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Continuous-time IO Systems Identification through Downsampled Models

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Abstract— **An indirect downsampling approach for continuous-time input/output system identification is proposed. This *modus operandi* was introduced to system identification through a subspace algorithm, where the input/output data set is partitioned into lower rate m subsets. Then, a state-space discrete-time model is identified by fusing the data subsets into a single one. In the present work the identification of the input/output downsampled model is performed by a least squares and a simplified refined instrumental variables (IV) procedures. In this approach, the inter-sample behaviour is preserved by the addition of fictitious inputs, leading to an increase of excitation requirements of the input signal. This over requirement is removed by directly estimating from the data the parameters of the transfer function numerator. The performance of the method is illustrated using the Rao-Garnier test system.**

I. Introduction

Identification of continuous-time (CT) systems is a fundamental research topic, since most physical systems are inherently continuous. However, due to the digital treatment of information, data is usually available in the discrete-time (DT) form, making the identification of DT systems more widespread. Nevertheless, difficulties arise when trying to recover the CT model parameters from the DT ones. See [10], [12], [16], [17] for a detailed discussion on indirect identification of CT systems.

During the last years, a new interest has raised to identify CT models directly from the DT data. Although direct identification methods avoid conversion between models, they have the drawback of having to handle the numerical evaluation of non-measurable time-derivatives in noisy environments. Extensive surveys of methods to solve this problem can be found in [3], [5], [13], [17], [20].

In [10] an indirect approach is proposed that circumvents the numerical problems of the model conversion between DT and CT due to fast sampling. The system is sampled as fast as it needs and then tuned to a convenient sampling period using a downsampling approach. Some of the parameters of the CT system may be directly estimated from the data. A prescribed number of zeros of the DT model is also secured. The downsampled (DS) algorithm was compared with the state-space (SS) methods of Contsid [3], [5] using a fast sampling example. Besides outperforming the Contsid SS methods, it also showed a performance comparable to the Simplified Refined Instrumental Variables for CT (SRIVC) method [19].

Contrary to the SS models, the conversion between the IO CT parameters and the IO DT parameters is not straightforward and the needs to be done through adequate SS realizations. In the present work, the DS approach is extended to the identification of transfer function (TF) models. The paper is organised in the following way: In Section II the indirect identification of CT input/output (IO) models is addressed using a least squares (LS) algorithm. In Section III the DS model is described and the LS algorithm is extended to estimate the parameters of the DS IO model. The SRIV [21] is adapted to the DS IO model identification in Section IV. In Section V, the performance of the proposed algorithm is assessed using the Rao-Garnier test system [12], a benchmark simulation example. Conclusions are drawn in the last section as well as some directions for the future work.

II. Indirect CT IO model identification

Consider the CT multiple-input single-output system

$$\frac{d^n y(t)}{dt}+a_1\frac{d^{n-1}y(t)}{dt^{n-1}}+\dots a_n y(t) = b_0\frac{d^n u(t)}{dt}+\cdots+b_n u(t), \tag{1}$$

where $y \in \mathbb{R}$, $u \in \mathbb{R}^{n_u}$, $a_i \in \mathbb{R}$, $i = 1,\dots n$, and $b_j \in \mathbb{R}^{1\times n_u}$, $j = 0,\dots,n$. Defining the derivative operator $p = \dfrac{d}{dt}$, equation (1) may be written in the compact form $A(p)y(t) = By(t)$, where

$$\begin{aligned} A(p) &= p^n + a_1 p^{n-1} + \cdots + a_n, & (2)\\ B(p) &= b_0 p^n + b_1 p^{n-1} + \cdots + b_n. & (3)\end{aligned}$$

The problem of system identification may now be stated as the estimation of the coefficients of the polynomials A and B from a record of the samples of the IO data $u(kT_s)$ and $y(kT_s)$, for $k = 0, 1, \dots, N-1$, where N is a suitable large number and T_s is the sampling period. The DT data may be modelled by

$$\alpha(q^{-1})y(k) = \beta(q^{-1})u(k), \tag{4}$$

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where $x(k) = x(kT_s)$, q^{-1} is the delay operator $q^{-1}x(k) = x(k-1)$ and

$$\alpha(q^{-1}) = 1 + \alpha_1 q^{-1} + \cdots + \alpha_n q^{-n}, \tag{5}$$
$$\beta(q^{-1}) = \beta_0 + \beta_1 q^{-1} + \cdots + \beta_n q^{-n}. \tag{6}$$

The coefficients of $\alpha(q^{-1})$ and $\beta(q^{-1})$ have a non-linear and non-straightforward relation with the coefficients of $A(p)$ and $B(p)$, that depends on the inter-sample behaviour of $u(t)$. However, there is a direct relation between SS realisations of (1) and (4). For instance, assuming a zero-order-hold (ZOH) inter-sample behaviour, the exact conversion of a CT system SS realisation

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t), \\ y(t) &= C_c x(t) + D_c u(t) \end{aligned} \tag{7}$$

to the DT system SS realisation

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k), \\ y(k) &= C_d u(k) + D_d u(k) \end{aligned} \tag{8}$$

is given by

$$A_d = e^{A_c T_s}, \tag{9}$$
$$B_d = \int_0^{T_s} e^{A_c(t-\tau)} d\tau B_c, \tag{10}$$
$$C_d = C_c, \tag{11}$$
$$D_d = D_c. \tag{12}$$

Since this is an injective relation (for an adequate T_s), the CT SS model can be recovered from the inverse relations of (9)–(12). Observe that the system matrix is recovered using $A_c = \frac{1}{T_s}\log A_d$. Consequently, A_d cannot have real negative eigenvalues with odd multiplicity because this would lead to a matrix A_c with complex entries. As a result, some care must be taken to avoid identified DT models with negative real poles.

If (8) is in the observable canonical form (OCF), i.e.,

$$A_d = \begin{bmatrix} 0 & \cdots & 0 & -\alpha_n \\ 1 & \cdots & 0 & -\alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\alpha_1 \end{bmatrix},$$
$$B_d = \begin{bmatrix} \beta_n - \beta_0\alpha_n \\ \beta_{n-1} - \beta_0\alpha_{n-1} \\ \vdots \\ \beta_1 - \beta_0\alpha_1 \end{bmatrix}, \tag{13}$$
$$C_d = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad D_d = \beta_0,$$

and T_{co} is the similarity transformation matrix between CT realisation (7) and the correspondent OCF, then the relation between the coefficients of $A(p)$ and $B(p)$ and of $\alpha(q^{-1})$ and $\beta(q^{-1})$ is forthright [7], [11]. Defining

$$\theta_A = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \end{bmatrix}^T \in \mathbb{R}^n, \tag{14}$$
$$\theta_B = \begin{bmatrix} b_n^T & b_{n-1}^T & \cdots & b_0^T \end{bmatrix}^T \in \mathbb{R}^{(n+1)\times n_u}, \tag{15}$$
$$\bar{\theta}_\alpha = \begin{bmatrix} \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 \end{bmatrix}^T \in \mathbb{R}^n, \tag{16}$$
$$\bar{\theta}_\beta = \begin{bmatrix} \beta_n^T & \beta_{n-1}^T & \cdots & \beta_0^T \end{bmatrix}^T \in \mathbb{R}^{(n+1)\times n_u}, \tag{17}$$

then

$$\theta_A = \frac{1}{T_s} T_{co}^{-1} \log\left\{\begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & -\bar{\theta}_\alpha \end{bmatrix}\right\} T_{co} e_n, \tag{18}$$

with $e_i,\ i = 1, \ldots n$, the vectors of the canonical basis of $\mathbb{R}^n$ and with $\mathcal{I}(\theta_A) = \int_0^{T_s} e^{A_c \tau} d\tau$ it becomes:

$$\theta_B = \begin{bmatrix} T_{co}^{-1}\mathcal{I}^{-1}(\theta_A) & -T_{co}^{-1}\mathcal{I}^{-1}(\theta_A)\bar{\theta}_\alpha \\ 0_{1\times n} & 1 \end{bmatrix} \bar{\theta}_\beta. \tag{19}$$

The available data is usually contaminated with random noise which is either produced by disturbances or introduced in data acquisition and measurements. As a result, a term accounting for noise must be included in both CT and DT models. The collected data is thus modelled by

$$\alpha(q^{-1})y(k) = \beta(q^{-1})u(k) + \varepsilon(k), \tag{20}$$

where $\varepsilon(k)$ is the noise term. Hence $y(k) = \varphi(k)\theta + \varepsilon(k)$, with

$$\varphi(k) = \begin{bmatrix} -y(k-n) & \cdots & -y(k-1) \\ & & u(k-n) & \cdots & u(k) \end{bmatrix}, \tag{21}$$
$$\theta = \begin{bmatrix} \theta_\alpha^T & \theta_\beta^T \end{bmatrix}^T, \tag{22}$$

and $\theta_\alpha = \bar{\theta}_\alpha$ and $\theta_\beta = \text{vec}(\bar{\theta}_\beta^T)$. The coefficients of $\alpha(q^{-1})$ and $\beta(q^{-1})$ can now be estimated by the well known LS estimator (LSE) given by $\hat{\theta} = \left(\Phi^T\Phi\right)^{-1}\Phi^T Y$, where

$$\Phi = \begin{bmatrix} \varphi(n)^T & \varphi(n+1)^T & \cdots & \varphi(N-1)^T \end{bmatrix}^T,$$
$$Y = \begin{bmatrix} y(n) & y(n+1) & \cdots & y(N+1) \end{bmatrix}^T.$$

The coefficients of $A(p)$ and $B(p)$ may be recovered through equations (18)–(19). If $\hat{\theta}_\alpha$ is calculated before $\hat{\theta}_\beta$, then θ_B may be directly estimated from the data. This can be done by reformulating the LS problem in a way that $\hat{\theta}_\alpha$ and $\hat{\theta}_\beta$ are calculated separately. Given that $\hat{\theta}$ is the solution of the LS problem $\hat{\theta} = \min_\theta \|Y - \Phi\theta\|_2^2$, it can be rewritten as

$$\hat{\theta} = \begin{bmatrix} \hat{\theta}_\alpha^T & \hat{\theta}_\beta^T \end{bmatrix}^T = \min_{\hat{\theta}_\alpha, \hat{\theta}_\beta} \|Y - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta\|_2^2. \tag{23}$$

Lemma 1: Let $\hat{\theta}_\alpha$ and $\hat{\theta}_\beta$ be the solutions of (23). Then they are also the solutions of

$$\hat{\theta}_\alpha = \min_{\theta_\alpha} \left\| Y\backslash_{\Phi_U^\perp} - \Phi_Y\backslash_{\Phi_U^\perp}\theta_Y \right\|_2^2, \tag{24}$$
$$\hat{\theta}_\beta = \min_{\theta_\beta} \left\| Y\backslash_{\Phi_Y^\perp} - \Phi_U\backslash_{\Phi_Y^\perp}\theta_\beta \right\|_2^2, \tag{25}$$

where $A\backslash_B$ stands for the orthogonal projection the column-space of A into the column-space of B and $B^\perp$ is a matrix whose columns span the orthogonal complement of the column-space of B.

Proof:

Define $E := Y - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta$ and decompose Y and Φ_Y in the following manner:

$$\begin{aligned} Y &= Y\backslash_{\Phi_U} + Y^T\backslash_{\Phi_U^\perp} \\ \Phi_Y &= \Phi_Y\backslash_{\Phi_U} + \Phi_Y\backslash_{\Phi_U^\perp}, \end{aligned}$$

Then

$$\begin{aligned} Y - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta &= \underbrace{Y\backslash_{\Phi_U} - (\Phi_Y\backslash_{\Phi_U}\theta_\alpha - \Phi_U\theta_\beta)}_{E\backslash_{\Phi_U}} \\ &+ \underbrace{\left(Y\backslash_{\Phi_U^\perp} - \Phi_Y\backslash_{\Phi_U^\perp}\theta_\alpha\right)}_{E\backslash_{\Phi_U^\perp}}. \end{aligned}$$

Given that the column-spaces of $E\backslash_{\Phi_U}$ and $E\backslash_{\Phi_U^\perp}$ are orthogonal to each other, we have

$$\left\|Y^T - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta\right\|_2^2 = \underbrace{\|E\backslash_{\Phi_U}\|_2^2}_{\Sigma_U} + \underbrace{\left\|E\backslash_{\Phi_U^\perp}\right\|_2^2}_{\Sigma_{U\perp}}$$

and

$$\begin{aligned} \min\left\|Y^T - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta\right\|_2^2 &= \min\{\Sigma_U + \Sigma_{U^\perp}\} \\ &= \min\{\Sigma_U\} + \min\{\Sigma_{U^\perp}\}. \end{aligned}$$

Since Σ_U belongs to the column-space of Φ_U, for a given θ_α there is always a θ_β such that $\Sigma_U = 0$. Consequently,

$$\begin{aligned} \min_{\theta_\alpha,\theta_\beta}\left\|Y^T - \Phi_Y\theta_\alpha - \Phi_U\theta_\beta\right\|_2^2 &= \min_{\theta_\alpha}\{\Sigma_{U^\perp}\} \\ &= \min_{\theta_\alpha}\left\|Y^T\backslash_{\Phi_U^\perp} - \left(\Phi_Y\backslash_{\Phi_U^\perp}\right)\theta_\alpha\right\|_2^2 \end{aligned}$$

and

$$\hat{\theta}_\alpha = \arg\min_{\theta_\alpha}\left\|Y^T\backslash_{\Phi_U^\perp} - \Phi_Y\backslash_{\Phi_U^\perp}\theta_\alpha\right\|_2^2.$$

Equality (25) can be proven in an identical way by decomposing $Y = Y\backslash_{\Phi_Y} + Y\backslash_{\Phi_Y^\perp}$ and $\Phi_U = \Phi_U\backslash_{\Phi_Y} + \Phi_U\backslash_{\Phi_Y^\perp}$. □

Using Lemma 1, $\hat{\theta}_\alpha$ is estimated first. Then, $\hat{\theta}_A$ is calculated through (18) and θ_B directly estimated from the data solving

$$Y\backslash_{\Phi_{Y^\perp}} = \bar{\Phi}_U\left(\hat{\theta}_\alpha, \hat{\theta}_A\right)\backslash_{\Phi_{Y^\perp}}\mathrm{vec}\left(\theta_B^T\right) + E\backslash_{\Phi_{Y^\perp}} \quad (26)$$

in the LS sense. From (19), (26) and after some calculations,

$$\bar{\Phi}_U\left(\hat{\theta}_\alpha, \hat{\theta}_A\right) = \begin{bmatrix} \left(\mathcal{I}(\hat{\theta}_A)T_{co}\right)\otimes I_{n_u} & \hat{\theta}_\alpha\otimes I_{n_u} \\ 0_{n_u\times nn_u} & I_{n_u} \end{bmatrix}\Phi_U. \quad (27)$$

$\hat{\theta}_A$ and $\hat{\theta}_B$ are not the solution of the LS problem

$$\min_{\theta_A,\theta_B}\left\|Y - \Phi_Y\theta_\alpha - \bar{\Phi}_U(\theta_\alpha,\theta_A)\theta_\beta\right\|_2^2, \quad (28)$$

because $\bar{\Phi}_U$ depends on θ_α. When the direct term b_0 is zero, the last rows of both θ_B and $\bar{\theta}_\beta$ disappear and $\bar{\Phi}_U$ becomes $\bar{\Phi}_U\left(\hat{\theta}_A\right) = \mathcal{I}(\hat{\theta}_A)T_{co}$. Moreover, for fast sampling $\lim_{T_s\to 0}\mathcal{I} = T_sI_n$ and $\bar{\Phi}_U$ becomes independent of $\hat{\theta}_A$ (and consequently of $\hat{\theta}_\alpha$). Under these conditions, $\hat{\theta}_A$ and $\hat{\theta}_B$ are approximate solutions of (28). The identification of DT fast sampled systems is well known to be an ill-conditioned problem. In the next section will try resolve this difficulty by using a downsampling approach.

III. Downsampled CT IO model subspace identification

To overcome the difficulties of fast sampling the same procedure described in [10] is adopted. A larger sampling period is considered, i.e., the sampling period is multiplied by a factor m. The output data is divided into m subsets $\mathcal{Y}_\ell = \{y(\ell T_s), \cdots, y[(km+\ell)T_s], \cdots, y[(N_\ell m+\ell)Ts]\}$, $N_\ell = int(N-1-\ell)/m$ where $int(\cdot)$ stands for the integer part. A DT IO model describing these subsets is built. In order to preserve the inter-sampling behaviour, fictitious inputs consisting of the input values at the discarded sampling instants are added to this model. Thence, every subset may be described by

$$\alpha_m(q^{-1})y_m(k,\ell) = \beta_m(q^{-1})u_m(k,\ell), \quad \ell = 0,\ldots,m-1, \quad (29)$$

where

$$y_m(k,\ell) = y\left[(km+\ell)T_s\right] \in \mathbb{R}, \quad (30)$$

$$u_m(k,\ell) = \begin{bmatrix} u[(km+\ell]T_s) \\ u[(km+\ell+1)T_s] \\ \cdots \\ u[(k+1)m+\ell-1)T_s] \end{bmatrix} \in \mathbb{R}^{mn_u} \quad (31)$$

and the polynomials are defined as:

$$\begin{aligned} \alpha_m(q^{-1}) &= 1 + \alpha_{m,1}q^{-1} + \cdots + \alpha_{m,n}q^{-1}, && (32) \\ \beta_m(q^{-1}) &= \beta_{m,0} + \beta_{m,1}q^{-1} + \cdots + \beta_{m,n}q^{-1}, && (33) \end{aligned}$$

with $\alpha_{m,i}\in\mathbb{R}, \beta_{m,j}\in\mathbb{R}^{1\times mn_u}, i = 1,\ldots,n,\ j = 0,\ldots,n$.

A SS system realisation equivalent to (29) is

$$\begin{aligned} x_m(k+1,\ell) &= A_mx_m(k,\ell) + B_mu_m(k,\ell), \\ y_m(k,\ell) &= C_mx_m(k,\ell) + D_mu_m(k,\ell), \end{aligned} \quad (34)$$

with $A_m\in\mathbb{R}^{n\times n}, B_m\in\mathbb{R}^{n\times mn_u}, C_m\in\mathbb{R}^{1\times n}$ and $D_m\in\mathbb{R}^{1\times mn_u}$. Also $x\left[(km+\ell)T_s\right] = x_m(k,l)$. The conversion of the CT system SS realisation (7) into (34) is given by:

$$\begin{aligned} A_m &:= e^{A_cmT_s}, && (35) \\ B_{m_i} &:= \psi_iB_c, && (36) \\ B_m &:= \begin{bmatrix} B_{m_0} & \cdots & B_{m_{m-1}} \end{bmatrix}, && (37) \\ C_m &:= C_c, && (38) \\ D_m &= \begin{cases} D_c, & m = 1 \\ \begin{bmatrix} D_c & 0 & \cdots & 0 \end{bmatrix}, & m > 1, \end{cases} && (39) \end{aligned}$$

with $\psi_i = e^{A(m-i-1)T_s}\mathcal{I}(\theta_A)\in\mathbb{R}^{n\times n}$. See [10] for more detail. As it happens for $m = 1$ in the previous section, this is an injective relation for every m. Thus, the CT SS model can be recovered from the inverse relations of (35)–(39). Again, the most straightforward relation between the coefficients of (34) and (29) is when the former is in the OCF. Now define

$$\theta_m = \begin{bmatrix} \theta_{\alpha_m}^T & \theta_{\beta_m}^T \end{bmatrix}^T, \quad (40)$$

with

$$\theta_{\alpha_m} = \begin{bmatrix}\alpha_{m,n} & \alpha_{m,n-1} & \cdots & \alpha_{m,1}\end{bmatrix}^T \in \mathbb{R}^n \tag{41}$$

$$\theta_{\beta_m} = \begin{bmatrix}\beta_{m,n} & \beta_{m,n-1} & \cdots & \beta_{m,0}\end{bmatrix}^T \in \mathbb{R}^{(n+1)mn_u}, \tag{42}$$

and

$$\varphi_m(k,\ell) \;=\; \begin{bmatrix} -y_m(k-n,\ell) & \cdots - y_m(k-1,\ell) \\ u_m^T(k-n,\ell) & \cdots \quad u_m^T(k,\ell) \end{bmatrix}. \tag{43}$$

Re-write IO model (29) as:

$$y_m(k,\ell) = \varphi(k,\ell)\theta_m + \varepsilon_m(k,\ell), k \;=\; n,\ldots,N_\ell, \quad \ell = 0,\ldots,m-1, \tag{44}$$

where $e_m(k,\ell)$ is the noise term. Stacking $y_m(k,\ell)$, $\varphi_m(k,\ell)$ and $\varepsilon_m(k,\ell)$ in Y_m, Φ_m and E_m as

$$\begin{aligned} Y_m &= [\; y_m(n,0) \quad y_m(n,1) \quad \cdots \quad y_m(n,m-1) \\ &\qquad y_m(n+1,0) \quad \cdots \quad y_m(n+1,m-1) \quad \cdots \;]^T, \\ \Phi_m &= [\; \varphi_m^T(n,0) \quad \varphi_m^T(n,1) \quad \cdots \quad \varphi_m^T(n,m-1) \\ &\qquad \varphi_m^T(n+1,0) \quad \cdots \quad \varphi_m^T(n+1,m-1) \quad \cdots \;]^T, \\ E_m &= [\; \varepsilon_m(n,0) \quad \varepsilon_m(n,1) \quad \cdots \quad \varepsilon_m(n,m-1) \\ &\qquad \varepsilon_m(n+1,0) \quad \cdots \varepsilon_m(n+1,m-1) \quad \cdots \;]^T, \end{aligned}$$

the whole data record is described by $Y_m \;=\; \Phi_m\theta_m \;+\; E_m$ and θ_m estimated by the LSE

$$\hat{\theta}_m = \left(\Phi_m^T \Phi_m\right)^{-1} \Phi_m^T Y_m. \tag{45}$$

Notice that $\beta_{m,0} = D_m$ and, from (39), the last $(m-1)$ blocks of this vector are zero. So, in the sequel, the corresponding blocks will be removed from θ_m and $\varphi_m(k,\ell)$. If (34) is in the OCF, then

$$\operatorname{vec}\left(\begin{bmatrix} B_m^T & D^T \end{bmatrix}\right) = M_\alpha \theta_\beta \tag{46}$$

with $M_\alpha := \left[\begin{array}{c|c} I_{mnn_u} & -\theta_\alpha \otimes \begin{bmatrix} I_{n_u} \\ 0_{(m-1)n_u\times n_u}\end{bmatrix} \\ \hline 0_{n_u\times mnn_u} & I_{n_u}\end{array}\right]$.

On the other hand, using (36)–(37) and after a few calculations

$$\operatorname{vec}\left(B_m^T\right) = \left(M_\psi \otimes I_{n_u}\right)\operatorname{vec}\left(B^T\right), \tag{47}$$

where $M_\psi = \begin{bmatrix}\Psi_1^T & \Psi_2^T & \cdots & \Psi_n^T\end{bmatrix}^T \in \mathbb{R}^{mn\times n}$ and

$$\Psi_i = \begin{bmatrix}\psi_0^{(i)T} & \psi_1^{(i)T} & \cdots & \psi_{m-1}^{(i)T}\end{bmatrix}^T \in \mathbb{R}^{m\times n}, \tag{48}$$

with $\psi_\ell^{(i)} \in \mathbb{R}^{1\times n}$ being the i^{th} row of ψ_ℓ, $\ell = 0,\ldots,m-1$. Using (47) in (46) and the fact that $\theta_B = \begin{bmatrix}B^T T_{co}^{-T} & D\end{bmatrix}$, where T_{co} the similarity transformation matrix between DT realisation (34) and the correspondent OCF, a relation between θ_B and θ_{β_m} can be found:

$$\operatorname{vec}\left(\theta_B^T\right)) = \mathcal{M}^\dagger \theta_{\beta_m}, \tag{49}$$

where

$$\mathcal{M} = \left[\begin{array}{c|c} M_\psi \otimes I_{n_u} & \left(\theta_\alpha \otimes \begin{bmatrix} I_{n_u} \\ 0_{(m-1)n_u\times n_u}\end{bmatrix}\right) \\ \hline 0_{n_u\times nn_u} & I_{n_u}\end{array}\right] \tag{50}$$

and consequently

$$\mathcal{M}^\dagger = \left[\begin{array}{c|c} M_\psi^\dagger \otimes I_{n_u} & -\left(M_\psi^\dagger \otimes I_{n_u}\right)\left(\theta_\alpha \otimes \begin{bmatrix} I_{n_u} \\ 0_{(m-1)n_u\times n_u}\end{bmatrix}\right) \\ \hline 0_{n_u\times mnn_u} & I_{n_u}\end{array}\right] \tag{51}$$

After estimating θ_m, $\hat{\theta}_A$ and $\hat{\theta}_B$ can be calculated through (18) and (49), respectively, with θ_α, θ_β and T_s replaced by $\hat{\theta}_\alpha$, $\hat{\theta}_\beta$ and mT_s, respectively.

The number of columns of the regression matrix $\varphi_m(k,\ell)$ of the downsampled model has increased to accommodate the inter-sample input values. As a result, the excitation requirements of the input signal have also increased. Since the downsampled model has $n+(mn+1)n_u$ parameters and the objective is to estimate a CT model with $n+(n+1)n_u$ parameters, there is clearly an over-excitation requirement. This over-requirement can be removed by directly estimating θ_B from the data. This means that θ_{α_m} is estimated first using Lemma 1 and then θ_B is found by solving

$$Y_m \backslash_{\Phi_{Y_m}^\perp} = \bar{\Phi}_{U_m} \backslash_{\Phi_{Y_m}^\perp} \operatorname{vec}\left(\theta_B^T\right) + E \backslash_{\Phi_{Y_m}^\perp}, \tag{52}$$

where $\bar{\Phi}_{U_m} = \bar{\Phi}_{U_m}\mathcal{M}$ and $\begin{bmatrix}\Phi_{Y_m} & \Phi_{U_m}\end{bmatrix} = \Phi_m$. When the CT system TF is strictly proper, i.e., when the direct term b_0 is zero, $\mathcal{M}$ reduces to $\mathcal{M} = M_\psi \otimes I_{n_u}$. Moreover, if the number of zeros, n_z, of the CT TF is previously known, $\mathcal{M}$ can be set to the first n_z columns of $M_\psi \otimes I_{n_u}$.

IV. Instrumental variables for downsampling

It is well known that the LSE for DT models is optimal only when the equation error $\varepsilon(t)$ (or $\varepsilon_m(k,\ell)$ for the DS model) is white noise. Unfortunately, this almost never happens in practice and the LSE is usually very inaccurate. Consequently, when the output data is contaminated with noise, the estimated DT models frequently cannot be converted to CT because they have negative real poles and the logarithm of the state matrix has complex entries [22]. A DS SRIV is proposed in this section to solve this problem. This estimator is essentially an extension of the well known SRIV [21] for DS DT model identification.

The LSE for the DS DT model solves m LS problems simultaneously. In order to increase the accuracy, the LS problems may be solved by the SRIV estimator. This leads to an iterative process where in every iteration-i θ_m is estimated by

$$\hat{\theta}_m^{IV(i)} = \left[\left(\Phi_{m_f}^{IV(i)}\right)^T \Phi_{m_f}\right]^{-\dagger} \left(\Phi_{m_f}^{IV(i)}\right)^T Y_{m_f}, \tag{53}$$

where Y_{m_f}, Φ_{m_f} are built in the same way as Y_m, Φ_{m_f} with $y_m(k,\ell)$ and $u_m(k,\ell)$ replaced by the filtered signals $y_{m_f}(k,\ell) \;=\; \dfrac{1}{\hat{\alpha}_m^{(i-1)}(q^{-1})} y_m(k,\ell)$ and $u_{m_f}(k,\ell) \;=\; \dfrac{1}{\hat{\alpha}_m^{(i-1)}(q^{-1})} u_m(k,\ell)$, where $\hat{\alpha}_m^{(i-1)}(q^{-1})$ is the estimate of $\alpha_m(q^{-1})$ in the previous iteration. $\Phi_{m_f}^{IV(i)}$ is similar to Φ_{m_f} but with $y_{m_f}(k,\ell)$ replaced by $y_{m_f}^{IV(i)}(k,\ell)$, the filtered outputs of the model estimated in the previous iteration, i.e.,

$$y_{m_f}^{IV(i)}(k,\ell) = \frac{1}{\hat{\alpha}_m^{(i-1)}(q^{-1})} y_m^{IV(i)}(k,\ell), \quad (54)$$

$$y_m^{IV(i)}(k,\ell) = \frac{\hat{\beta}_m^{(i-1)}(q^{-1})}{\hat{\alpha}_m^{(i-1)}(q^{-1})} u_m(k,\ell). \quad (55)$$

This process is initialised with the LS estimates. The IV estimator $\hat{\theta}_{m_f}^{IV(i)}$ in (53) may be seen as the solution of

$$\left(\Phi_m^{IV(i)}\right)^T Y_{m_f} = \left(\Phi_{m_f}^{IV(i)}\right)^T \Phi_{m_f}\theta_m + \left(\Phi_{m_f}^{IV(i)}\right)^T E_{m_f} \quad (56)$$

in the LS sense. In order to reduce the excitation requirements, θ_B can be estimated from the data. Again, using Lemma 1, θ_α is estimated first and θ_B is the solution of

$$\left(\Phi_{m_f}^{IV(i)}\right)^T Y_{m_f} \backslash_{\Phi_{Y_{m_f}}^{\perp}} = \left(\Phi_{m_f}^{IV(i)}\right)^T \bar{\Phi}_{U_{m_f}} \backslash_{\Phi_{Y_{m_f}^{\perp}}} \text{vec}\left(\theta_B^T\right) + \left(\Phi_{m_f}^{IV(i)}\right)^T E \backslash_{\Phi_{Y_m}^{\perp}} \quad (57)$$

where $\bar{\Phi}_{U_{m_f}} = \Phi_{U_{m_f}}\mathcal{M}$ and $\begin{bmatrix}\Phi_{Y_{m_f}} & \Phi_{U_{m_f}}\end{bmatrix} = \Phi_{m_f}$. But, this can only be done if $\hat{\alpha}_m^{(i)}$ has no negative real zeros with odd multiplicity. To solve this problem a two stage DS-SRIV algorithm is used. A DS DT time model is identified at the first stage and refined at the second stage with θ_B being directly estimated from the data and $\hat{\theta}_{\beta_m}$ then calculated from θ_B. Since the goal at the first stage is to obtain $\alpha_m(q^{-1})$ that can be converted to $A(p)$, its main requirement is to generate convenient sets of IV, $y_m^{IV(i)}$, $i = 1, \ldots,$ such that $\alpha_m^{(i)}$ converges to an appropriate value. Therefore, becomes irrelevant whether $u(t)$ has sufficient excitation to estimate all parameters of $\beta_m(q^{-1})$. Even if this does not happen, multiple solutions of this polynomial exist and any one of those can be chosen. Under these conditions, estimate (56) is well known to be the minimal norm solution. At stage two the process is repeated but with θ_{α_m} converted to θ_A and θ_B estimated directly from the data. The polynomial coefficients, $\hat{\beta}_m^{(i-1)}$, of the IV filter (55) are calculated from $\hat{\theta}^{(i-1)}$ using $\hat{\theta}\beta_m = \mathcal{M}\text{vec}\left(\theta_B^T\right)$, where $\mathcal{M}$ is defined in (50).

V. Case Study

In this section, the two-stage algorithm is appraised *via* a fourth order non minimal phase system with high rate sampled data from the *idcdemo* of Contsid Toolbox [3], [6] and appearing often in the literature, e.g. [9], [12], [17]:

$$G(s) = \frac{-6400s + 1600}{s^4 + 5s^3 + 408s^2 + 416s + 1600}. \quad (58)$$

System (58) has two pairs of complex poles, $p_{c_{1,2}} = -2 \pm j19.90$ and $p_{c_{3,4}} = -0.5 \pm j1.94$, and a zero, $z = 0.25$. Since the poles and zeros are well apart, it should be feasible to identify the system.

This CT system was simulated using $T_s = 0.01$ sec, a ZOH inter-sample behaviour and a pseudo random binary sequence (PRBS) with 10 stages and a switching time $T_{sw} = 7T_s = 0.07$ sec as input.

TABLE I

Monte Carlo simulation results for SNR=20dB.

	a_1 5.0000	a_2 408.00	a_3 416.00	a_4 1600.0	b_3 -6400.0	b_4 1600.0
m=6	5.0037 (0.0303)	408.00 (0.58)	416.00 (1.41)	1599.5 (2.9)	-6399.6 (16.9)	1608.3 (19.4)
m=7	5.0043 (0.0301)	408.01 (0.58)	415.99 (1.42)	1599.5 (2.9)	-6399.7 (16.9)	1608.5 (19.4)
m=8	5.0049 (0.0297)	408.01 (0.57)	415.98 (1.42)	1599.5 (2.9)	-6399.6 (16.8)	1608.6 (19.4)
m=9	5.0044 (0.0300)	408.00 (0.58)	415.98 (1.43)	1599.5 (2.9)	-6399.6 (16.9)	1608.5 (19.4)
m=10	5.0041 (0.0301)	408.00 (0.59)	415.98 (1.42)	1599.4 (2.9)	-6399.5 (17.1)	1608.4 (19.6)
m=11	5.0049 (0.0305)	408.02 (0.59)	415.98 (1.42)	1599.5 (2.9)	-6399.8 (17.0)	1608.6 (19.4)
m=12	5.0054 (0.0326)	408.02 (0.60)	415.99 (1.43)	1599.5 (3.0)	-6399.9 (17.2)	1608.7 (19.9)
m=13	5.0071 (0.0340)	408.03 (0.66)	415.97 (1.44)	1599.4 (3.2)	-6399.8 (17.7)	1609.1 (19.7)
m=14	5.0042 (0.0354)	408.01 (0.71)	415.97 (1.46)	1599.5 (3.4)	-6399.4 (18.2)	1608.7 (19.7)
m=15	5.0060 (0.0579)	408.12 (1.08)	416.08 (1.68)	1599.9 (4.7)	-6401.3 (23.8)	1609.0 (20.4)
SRIVC	5.0024 (0.0223)	408.07 (0.38)	416.15 (1.19)	1599.8 (2.2)	-6401.3 (13.2)	1607.2 (19.9)

However, the TF of the discretised system for ZOH and a sampling period of $T_s = 0.01$ sec has a pair of poles $p_{d_{1,1}} = 0.995 \pm j0.01$ and a discrete zero $z_{d1} = 1.003$. That is, the pair of poles and the zero cluster around the point $1+j0$ and the zero almost cancels one pole. This is due to the very small sampling period and translates into numerical difficulties for the DT identification algorithm.

For the deterministic case, the system was identified with algorithms DS-LS and DS-SRIV for $m = 1, \ldots, 15$, since a choice of $m > 15$ results into aliasing. The estimated model always coincided with the true model.

Monte Carlo Simulations (MCS) with 100 experiments for SNR of 20dB and 0dB, respectively, were run to evaluate the effect of noise. The simulations were done keeping the above mentioned conditions and adding a bandwidth limited noise at the Nyquist frequency to the output.

As expected, the models of the DS-SRIV were much more accurate than those of the DS-LS. The DS-LS was used to initialise the DS-SRIV. For $SNR = 20$dB convergence was achieved for every $m \geq 6$. For $m < 6$ the DS-LS often estimated unstable models which caused the DS-SRIV algorithm to crash. This later situation also happened for $SNR = 0$dB but for a wider range of m, where the algorithm converged only for $m \geq 12$. Due to aliasing, sometimes it also crashed for $13 \leq m \leq 15$. The DS-SRIV was compared with the SRIVC implemented in the Matlab Contsid Toolbox.

Table I displays the MCS results for SNR= 20dB. The first row presents the true values of the parameters of TF (58).

TABLE II

MONTE CARLO SIMULATION RESULTS FOR SNR$=0$dB.

	a_1 5.0000	a_2 408.00	a_3 416.00	a_4 1600.0	b_3 -6400.0	b_4 1600.0
m=12	4.9882 (0.3326)	407.72 (5.23)	414.33 (13.46)	1597.3 (27.9)	-6390.3 (162.9)	1604.6 (233.3)
SRIVC	5.0168 (0.2015)	408.10 (3.25)	417.27 (12.24)	1599.9 (21.8)	-6422.0 (139.7)	1601.0 (232.2)

The other rows show the respective sample mean value and standard deviation (between brackets) for the DS-SRIV, with $m = 6, \ldots, 15$, and SRIVC algorithms. In every case, the mean values are very close to the true parameters; the standard deviations are generally small, although they are smaller for the SRIVC. This is due to the fact that the DS-SRIV only solves an approximation of the LS problem (28). For high values of m, the observed larger standard deviations are due to aliasing.

For SNR$=0$dB similar results can be seen in Table II. Here, the standard deviation is increased by a factor of 10 which corresponds to the increase of the noise.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, indirect LS and SRIV estimators for CT systems based on a downsampled approach is been proposed. The CT system is re-sampled at a rate decreased by a factor m, thus splitting the IO data records into m subsets. These subsets are described by a DT DS model where the input values at the discarded sampling instants are treated as fictitious inputs in order to preserve the inter-sampling behaviour. The DS model is estimated using simultaneously the m data subsets. The problem of the over-excitation requirement originated by the DS approach is resolved by the direct estimation of the $B(p)$ parameters. The effectiveness of this approach is illustrated using a case study often adopted in the literature.

In the future, to enhance the robustness of the algorithm, the instability of the intermediate models needs to be investigated. Considerable strategies to tackle this problem such as the restriction of the estimates of θ_{α_m} to be stable, the reflection of the unstable poles into the unit circle and the generation of the IVs and the filtered signals in the frequency domain whenever instability occurs will be considered. Also, the case of coloured output noise needs to be further analysed. To do this, both the extension of the Refined IV and of the Prediction Error Methods to downsampling will be considered.

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