d$  tends to be normal to both the transverse  $\bar{o}$  and vertical  $\bar{t}$  axis of the AUV and consequently aligned to its longitudinal  $\bar{n}$  axis). However, notice that the orientation errors  $e_o, e_t$  are well-defined only for nonzero values of  $e_d$ , since for  $e_d = 0$  the angles  $\theta_o, \theta_t$  are unidentified. Thus, in this work, we will design a controller that further guarantees that  $e_d(t) > \underline{\rho}_d > 0, \forall t \geq 0$  for an arbitrarily small positive

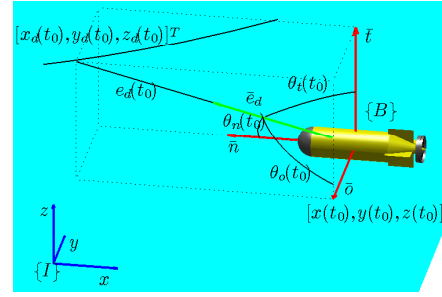


Fig. 2. Graphical illustration of the error definition at a time instant  $t_0$ .

design constant  $\underline{\rho}_d$ , to avoid the aforementioned singularity issue. Finally, to solve the tracking problem stated above, we make the following assumption regarding the position error  $e_d(t)$  and the heading angle  $\theta_n(t)$  measured from the vector  $\bar{e}_d$  to the longitudinal  $\bar{n}$  axis of the AUV.

*Assumption 1:* The initial position error  $e_d(0)$  and orientation heading  $\theta_n(0)$  satisfy: a)  $e_d(0) > \underline{\rho}_d$  and b)  $|\theta_n(0)| < \frac{\pi}{2}$ .

#### IV. MAIN RESULTS

In this section, we shall first present a model-free and low-complexity control scheme and subsequently we shall prove that it leads to the solution of the tracking control problem stated in the previous section.

##### A. Control Scheme

Given a smooth, bounded desired trajectory  $x_d(t), y_d(t), z_d(t)$  with bounded time derivatives, and any initial system configuration satisfying Assumption 1 for an arbitrarily small positive design constant  $\underline{\rho}_d$ :

##### I. Kinematic Controller

Select position/orientation performance functions  $\rho_d(t), \rho_o(t), \rho_t(t)$  that i) satisfy:

a. $e_d(0) < \rho_d(0)$	$\underline{\rho}_d < \rho_d(t)$	$\underline{\rho}_d < \lim_{t \rightarrow \infty} \rho_d(t)$
b. $ e_o(0)  < \rho_o(0) < 1$	$0 < \rho_o(t)$	$0 < \lim_{t \rightarrow \infty} \rho_o(t)$
c. $ e_t(0)  < \rho_t(0) < 1$	$0 < \rho_t(t)$	$0 < \lim_{t \rightarrow \infty} \rho_t(t)$
d. $\rho_o^2(t) + \rho_t^2(t) \leq \bar{\rho} < 1, \forall t \geq 0$		

for a positive constant  $\bar{\rho} < 1$  and ii) incorporate the desired performance specifications regarding the steady state error and the speed of convergence. Subsequently, design the desired velocities:

$$\begin{aligned} u_d &= k_d \ln \left( \left( 1 + \frac{e_d - \frac{\rho_d(t) + \underline{\rho}_d}{2}}{\frac{\rho_d(t) - \underline{\rho}_d}{2}} \right) / \left( 1 - \frac{e_d - \frac{\rho_d(t) + \underline{\rho}_d}{2}}{\frac{\rho_d(t) - \underline{\rho}_d}{2}} \right) \right) \\ q_d &= -k_t \ln \left( \left( 1 + \frac{e_t}{\rho_t(t)} \right) / \left( 1 - \frac{e_t}{\rho_t(t)} \right) \right) \\ r_d &= k_o \ln \left( \left( 1 + \frac{e_o}{\rho_o(t)} \right) / \left( 1 - \frac{e_o}{\rho_o(t)} \right) \right) \end{aligned} \quad (15)$$

with positive control gains  $k_d, k_o, k_t$ .

##### II. Dynamic Controller

Select velocity performance functions  $\rho_u(t), \rho_q(t), \rho_r(t)$  that satisfy:

a. $ u(0) - u_d(0)  < \rho_u(0)$	$0 < \rho_u(t)$	$0 < \lim_{t \rightarrow \infty} \rho_u(t)$
b. $ q(0) - q_d(0)  < \rho_q(0)$	$0 < \rho_q(t)$	$0 < \lim_{t \rightarrow \infty} \rho_q(t)$
c. $ r(0) - r_d(0)  < \rho_r(0)$	$0 < \rho_r(t)$	$0 < \lim_{t \rightarrow \infty} \rho_r(t)$

and design the external force in the surge as well as the external torques in pitch and yaw as:

$$\begin{aligned} X &= -k_u \ln \left( \left( 1 + \frac{u-u_d}{\rho_u(t)} \right) / \left( 1 - \frac{u-u_d}{\rho_u(t)} \right) \right) \\ M &= -k_q \ln \left( \left( 1 + \frac{q-q_d}{\rho_q(t)} \right) / \left( 1 - \frac{q-q_d}{\rho_q(t)} \right) \right) \\ N &= -k_r \ln \left( \left( 1 + \frac{r-r_d}{\rho_r(t)} \right) / \left( 1 - \frac{r-r_d}{\rho_r(t)} \right) \right) \end{aligned} \quad (16)$$

with positive control gains  $k_u, k_q, k_r$ .

*Remark 1:* The proposed control scheme does not incorporate any prior knowledge of the AUV dynamic parameters as well as of the external disturbances, or even of some corresponding upper bounding constants. Furthermore, no estimation (i.e., adaptive control) has been employed to acquire such knowledge. Moreover, compared with the traditional backstepping-like approaches, the proposed methodology proves significantly less complex. Notice that no hard calculations are required to output the proposed control signals thus making its implementation straightforward. Additionally, note that no velocity measurement of the unactuated degrees of freedom (sway, heave, roll) is incorporated in the control scheme. Such issue increases greatly the robustness against possible noises that corrupt the specific measurements and simplifies further the implementation. Finally, the proposed control scheme is independent of the time derivatives of  $x_d(t), y_d(t), z_d(t)$ , which is very important in practice where the velocity of the desired target is difficult or impossible to be measured.

## B. Stability Analysis

The main results of this work are summarized in the following theorem where it is proven that the aforementioned control scheme solves the tracking problem stated in the previous section.

*Theorem 2:* Consider the AUV model (3)-(10) in any initial configuration satisfying Assumption 1 as well as any smooth, bounded desired trajectory  $x_d(t), y_d(t), z_d(t)$  with bounded derivatives. The proposed control scheme (15)-(16) solves the tracking problem stated in Section III.

*Proof:* Let us first define the normalized errors:

$$\xi_d = \frac{e_d - \frac{\rho_d(t) + \rho_d}{2}}{\frac{\rho_d(t) - \rho_d}{2}}, \xi_o = \frac{e_o}{\rho_o(t)}, \xi_t = \frac{e_t}{\rho_t(t)} \quad (17)$$

$$\xi_u = \frac{u - u_d}{\rho_u(t)}, \xi_q = \frac{q - q_d}{\rho_q(t)}, \xi_r = \frac{r - r_d}{\rho_r(t)}. \quad (18)$$

In this respect, the desired velocities (15) and the control law (16) may be written as functions of the normalized errors  $\xi_i, i \in I$  with  $I = \{d, o, t, u, q, r\}$ , as follows:

$$u_d = k_d \ln \left( \frac{1 + \xi_d}{1 - \xi_d} \right), q_d = -k_o \ln \left( \frac{1 + \xi_o}{1 - \xi_o} \right), r_d = k_t \ln \left( \frac{1 + \xi_t}{1 - \xi_t} \right) \quad (19)$$

$$X = -k_u \ln \left( \frac{1 + \xi_u}{1 - \xi_u} \right), M = -k_q \ln \left( \frac{1 + \xi_q}{1 - \xi_q} \right), N = -k_r \ln \left( \frac{1 + \xi_r}{1 - \xi_r} \right). \quad (20)$$

Let us now define the overall closed loop system state vector:

$$\xi = [\xi_d \ \xi_o \ \xi_t \ \xi_u \ \xi_q \ \xi_r \ v \ w \ p]^T.$$

Differentiating the normalized errors with respect to time and employing (3)-(14) as well as (19)-(20), we obtain in

a compact form, the dynamical system of the overall state vector:

$$\dot{\xi} = h(t, \xi) \quad (21)$$

where the function  $h(t, \xi)$  includes all terms found at the right hand side after the differentiation of  $\xi$ . Let us also define the open set:

$$\Omega_\xi = \underbrace{(-1, 1) \times \dots \times (-1, 1)}_{6\text{-times}} \times (-\bar{v}, \bar{v}) \times (-\bar{w}, \bar{w}) \times (-\bar{p}, \bar{p})$$

where  $\bar{v}, \bar{w}, \bar{p}$  are positive constants to be specified later, for analysis purposes only. In the sequel, we proceed in two phases. First, the existence of a maximal solution  $\xi(t)$  of (21) over the set  $\Omega_\xi$  for a time interval  $[0, \tau_{\max})$  (i.e.,  $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$ ) is ensured. Then, we prove that the proposed control scheme guarantees, for all  $t \in [0, \tau_{\max})$ : a) the boundedness of all closed loop signals of (21) as well as that b)  $\xi(t)$  remains strictly within a compact subset of  $\Omega_\xi$ , which subsequently will lead by contradiction to  $\tau_{\max} = \infty$  and consequently to the solution of the tracking problem stated in Section III. Moreover, it should be noticed that the design parameter  $\rho_d$  and the performance functions  $\rho_d(t), \rho_o(t), \rho_t(t)$  were selected (see Subsection IV-A) such that  $e_d > \rho_d > 0$  and  $|\theta_n| \leq \bar{\theta}_n < \frac{\pi}{2}, \forall \xi \in \Omega_\xi$ . In this respect, the proposed scheme avoids any singularity issues that have been the main drawbacks in similar control approaches for AUVs, via guaranteeing that  $\xi(t)$  remains strictly within a compact subset of  $\Omega_\xi$ .

*Phase A.* The set  $\Omega_\xi$  is nonempty and open. Moreover, owing to the selection of the performance functions  $\rho_i(t), i \in I$  as well as to Assumption 1 we conclude that  $\xi(0) \in \Omega_\xi$  for sufficiently large  $\bar{v}, \bar{w}, \bar{p}$  satisfying  $|v(0)| < \bar{v}, |w(0)| < \bar{w}, |p(0)| < \bar{p}$ . Additionally, due to the smoothness of: a) the system nonlinearities, b) the desired trajectory and c) the proposed control scheme over  $\Omega_\xi$ , it can be easily verified that  $h(t, \xi)$  is continuous on  $t$  and continuous for all  $\xi \in \Omega_\xi$ . Therefore, the hypotheses of Theorem 1 stated in Subsection II-B hold and the existence of a maximal solution  $\xi(t)$  of (21) on a time interval  $[0, \tau_{\max})$  such that  $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$  is ensured.

*Phase B.* We have proven in Phase A that  $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$  or equivalently that:

$$\xi_i(t) \in (-1, 1), i \in I \quad (22)$$

and  $|v(t)| < \bar{v}, |w(t)| < \bar{w}, |p(t)| < \bar{p}$  for all  $t \in [0, \tau_{\max})$ . Therefore, the signals:

$$\varepsilon_i(t) = \ln \left( \frac{1 + \xi_i(t)}{1 - \xi_i(t)} \right), i \in I \quad (23)$$

are well defined for all  $t \in [0, \tau_{\max})$ . Consider now the positive definite and radially unbounded function  $V_d = \frac{1}{2} \varepsilon_d^2$ . Differentiating with respect to time and employing (11), (23), we obtain:

$$\begin{aligned} \dot{V}_d &= \frac{4\varepsilon_d}{(1 - \xi_d^2) (\rho_d(t) - \rho_d)} (-u \cos(\theta_n) - v \cos(\theta_o) \\ &\quad - w \cos(\theta_t) + \frac{e_x}{e_d} (\dot{x}_d - \delta_x) + \frac{e_y}{e_d} (\dot{y}_d - \delta_y) \\ &\quad + \frac{e_z}{e_d} (\dot{z}_d - \delta_z) - (1 + \xi_d) \frac{\dot{\rho}_d(t)}{2} \end{aligned} \quad (24)$$

Incorporating  $u = u_d + \xi_u \rho_u(t)$  from (18) and substituting  $u_d$  from (19) and  $\varepsilon_d$  from (23),  $\dot{V}_d$  becomes:

$$\begin{aligned} \dot{V}_d = & \frac{4\varepsilon_d}{(1 - \xi_d^2)(\rho_d(t) - \underline{\rho}_d)} (-\xi_u \rho_u(t) \cos(\theta_n) - v \cos(\theta_o) \\ & - w \cos(\theta_t) + \frac{e_x}{e_d} (\dot{x}_d - \delta_x) + \frac{e_y}{e_d} (\dot{y}_d - \delta_y) \\ & + \frac{e_z}{e_d} (\dot{z}_d - \delta_z) - (1 + \xi_d) \frac{\dot{\rho}_d(t)}{2} - k_d \varepsilon_d \cos(\theta_n) \Big). \end{aligned}$$

Furthermore, utilizing: i) (22), ii) the fact that  $\dot{\rho}_d(t)$ ,  $\rho_u(t)$ ,  $\dot{x}_d(t)$ ,  $\dot{y}_d(t)$ ,  $\dot{z}_d(t)$ ,  $\delta_x(t)$ ,  $\delta_y(t)$ ,  $\delta_z(t)$  are bounded by construction and by assumption and iii)  $|v(t)| \leq \bar{v}$ ,  $|w(t)| < \bar{w}$ ,  $\left| \frac{e_x}{e_d} \right| \leq 1$ ,  $\left| \frac{e_y}{e_d} \right| \leq 1$ ,  $\left| \frac{e_z}{e_d} \right| \leq 1$ , we arrive at:

$$\begin{aligned} & |-\xi_u \rho_u(t) \cos(\theta_n) - v \cos(\theta_o) - w \cos(\theta_t) \\ & + \frac{e_x}{e_d} (\dot{x}_d - \delta_x) + \frac{e_y}{e_d} (\dot{y}_d - \delta_y) + \frac{e_z}{e_d} (\dot{z}_d - \delta_z) \\ & - (1 + \xi_d) \frac{\dot{\rho}_d(t)}{2} \Big| \leq \bar{F}_d, \end{aligned}$$

for an unknown positive constant  $\bar{F}_d$ . Moreover,  $\frac{1}{(1 - \xi_d^2)} > 1$  and as mentioned earlier  $\cos(\theta_n) \geq \cos(\bar{\theta}_n) > 0$  and  $\rho_d(t) - \underline{\rho}_d > 0$  for all  $t \in [0, \tau_{\max})$ . Therefore, we conclude that  $\dot{V}_d$  is negative when  $|\varepsilon_d(t)| > \frac{\bar{F}_d}{k_d \cos(\bar{\theta}_n)}$  and consequently that:

$$|\varepsilon_d(t)| \leq \bar{\varepsilon}_d = \max \left\{ |\varepsilon_d(0)|, \frac{\bar{F}_d}{k_d \cos(\bar{\theta}_n)} \right\}, \quad (25)$$

for all  $t \in [0, \tau_{\max})$ , which from (23), taking the inverse logarithmic function, leads to:

$$-1 < \frac{e^{-\bar{\varepsilon}_d} - 1}{e^{-\bar{\varepsilon}_d} + 1} = \underline{\xi}_d \leq \xi_d(t) \leq \bar{\xi}_d = \frac{e^{\bar{\varepsilon}_d} - 1}{e^{\bar{\varepsilon}_d} + 1} < 1 \quad (26)$$

for all  $t \in [0, \tau_{\max})$ . As a result, the desired velocity  $u_d$  remains bounded (i.e.,  $|u_d(t)| \leq k_d \bar{\varepsilon}_d$ ) for all  $t \in [0, \tau_{\max})$ . Following similar analysis with  $V_o = \frac{1}{2} \varepsilon_o^2$ ,  $V_t = \frac{1}{2} \varepsilon_t^2$ , we arrive at  $|\varepsilon_o(t)| \leq \bar{\varepsilon}_o = \max \left\{ |\varepsilon_o(0)|, \frac{\bar{F}_o}{k_o \cos(\bar{\theta}_n)} \right\}$ ,

$|\varepsilon_t(t)| \leq \bar{\varepsilon}_t = \max \left\{ |\varepsilon_t(0)|, \frac{\bar{F}_t}{k_t \cos(\bar{\theta}_n)} \right\}$ ,  $\forall t \in [0, \tau_{\max})$  for some positive constants  $\bar{F}_o$  and  $\bar{F}_t$ , which further lead in:

$$-1 < \frac{e^{-\bar{\varepsilon}_o} - 1}{e^{-\bar{\varepsilon}_o} + 1} = \underline{\xi}_o \leq \xi_o(t) \leq \bar{\xi}_o = \frac{e^{\bar{\varepsilon}_o} - 1}{e^{\bar{\varepsilon}_o} + 1} < 1 \quad (27)$$

$$-1 < \frac{e^{-\bar{\varepsilon}_t} - 1}{e^{-\bar{\varepsilon}_t} + 1} = \underline{\xi}_t \leq \xi_t(t) \leq \bar{\xi}_t = \frac{e^{\bar{\varepsilon}_t} - 1}{e^{\bar{\varepsilon}_t} + 1} < 1. \quad (28)$$

Moreover, invoking (18), it is straightforward to deduce the boundedness of  $u(t)$ ,  $q(t)$ ,  $r(t)$  and consequently, differentiating (19) with respect to time and after some algebraic manipulations, the boundedness of  $\dot{u}_d(t)$ ,  $\dot{q}_d(t)$ ,  $\dot{r}_d(t)$ ,  $\forall t \in [0, \tau_{\max})$ .

Applying the aforementioned line of proof for the dynamic part (5)-(10) of the AUV, considering  $V_u = \frac{1}{2} \varepsilon_u^2$ ,  $V_q = \frac{1}{2} \varepsilon_q^2$ ,  $V_r = \frac{1}{2} \varepsilon_r^2$  and the proposed control law (16), we conclude that:

$$|\varepsilon_i(t)| \leq \bar{\varepsilon}_i = \max \left\{ |\varepsilon_i(0)|, \frac{m_i \bar{F}_i}{k_i} \right\}, \quad \forall t \in [0, \tau_{\max}) \quad (29)$$

for some positive constants  $\bar{F}_i$ ,  $i \in \{u, q, r\}$ . Correspondingly, we arrive at:

$$-1 < \frac{e^{-\bar{\varepsilon}_i} - 1}{e^{-\bar{\varepsilon}_i} + 1} = \underline{\xi}_i \leq \xi_i(t) \leq \bar{\xi}_i = \frac{e^{\bar{\varepsilon}_i} - 1}{e^{\bar{\varepsilon}_i} + 1} < 1 \quad (30)$$

for  $i \in \{u, q, r\}$ ,  $\forall t \in [0, \tau_{\max})$  as well as at the boundedness of the control law (16). In the sequel, special attention will be paid on the stability of the unactuated (sway, heave, roll) degrees of freedom. Initially notice that  $u$ ,  $q$  and  $r$  have been proven bounded for all  $t \in [0, \tau_{\max})$ . Moreover,  $\delta_v(t)$ ,  $\delta_w(t)$  and  $\delta_p(t)$  are assumed bounded. Thus, owing to the negativity of the hydrodynamic damping coefficients  $Y_v$ ,  $Z_w$ ,  $K_p$  in (6)-(8) there exist positive constants  $\bar{v}'$ ,  $\bar{w}'$ ,  $\bar{p}'$  which depend on the magnitude of  $\bar{u}$ ,  $\bar{q}$ ,  $\bar{r}$ ,  $m_u$ ,  $m_v$ ,  $m_w$ ,  $m_p$ ,  $m_q$ ,  $m_r$ ,  $Y_v$ ,  $Z_w$ ,  $K_p$ ,  $\delta_v(t)$ ,  $\delta_w(t)$ ,  $\delta_p(t)$  as well as of the control gains  $k_d$ ,  $k_o$ ,  $k_t$  implicitly, such that  $|v(t)| \leq \bar{v}'$ ,  $|w(t)| \leq \bar{w}'$ ,  $|p(t)| \leq \bar{p}'$ ,  $\forall t \in [0, \tau_{\max})$ .

Up to this point, what remains to be shown is that  $t_{\max} = \infty$ . Notice that (26), (28) and (30) imply that  $\xi(t) \in \Omega'_\xi$ ,  $\forall t \in [0, \tau_{\max})$ , where:

$$\Omega'_\xi = \prod_{i \in I} \left[ \frac{e^{-\bar{\varepsilon}_i} - 1}{e^{-\bar{\varepsilon}_i} + 1}, \frac{e^{\bar{\varepsilon}_i} - 1}{e^{\bar{\varepsilon}_i} + 1} \right] \times [-\bar{v}', \bar{v}'] \times [-\bar{w}', \bar{w}'] \times [-\bar{p}', \bar{p}']$$

is a nonempty and compact set. Moreover, it can be easily verified that  $\Omega'_\xi \subset \Omega_\xi$  for certain control gain values, system parameters and external disturbances satisfying  $\bar{v}' < \bar{v}$ ,  $\bar{w}' < \bar{w}$ ,  $\bar{p}' < \bar{p}$ . Hence, assuming  $\tau_{\max} < \infty$  and since  $\Omega'_\xi \subset \Omega_\xi$ , Proposition 1 in Subsection II-B dictates the existence of a time instant  $t' \in [0, \tau_{\max})$  such that  $\xi(t') \notin \Omega'_\xi$ , which is a clear contradiction. Therefore,  $\tau_{\max} = \infty$ . As a result, all closed loop signals remain bounded and moreover  $\xi(t) \in \Omega'_\xi \subset \Omega_\xi$ ,  $\forall t \geq 0$ . Finally, from (17), (26), (27) and (28), we conclude that:

$$\begin{aligned} \underline{\rho}_d < \frac{e^{-\bar{\varepsilon}_d} - 1}{e^{-\bar{\varepsilon}_d} + 1} \frac{\rho_d(t) - \underline{\rho}_d}{2} + \frac{\rho_d(t) + \underline{\rho}_d}{2} \\ & \leq e_d(t) \leq \\ & \frac{e^{\bar{\varepsilon}_d} - 1}{e^{\bar{\varepsilon}_d} + 1} \frac{\rho_d(t) - \underline{\rho}_d}{2} + \frac{\rho_d(t) + \underline{\rho}_d}{2} < \rho_d(t) \end{aligned}$$

$$-\rho_o(t) < \frac{e^{-\bar{\varepsilon}_o} - 1}{e^{-\bar{\varepsilon}_o} + 1} \rho_o(t) \leq e_o(t) \leq \frac{e^{\bar{\varepsilon}_o} - 1}{e^{\bar{\varepsilon}_o} + 1} \rho_o(t) < \rho_o(t)$$

$$-\rho_t(t) < \frac{e^{-\bar{\varepsilon}_t} - 1}{e^{-\bar{\varepsilon}_t} + 1} \rho_t(t) \leq e_t(t) \leq \frac{e^{\bar{\varepsilon}_t} - 1}{e^{\bar{\varepsilon}_t} + 1} \rho_t(t) < \rho_t(t)$$

for all  $t \geq 0$  and consequently that tracking with prescribed performance is achieved, as presented in Eq. (1) of Subsection II-A, which completes the proof.  $\blacksquare$

*Remark 2:* From the aforementioned proof, it is worth noticing that the proposed control scheme achieves its goals without residing to the need of rendering  $\bar{\varepsilon}_i$ ,  $i \in I$  arbitrarily small, through extreme values of the control gains  $k_i$ ,  $i \in I$ . In this spirit, large model uncertainties can be compensated, as they affect only the size of  $\bar{\varepsilon}_i$ ,  $i \in I$  but leave unaltered the achieved stability properties. In fact, the actual tracking performance, which is solely determined by the performance functions  $\rho_d(t)$ ,  $\rho_o(t)$ ,  $\rho_t(t)$ , becomes isolated against model uncertainties thus extending greatly the robustness of the proposed control scheme. Moreover, the selection of the control gains  $k_i$ ,  $i \in I$  is significantly simplified to adopting those values that lead to reasonable control effort.

## V. SIMULATION RESULTS

To illustrate the tracking performance and the robustness of the proposed scheme against external disturbances

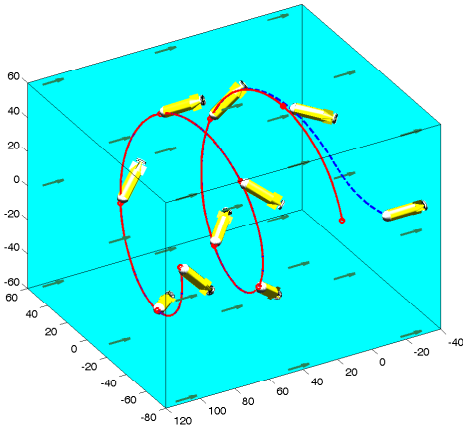


Fig. 3. Trajectory tracking in 3D space. The red solid line indicates the desired trajectory and the blue dashed line indicates the AUV trajectory. The green arrows illustrate the ocean current.

representing ocean currents and waves, a simulation study was carried out on a 6 DOF torpedo-like AUV actuated in surge, pitch and yaw. The vehicle dynamic model used in the simulation can be found in [19]. The AUV starts at rest from initial configuration:  $x(0) = -65$  m,  $y(0) = -15$  m,  $z(0) = 5$  m,  $R(0) = [0 \ 1 \ 0; 1 \ 0 \ 0; 0 \ 0 \ 1]$  and is requested to track the trajectory:  $x_d(t) = -50 \cos(0.02\pi t)$  m,  $y_d(t) = 0.15\pi t$  m,  $z_d(t) = 50 \sin(0.02\pi t)$  m, with maximum steady state errors  $e_{d_{\max}} = 0.1$ ,  $e_{o_{\max}} = 0.01$ ,  $e_{t_{\max}} = 0.01$  and minimum convergence rate as obtained by the exponentials  $e^{-0.1t}$ ,  $e^{-t}$  and  $e^{-t}$ . Notice that the aforementioned initial configuration satisfies Assumption 1 for  $\underline{\rho}_d = 0.05$ . Hence, following Subsection IV-A, we select the performance functions:  $\rho_d(t) = (40 - 0.1)e^{-0.1t} + 0.1$ ,  $\rho_o(t) = (0.45 - 0.01)e^{-t} + 0.01$ ,  $\rho_t(t) = (0.45 - 0.01)e^{-t} + 0.01$ ,  $\rho_u(t) = (2 - 0.1)e^{-t} + 0.1$ ,  $\rho_q(t) = (2 - 0.1)e^{-t} + 0.1$ ,  $\rho_r(t) = (2 - 0.1)e^{-t} + 0.1$  and the control gains:  $k_d = 5$ ,  $k_o = 1$ ,  $k_t = 1$ ,  $k_u = 500$ ,  $k_q = 100$ ,  $k_r = 100$  such that (15)-(16) yield reasonable control effort. Moreover, we considered a case where the ocean current is  $d_x(t) = 0$ ,  $d_y(t) = -2.5$ ,  $d_z(t) = 0$  and the wave terms include three sinusoids with amplitude and frequency equally distributed in  $[0.1, 1]$  and  $[0.01, 0.1]$  respectively. The trajectory tracking is shown in Fig. 3 and the evolution of the output errors along with the performance bounds is pictured in Fig. 4. As it was predicted by the theoretical analysis, tracking with prescribed performance is achieved despite the presence of external disturbances and the lack of knowledge of the AUV model parameters.

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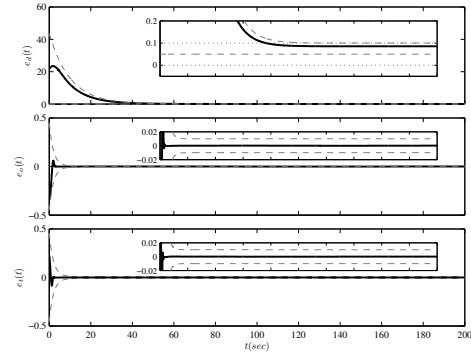


Fig. 4. Tracking error evolution. The grey dashed lines indicate the desired performance bounds. The black solid lines indicate the evolution of  $e_d(t)$ ,  $e_o(t)$  and  $e_t(t)$ . The subplots give details at the steady state.

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