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Measurable Disturbance Rejection with Quadratic Stability in Continuous-Time Linear Switching Systems

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Abstract— **It is well-known that disturbances with different features (i.e., inaccessible, measurable, or previewed disturbances) must be handled with the appropriate compensation schemes: namely, those which better exploit the available information. In particular, this work is focused on rejection of disturbances accessible for measurement in continuous-time linear switching systems, with the requirement that the compensated system be quadratically stable under arbitrary switching. A dynamic feedforward switching compensator is designed on the assumption that the plant be quadratically stable under arbitrary switching. This assumption can be relaxed to quadratic stabilizability by linear state feedback and by linear output injection, provided that a measurement dynamic feedback stabilizer is also devised. The proposed techniques apply to linear switching systems whose modes may be either left-invertible or not. The methodology adopted is based on the use of the geometric approach enhanced with stability notions which are typically considered in linear switching systems.**

I. INTRODUCTION

In recent years, linear switching systems have been used to tackle modelling, control, and observation problems in many fields, such as aerospace, automotive technologies, electrical networks, networked control systems, power electronics, and robotics. Hence, the development of methodologies aimed at investigating analysis and synthesis problems involving these systems have attracted a considerable amount of research effort. At present, a wide literature is available on fundamental issues like stability and stabilizability of linear switching systems. However, a small number of contributions can be found, dealing with performance of linear switching systems: e.g., output regulation was considered in [1]–[4], disturbance decoupling was discussed in [5]–[7], linear quadratic optimal control was treated in [8]–[10].

As to signal rejection, the problem is far from being completely solved, due to the relevant questions that arise when switching systems are addressed. As was highlighted in [11] (though with reference to the basic case of linear time-invariant systems), disturbance input signals with different properties (i.e., inaccessible, measurable, or previewed signals) must be handled with different compensation schemes, capable of best exploiting the information available on the signals: i.e., feedback compensation schemes, dynamic feedforward compensation schemes, or combined dynamic feedforward compensation schemes and finite impulse response compensation schemes, respectively. To the best of the authors' knowledge, only rejection of inaccessible disturbances has been investigated so far, and the sole compensators devised are state feedback compensators [5]–[7].

Since circumstances where disturbances are accessible for measurement are not rare (e.g., in trajectory tracking or machine tool control), this work is focused on the rejection of this class of disturbances in continuous-time linear switching systems, with the requirement that the compensated systems be quadratically stable under arbitrary switching. A dynamic feedforward compensation scheme is devised, on the assumption that the given system is quadratically stable under arbitrary switching. However, the extension to systems satisfying the weaker assumptions of being quadratically stabilizable under arbitrary switching by linear state feedback and by linear output injection can be obtained by including a measurement dynamic feedback stabilizer. The proposed methodology applies to linear switching systems whose modes may be either left-invertible or not. Earlier contributions to decoupling of measurable signals in continuous-time and discrete-time linear switching systems were given in [12] and [13], respectively. However, in these works, the treatment was restricted to switching systems satisfying a kind of left-invertibility assumption. Nonetheless, in [13], the design of the measurement dynamic feedback stabilizer was discussed.

In this work, both classical objects of the geometric approach [14] and novel geometric notions, specifically oriented to linear switching systems, are enhanced with the concept of quadratic stability under arbitrary switching. In particular, the aim of developing techniques that can be applied to linear switching systems whose modes may be nonleft-invertible requires the introduction of the novel idea of reachability subspaces constrained to the maximal robust controlled invariant subspace for the set of the modes of the switching system. Consequently, a deeper investigation of the property of internal quadratic stabilizability under arbitrary switching of the maximal robust controlled invariant subspace, also taking into account the presence of such constrained reachability subspaces, is presented. In this work, quadratic stability is privileged mainly because, by contrast with other forms of stability (like, e.g., asymptotic stability or exponential stability), it can be studied by means of linear matrix inequalities, which, in turn, can be solved with powerful computational tools [15], [16].

Notation: $\mathbb{R}$ and $\mathbb{R}_0^+$ stand for the sets of real numbers and nonnegative real numbers, respectively. Matrices and linear maps are denoted by upper-case letters, like A. The image and the kernel of A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. The transpose of A and the Moore-Penrose inverse of A are denoted by $A^\top$ and $A^\dagger$, respectively. For

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a symmetric matrix A, the symbols $A>0$ and $A<0$ mean that A is positive-definite and negative-definite, respectively, while the symbols $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The symbol $\{0\}$ denotes the origin of a vector space. The quotient space of a subspace $\mathcal{V}\subseteq\mathcal{X}$ over a subspace $\mathcal{W}\subseteq\mathcal{V}$ is denoted by $\mathcal{V}/\mathcal{W}$. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The dimension of a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\dim\mathcal{V}$. The symbols I_n and $O_{m\times n}$ are respectively used for the n-dimensional identity matrix and the $m\times n$ zero matrix (the subscripts are omitted when the dimensions can be inferred from the context).

II. PROBLEM STATEMENT

Let $\Sigma_{\sigma(t)}$ be a continuous-time linear switching system defined by

$$\Sigma_{\sigma(t)} \equiv \begin{cases} \dot{x}(t) = A_{\sigma(t)}\,x(t) + B_{\sigma(t)}\,u(t) + H_{\sigma(t)}\,h(t), \\ e(t) = E_{\sigma(t)}\,x(t), \end{cases} \tag{1}$$

where $t\in\mathbb{R}_0^+$ is the time variable, $x\in\mathcal{X}=\mathbb{R}^n$ is the state, $u\in\mathbb{R}^p$ is the control input, $h\in\mathbb{R}^m$ is the measurable disturbance input, $e\in\mathbb{R}^q$ is the to-be-controlled output, with $p,m,q\leq n$. Let the modes of $\Sigma_{\sigma(t)}$ be defined as the linear time-invariant systems of the finite set $\{\Sigma_i,\, i\in\mathcal{I}\}$, where $\mathcal{I}=\{1,2,\ldots,N\}$ and

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) = A_i\,x(t) + B_i\,u(t) + H_i\,h(t), \\ e(t) = E_i\,x(t), \end{cases} \quad i\in\mathcal{I}, \tag{2}$$

with A_i, B_i, H_i, E_i constant real matrices of suitable dimensions. The matrices B_i, H_i, E_i, with $i\in\mathcal{I}$, are assumed to be full-rank. The input functions $u(t)$ and $h(t)$, with $t\in\mathbb{R}_0^+$, are assumed to be piecewise-continuous. Let the switching signal $\sigma(t)$ be defined as the function $\sigma:\mathbb{R}_0^+\to\mathcal{I}$, $t\to i$, so that the active mode Σ_i at the time $t\in\mathbb{R}_0^+$ is given by $i\in\mathcal{I}$, such that $i=\sigma(t)$. The switching signal $\sigma(t)$ is assumed to be arbitrary and not a-priori known. Let $\Sigma_{C,\sigma(t)}$ be a continuous-time linear switching feedforward compensator defined by

$$\Sigma_{C,\sigma(t)} \equiv \begin{cases} \dot{x}_C(t) = A_{C,\sigma(t)}\,x_C(t) + B_{C,\sigma(t)}\,h(t), \\ u(t) = C_{C,\sigma(t)}\,x_C(t) + D_{C,\sigma(t)}\,h(t), \end{cases} \tag{3}$$

where $x_C\in\mathcal{X}_C=\mathbb{R}^{n_C}$ is the state. Let the modes $\{\Sigma_{C,i},\, i\in\mathcal{I}\}$ of $\Sigma_{C,\sigma(t)}$ be defined by

$$\Sigma_{C,i} \equiv \begin{cases} \dot{x}_C(t) = A_{C,i}\,x_C(t) + B_{C,i}\,h(t), \\ u(t) = C_{C,i}\,x_C(t) + D_{C,i}\,h(t), \end{cases} \quad i\in\mathcal{I}, \tag{4}$$

with $A_{C,i}$, $B_{C,i}$, $C_{C,i}$, $D_{C,i}$ constant real matrices of suitable dimensions. Let $\hat{\Sigma}_{\sigma(t)}$ be the continuous-time linear switching system resulting from the connection of $\Sigma_{\sigma(t)}$ and $\Sigma_{C,\sigma(t)}$ illustrated in Fig. 1. Hence, $\hat{\Sigma}_{\sigma(t)}$ is described by

$$\hat{\Sigma}_{\sigma(t)} \equiv \begin{cases} \dot{\hat{x}}(t) = \hat{A}_{\sigma(t)}\,\hat{x}(t) + \hat{H}_{\sigma(t)}\,h(t), \\ e(t) = \hat{E}_{\sigma(t)}\,\hat{x}(t), \end{cases} \tag{5}$$

with the modes

$$\hat{\Sigma}_i \equiv \begin{cases} \dot{\hat{x}}(t) = \hat{A}_i\,\hat{x}(t) + \hat{H}_i\,h(t), \\ e(t) = \hat{E}_i\,\hat{x}(t), \end{cases} \quad i\in\mathcal{I}, \tag{6}$$

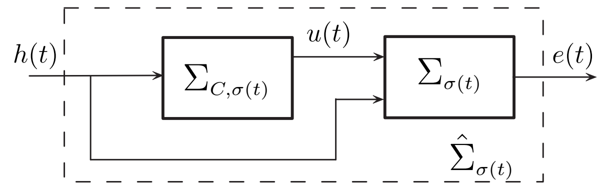

Fig. 1. Measurable disturbance rejection with quadratic stability under arbitrary switching in continuous-time linear switching systems.

where

$$\hat{A}_i = \begin{bmatrix} A_i & B_i\,C_{C,i} \\ O & A_{C,i} \end{bmatrix}, \tag{7}$$

$$\hat{H}_i = \begin{bmatrix} B_i\,D_{C,i} + H_i \\ B_{C,i} \end{bmatrix}, \tag{8}$$

$$\hat{E}_i = \begin{bmatrix} E_i & O \end{bmatrix}. \tag{9}$$

The following assumptions are made. Their impact is described in the subsequent remarks.

$\mathcal{A}.1$ The switching signal $\sigma(t)$ is such that the number of switches is finite in any finite time interval.

$\mathcal{A}.2$ The system $\Sigma_{\sigma(t)}$ is in the zero state at the initial time $t=0$.

$\mathcal{A}.3$ The system $\Sigma_{\sigma(t)}$ is quadratically stable under arbitrary switching.

Remark 1: Assumption $\mathcal{A}.1$ guarantees well-posedness of the switching signal: i.e., cases where the switching times have an accumulation point are excluded.

Remark 2: Assumption $\mathcal{A}.2$ is a standing assumption in disturbance decoupling problems, in general. In the light of the succeeding Assumption $\mathcal{A}.3$, Assumption $\mathcal{A}.2$ is not to be considered as severely restrictive, since the zero state, being the equilibrium point with the control input and the disturbance input equal to zero, can be reached with arbitrary accuracy by the switching system, provided that the inputs remain equal to zero for a sufficiently long time.

Remark 3: Assumption $\mathcal{A}.3$ also is not to be regarded as critically limitative. In fact, quadratic stability under arbitrary switching can be attained by means of switching output dynamic feedback, provided that the plant be quadratically stabilizable under arbitrary switching by linear state feedback and by linear output injection.

Then, the problem of rejecting disturbance input signals accessible for measurement by means of a switching dynamic feedforward compensator, with the requirement that the cascade system be quadratically stable under arbitrary switching, is stated as follows.

Problem 1: Given the continuous-time linear switching system $\Sigma_{\sigma(t)}$, defined by (1), with the modes (2), find a continuous-time linear switching system $\Sigma_{C,\sigma(t)}$, defined by (3), with the modes (4), such that, on Assumptions $\mathcal{A}.1$–$\mathcal{A}.3$, the following conditions are satisfied:

$\mathcal{C}.1$ the output signal $e(t)$ is equal to zero for all $t\in\mathbb{R}_0^+$, for any piecewise-continuous measurable input signal $h(t)$, with $t\in\mathbb{R}_0^+$;

$\mathcal{C}.2$ the overall system $\hat{\Sigma}_{\sigma(t)}$, defined by (5), with the modes (6), is quadratically stable under arbitrary switching.

III. NEW GEOMETRIC-APPROACH TOOLS FOR LINEAR SWITCHING SYSTEMS

The aim of this section is collecting the tools of the geometric approach needed to solve Problem 1. For the reader's convenience, some fundamental notions of the geometric approach are reviewed [14]. Original geometric notions, specifically oriented to linear switching systems, like that of reachability subspace constrained to the maximal robust controlled invariant subspace and that of internal quadratic stabilizability under arbitrary switching of the maximal robust controlled invariant subspace, are introduced.

The following definitions and properties refer to the linear time-invariant systems of a set $\{\Sigma_i,\, i\in\mathcal{I}\}$, like that defined by (2). Short notations for images and null spaces of input and output distribution matrices, respectively, are adopted: $\mathcal{B}_i=\operatorname{im} B_i$, $\mathcal{H}_i=\operatorname{im} H_i$, and $\mathcal{E}_i=\ker E_i$, with $i\in\mathcal{I}$. The subspace $\mathcal{E}\subseteq\mathcal{X}$ is introduced by the following definition

$$\mathcal{E}=\bigcap_{i\in\mathcal{I}}\mathcal{E}_i. \tag{10}$$

A subspace $\mathcal{J}\subseteq\mathcal{X}$ is said to be a robust invariant subspace if $A_i\,\mathcal{J}\subseteq\mathcal{J}$, for all $i\in\mathcal{I}$. A subspace $\mathcal{V}\subseteq\mathcal{X}$ is said to be a robust controlled invariant subspace if

$$A_i\,\mathcal{V}\subseteq\mathcal{V}+\mathcal{B}_i,\quad \forall\, i\in\mathcal{I}. \tag{11}$$

A subspace $\mathcal{V}\subseteq\mathcal{X}$ is a robust controlled invariant subspace if and only if a set of linear maps $\{F_i\in\mathbb{R}^{p\times n},\, i\in\mathcal{I}\}$ exists, such that $(A_i+B_i\,F_i)\,\mathcal{V}\subseteq\mathcal{V}$, for all $i\in\mathcal{I}$. The set of all robust controlled invariant subspaces contained in a given subspace is an upper semilattice with respect to the inclusion of subspaces and the sum of subspaces. The maximum of the set of all robust controlled invariant subspaces defined according to (11), contained in the subspace $\mathcal{E}$, defined by (10), is denoted by $\max\mathcal{V}_R(A_i,\mathcal{B}_i,\mathcal{E})$ or, briefly, by $\mathcal{V}_R^*$: i.e.,

$$\mathcal{V}_R^*=\max\mathcal{V}_R\left(A_i,\mathcal{B}_i,\mathcal{E}\right). \tag{12}$$

The subspace $\mathcal{V}_R^*$ can be computed with the algorithms first presented in [17], [18].

The following new definitions are set forth. Although they can be referred to a generic robust $(A_i,\mathcal{B}_i)$-controlled invariant subspace $\mathcal{V}\subseteq\mathcal{X}$, they will be introduced with reference to the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, in a fashion which is strictly functional to the subsequent developments. For the sake of brevity, the properties are presented without proof.

Definition 1: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), and the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12). For any $i\in\mathcal{I}$, the set of all $(A_i,\mathcal{V}_R^*)$-conditioned invariant subspaces containing $\mathcal{B}_i$ is a lower semilattice with respect to the inclusion of subspaces and the intersection of subspaces. Hence, it has an infimum, the so-called minimal $(A_i,\mathcal{V}_R^*)$-conditioned invariant subspace containing $\mathcal{B}_i$, also denoted by

$$\mathcal{S}_{\mathcal{V}_R^*,i}=\min\mathcal{S}(A_i,\mathcal{V}_R^*,\mathcal{B}_i). \tag{13}$$

Property 1: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), and the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12). For any $i\in\mathcal{I}$, the subspace $\mathcal{S}_{\mathcal{V}_R^*,i}$, defined by (13), is the last term of the sequence $\mathcal{S}^0_{\mathcal{V}_R^*,i}=\mathcal{B}_i$, $\mathcal{S}^j_{\mathcal{V}_R^*,i}=A_i\,(\mathcal{S}^{j-1}_{\mathcal{V}_R^*,i}\cap\mathcal{V}_R^*)+\mathcal{B}_i$, with $j=1,2,\dots,k$, where $k<n$ is the least integer such that $\mathcal{S}^{k+1}_{\mathcal{V}_R^*,i}=\mathcal{S}^{k}_{\mathcal{V}_R^*,i}$.

Definition 2: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), and the minimal $(A_i,\mathcal{V}_R^*)$-conditioned invariant subspaces containing $\mathcal{B}_i$, defined by (13). For any $i\in\mathcal{I}$, the reachability subspace constrained to $\mathcal{V}_R^*$ is defined by

$$\mathcal{R}_{\mathcal{V}_R^*,i}=\mathcal{V}_R^*\cap\min\mathcal{S}(A_i,\mathcal{V}_R^*,\mathcal{B}_i). \tag{14}$$

Property 2: Consider the set of linear time-invariant systems $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), and the constrained reachability subspaces $\mathcal{R}_{\mathcal{V}_R^*,i}$, defined by (14). Then, for any $i\in\mathcal{I}$, $\mathcal{V}_R^*\cap\mathcal{B}_i=\mathcal{R}_{\mathcal{V}_R^*,i}\cap\mathcal{B}_i$.

Property 3: Consider the set of linear time-invariant systems $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), and the constrained reachability subspaces $\mathcal{R}_{\mathcal{V}_R^*,i}$, defined by (14). Then, for any $i\in\mathcal{I}$, $\mathcal{R}_{\mathcal{V}_R^*,i}$ is the minimal $(A_i,\mathcal{B}_i)$-controlled invariant subspace self-bounded with respect to $\mathcal{V}_R^*$ (where self-boundedness is characterized by $\mathcal{R}_{\mathcal{V}_R^*,i}\subseteq\mathcal{V}_R^*$ and $\mathcal{R}_{\mathcal{V}_R^*,i}\supseteq\mathcal{V}_R^*\cap\mathcal{B}_i$).

Lemma 1: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), the minimal conditioned invariant subspaces $\mathcal{S}_{\mathcal{V}_R^*,i}$, defined by (13), and the constrained reachability subspaces $\mathcal{R}_{\mathcal{V}_R^*,i}$, defined by (14). For any $i\in\mathcal{I}$, apply the state-space basis transformation

$$T_i=\left[\begin{array}{cccc} T_{1,i} & T_{2,i} & T_{3,i} & T_{4,i}\end{array}\right], \tag{15}$$

where $T_{1,i}$ is a basis matrix of $\mathcal{R}_{\mathcal{V}_R^*,i}$, $T_{2,i}$ is such that $[\,T_{1,i}\ T_{2,i}\,]$ is a basis matrix of $\mathcal{V}_R^*$, and $T_{3,i}$ is such that $[\,T_{1,i}\ T_{3,i}\,]$ is a basis matrix of $\mathcal{S}_{\mathcal{V}_R^*,i}$. Then, with respect to new coordinates, the matrices of the i-th system of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$ have the structure:

$$A_i'=T_i^{-1}\,A_i\,T_i=\left[\begin{array}{cccc} A_{11,i}' & A_{12,i}' & A_{13,i}' & A_{14,i}'\\ O & A_{22,i}' & A_{23,i}' & A_{24,i}'\\ A_{31,i}' & A_{32,i}' & A_{33,i}' & A_{34,i}'\\ O & O & A_{43,i}' & A_{44,i}'\end{array}\right], \tag{16}$$

$$B_i'=T_i^{-1}\,B_i=\left[\begin{array}{c} B_{1,i}'\\ O\\ B_{3,i}'\\ O\end{array}\right], \tag{17}$$

$$H_i'=T_i^{-1}\,H_i=\left[\begin{array}{c} H_{1,i}'\\ H_{2,i}'\\ H_{3,i}'\\ H_{4,i}'\end{array}\right], \tag{18}$$

$$E_i'=E_i\,T_i=\left[\begin{array}{cccc} O & O & E_{3,i}' & E_{4,i}'\end{array}\right]. \tag{19}$$

Proof: The zero matrices in the first block of columns of A_i' are due to $\mathcal{R}_{\mathcal{V}_R^*,i}$ being an $(A_i,\mathcal{B}_i)$-controlled invariant subspace. The zero matrices in the last block of rows of A_i' are due to $\mathcal{V}_R^*$ being an $(A_i, \mathcal{B}_i)$-controlled invariant subspace. The zero matrices in the second and fourth blocks of rows of B_i' are due to $\mathcal{B}_i \subseteq \mathcal{S}_{\mathcal{V}_R^*,i}$. The zero matrices in the first and second blocks of columns of E_i' are due to $\mathcal{V}_R^* \subseteq \mathcal{E}$. ■

Lemma 2: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), and the constrained reachability subspaces $\mathcal{R}_{\mathcal{V}_R^*,i}$, defined by (14). Let $\{F_i \in \mathbb{R}^{p\times n},\, i\in\mathcal{I}\}$ be a set of linear maps such that

$$(A_i + B_i\, F_i)\,\mathcal{V}_R^* \subseteq \mathcal{V}_R^*, \quad \forall\, i\in\mathcal{I}. \tag{20}$$

For any $i\in\mathcal{I}$, refer to the coordinates introduced with the state-space basis transformation T_i, defined by (15). Let $F_i' = F_i\, T_i$ be accordingly partitioned as

$$F_i' = \left[\begin{array}{cccc} F_{1,i}' & F_{2,i}' & F_{3,i}' & F_{4,i}' \end{array}\right]. \tag{21}$$

Then, with respect to new coordinates, the matrix $A_{F,i}' = T_i^{-1}\,(A_i' + B_i'\, F_i')\, T_i$ has the structure shown in (22).

Proof: The zero matrices in first block of columns of $A_{F,i}'$ are due to $\mathcal{R}_{\mathcal{V}_R^*,i}$ being an $(A_i + B_i\, F_i)$-invariant subspace. The zero matrices in the last two blocks of rows of $A_{F,i}'$ are due to $\mathcal{V}_R^*$ being an $(A_i + B_i\, F_i)$-invariant subspace. ■

Remark 4: For any $i\in\mathcal{I}$, the restricted linear map $(A_i + B_i\, F_i)|_{\mathcal{R}_{\mathcal{V}_R^*,i}}$ is represented by the matrix $A_{11,i}' + B_{1,i}'\, F_{1,i}'$. The eigenvalues of $A_{11,i}' + B_{1,i}'\, F_{1,i}'$ are assignable by the linear map F_i. For any $i\in\mathcal{I}$, the restriction $(A_i + B_i\, F_i)|_{\mathcal{V}_R^*}$ is represented by the matrix

$$X_i = \left[\begin{array}{cc} A_{11,i}' + B_{1,i}'\, F_{1,i}' & A_{12,i}' + B_{1,i}'\, F_{2,i}' \\ O & A_{22,i}' \end{array}\right]. \tag{23}$$

The eigenvalues of $A_{22,i}'$ are not assignable by F_i. For any $i\in\mathcal{I}$, the matrix X_i is coordinate-free [14, Section 3.2.1].

If the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$ considered above are the modes (2) of the linear switching system $\Sigma_{\sigma(t)}$, defined by (1), the following nomenclature can be used with the aim of enhancing the structural geometric notions previously considered, with concepts concerning quadratic stabilizability under arbitrary switching. A robust invariant subspace $\mathcal{J}\subseteq\mathcal{X}$ is also called an $A_{\sigma(t)}$-invariant subspace. A robust controlled invariant subspace $\mathcal{V}\subseteq\mathcal{X}$ is also called an $(A_{\sigma(t)}, \mathcal{B}_{\sigma(t)})$-controlled invariant subspace. The switching linear map $F_{\sigma(t)}$ is associated with the set of linear maps $\{F_i \in \mathbb{R}^{p\times n},\, i\in\mathcal{I}\}$. The switching dynamics $A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)}$ restricted to an $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})$-invariant subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})|_{\mathcal{V}}$. The subspace $\mathcal{V}_R^*$ is also denoted by $\max\mathcal{V}_R\left(A_{\sigma(t)}, \mathcal{B}_{\sigma(t)}, \mathcal{E}\right)$. With this notation, the following definition can be set.

Definition 3: The maximal $(A_{\sigma(t)}, \mathcal{B}_{\sigma(t)})$-controlled invariant subspace $\mathcal{V}_R^*$ is said to be internally quadratically stabilizable under arbitrary switching by the linear map $F_{\sigma(t)}$ if

i) $\mathcal{V}_R^*$ is an $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})$-invariant subspace;
ii) $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})|_{\mathcal{V}_R^*}$ is quadratically stable under arbitrary switching.

IV. A CONSTRUCTIVE CONDITION FOR MEASURABLE SIGNAL REJECTION WITH QUADRATIC STABILITY

This section is focused on the discussion of a constructive condition for solving Problem 1.

Lemma 3: Consider the linear time-invariant systems of the set $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12), the images of the input distribution matrices $\mathcal{B}_i$, with $i\in\mathcal{I}$. Let the following hypothesis hold

$\mathcal{H}.1$ $\mathcal{H}_i \subseteq \mathcal{V}_R^* + \mathcal{B}_i, \quad \forall\, i\in\mathcal{I}.$

For any $i\in\mathcal{I}$, with respect to the coordinates introduced in Lemma 1, let

$$W_i' = \left[\begin{array}{c} W_{1,i}' \\ W_{2,i}' \\ O \\ O \end{array}\right] \tag{24}$$

be a basis matrix of the subspace $\mathcal{V}_R^*/\mathcal{V}_R^*\cap\mathcal{B}_i$. Let $\Gamma_i \in \mathbb{R}^{p\times m}$ and $\Lambda_i \in \mathbb{R}^{s\times m}$, where $s = \dim\,(\mathcal{V}_R^*/\mathcal{V}_R^*\cap\mathcal{B}_i)$, be defined by

$$\left[\begin{array}{c} \Gamma_i \\ \Lambda_i \end{array}\right] = \left[\begin{array}{cc} B_i' & W_i' \end{array}\right]^{\dagger} H_i', \tag{25}$$

where B_i', H_i' are given by (17), (18), respectively. Then,

$$H_i' = \left[\begin{array}{c} B_{1,i}' \\ O \\ B_{3,i}' \\ O \end{array}\right] \Gamma_i + \left[\begin{array}{c} W_{1,i}' \\ W_{2,i}' \\ O \\ O \end{array}\right] \Lambda_i, \tag{26}$$

Proof: Equation (26) follows from (17), (18), where $H_{4,i}' = O$ by virtue of Hypothesis $\mathcal{H}.1$, (24) and (25). ■

Lemma 4: Consider the linear switching system $\Sigma_{\sigma(t)}$, defined by (1), with the modes $\{\Sigma_i,\, i\in\mathcal{I}\}$, defined by (2), and the maximal robust controlled invariant subspace $\mathcal{V}_R^*$, defined by (12). Let the following hypothesis hold

$\mathcal{H}.2$ $\mathcal{V}_R^*$ be internally quadratically stabilizable under arbitrary switching by a switching linear map $F_{\sigma(t)}$.

For any $i\in\mathcal{I}$, refer to the coordinates introduced in Lemma 1. Consider the set $\{F_i',\, i\in\mathcal{I}\}$ of linear maps, where F_i' is partitioned as in (21). Let the blocks $F_{1,i}'$ and $F_{2,i}'$ be such that:

i)

$$F_{1,i}' = -(B_{3,i}')^{\dagger}\, A_{31,i}' + \Omega_i\, \alpha_i, \tag{27}$$

$$F_{2,i}' = -(B_{3,i}')^{\dagger}\, A_{32,i}' + \Omega_i\, \beta_i, \tag{28}$$

where Ω_i denotes a basis matrix of $\ker B_{3,i}'$ and α_i, β_i are parameter matrices of suitable dimensions;

ii) a positive definite symmetric matrix $P \in \mathbb{R}^{n_{\mathcal{V}}\times n_{\mathcal{V}}}$, where $n_{\mathcal{V}} = \dim\mathcal{V}_R^*$, exists, satisfying the LMIs

$$P\, X_i^{\top} + X_i\, P < 0, \quad \forall\, i\in\mathcal{I}, \tag{29}$$

where X_i, with $i\in\mathcal{I}$, are defined by (23).

Then, the following conditions hold:

i) $\mathcal{V}_R^*$ is an $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})$-invariant subspace;
ii) the switching dynamics $(A_{\sigma(t)} + B_{\sigma(t)}\, F_{\sigma(t)})|_{\mathcal{V}_R^*}$ is quadratically stable under arbitrary switching.

$$A'_{F,i} = T_i^{-1}(A_i + B_i F_i) T_i = \begin{bmatrix} A'_{11,i} + B'_{1,i} F'_{1,i} & A'_{12,i} + B'_{1,i} F'_{2,i} & A'_{13,i} + B'_{1,i} F'_{3,i} & A'_{14,i} + B'_{1,i} F'_{4,i} \\ O & A'_{22,i} & A'_{23,i} & A'_{24,i} \\ O & O & A'_{33,i} + B'_{3,i} F'_{3,i} & A'_{34,i} + B'_{3,i} F'_{4,i} \\ O & O & A'_{43,i} & A'_{44,i} \end{bmatrix}. \quad (22)$$

Proof: First, note that $F'_{1,i}$ and $F'_{2,i}$, respectively defined by (27) and (28), solve

$$A'_{31,i} + B'_{3,i} F'_{1,i} = 0, \quad (30)$$
$$A'_{32,i} + B'_{3,i} F'_{2,i} = 0, \quad (31)$$

respectively. In fact, the existence of a solution of (30) and (31) is guaranteed by robust controlled invariance of $\mathcal{V}_R^*$. Also note that, if $\ker B'_{3,i} \neq \{0\}$, then there are degrees of freedom in the choice of $F'_{1,i}$ and $F'_{2,i}$, which may affect the solution of (29). Nonetheless, the existence of $F'_{1,i}$ and $F'_{2,i}$, of the form (27) and (28), such that the LMIs (29) are satisfied is guaranteed by Hypothesis $\mathcal{H}.2$. In the light of these considerations, propositions i) and ii) of the statement can be proved as follows.

Proposition i). From (16), (17), (21), with (27) and (28), it follows that the matrices $A'_i + B'_i F'_i$, with $i \in \mathcal{I}$, have the structure of matrices $A'_{F,i}$, with $i \in \mathcal{I}$, shown in (22). Hence, $\mathcal{V}_R^*$ satisfies (20), which is equivalent to proposition i) of the thesis.

Proposition ii) is a consequence of proposition i), and equations (23) and (29), in view of the definition of quadratic stability under arbitrary switching of a switching dynamics (see, e.g., [19, Chapter 5]). ■

Lemma 5: Consider the continuous-time linear switching system $\Sigma_{\sigma(t)}$, defined by (1), with the modes (2). Let Assumptions $\mathcal{A}.1$–$\mathcal{A}.3$ hold. Consider the continuous-time linear switching compensator $\Sigma_{C,\sigma(t)}$, defined by (3), with the modes (4). Let $\Sigma_{C,\sigma(t)}$ be quadratically stable under arbitrary switching. Then, the continuous-time linear switching system $\hat{\Sigma}_{\sigma(t)}$, defined by (5), with the modes (6), where (7)–(9) hold, is quadratically stable under arbitrary switching.

Proof: The dynamics of the switching system $\hat{\Sigma}_{\sigma(t)}$ is quadratically stable under arbitrary switching if and only if a symmetric positive-definite matrix $\hat{P} \in \mathbb{R}^{(n+n_C)\times(n+n_C)}$ exists, such that

$$\hat{A}_i^\top \hat{P} + \hat{P} \hat{A}_i < 0, \quad \forall i \in \mathcal{I}. \quad (32)$$

By Assumption $\mathcal{A}.3$, a symmetric positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ exists, such that

$$A_i^\top Q + Q A_i < 0, \quad \forall i \in \mathcal{I}. \quad (33)$$

Since $\Sigma_{C,\sigma(t)}$ is quadratically stable under arbitrary switching by assumption, a symmetric positive-definite matrix $P \in \mathbb{R}^{n_C\times n_C}$ exists, such that

$$A_{C,i}^\top P + P A_{C,i} < 0, \quad \forall i \in \mathcal{I}. \quad (34)$$

Hence, it will be shown that $\gamma > 0$ exists, such that (32) is satisfied with

$$\hat{P} = \begin{bmatrix} Q & O \\ O & \gamma P \end{bmatrix}. \quad (35)$$

With the partition considered in (7) and (35), the left-hand side argument of (32) can be written as

$$\hat{A}_i^\top \hat{P} + \hat{P} \hat{A}_i = \begin{bmatrix} A_i^\top Q + Q A_i & Q B_i C_{C,i} \\ (B_i C_{C,i})^\top Q & \gamma \left(A_{C,i}^\top P + P A_{C,i}\right) \end{bmatrix}, \ \forall i \in \mathcal{I}. \quad (36)$$

By virtue of (33), (36), and the Schur complement, (32) can also be written as

$$\gamma \left(A_{C,i}^\top P + P A_{C,i}\right) - (B_i C_{C,i})^\top Q \left(A_i^\top Q + Q A_i\right)^{-1} Q B_i C_{C,i} < 0, \ \forall i \in \mathcal{I}.$$

For any $i \in \mathcal{I}$, let

$$\mu_i = \lambda_{\min}\Big((B_i C_{C,i})^\top Q \left(A_i^\top Q + Q A_i\right)^{-1} Q B_i C_{C,i}\Big),$$
$$\nu_i = \lambda_{\max}\left(A_{C,i}^\top P + P A_{C,i}\right).$$

Note that, by virtue of (34), $\nu_i < 0$ for all $i \in \mathcal{I}$. Hence, (32) is satisfied for any $\gamma > 0$ such that $\gamma \max_{i\in\mathcal{I}} \nu_i < \min_{i\in\mathcal{I}} \mu_i$. ■

In the light of the ideas introduced above, a constructive condition for solvability of Problem 1 is stated as follows.

Theorem 1: Consider the continuous-time linear switching system $\Sigma_{\sigma(t)}$, defined by (1), with the modes (2). Let Assumptions $\mathcal{A}.1$–$\mathcal{A}.3$ and Hypotheses $\mathcal{H}.1$–$\mathcal{H}.2$ hold. Let the switching compensator $\Sigma_{C,\sigma(t)}$, defined by (3), with the modes (4), have the following matrices, with respect to the coordinates introduced in Lemma 1,

$$A'_{C,i} = \begin{bmatrix} A'_{11,i} + B'_{1,i} F'_{1,i} & A'_{12,i} + B'_{1,i} F'_{2,i} \\ O & A'_{22,i} \end{bmatrix}, \quad (37)$$

$$B'_{C,i} = \begin{bmatrix} W'_{1,i} \\ W'_{2,i} \end{bmatrix} \Lambda_i \quad (38)$$

$$C'_{C,i} = \begin{bmatrix} F'_{1,i} & F'_{2,i} \end{bmatrix}, \quad (39)$$

$$D'_{C,i} = -\Gamma_i, \quad (40)$$

where $F'_{1,i}$, $F'_{2,i}$, $W'_{1,i}$, $W'_{2,i}$, Λ_i, Γ_i are respectively defined by (27), (28), (24), (25). Then, $\Sigma_{C,\sigma(t)}$ thus defined, with the initial condition

$$x_C(0) = 0, \quad (41)$$

solves Problem 1.

Proof: With respect to the coordinates introduced in Lemma 1, the modes (2) of the switching system $\Sigma_{\sigma(t)}$ are

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) = A'_i x(t) + B'_i u(t) + H'_i h(t), \\ e(t) = E'_i x(t), \end{cases} i \in \mathcal{I}, \quad (42)$$

where A'_i, B'_i, H'_i, E'_i are given by (16), (17), (26), (19), respectively. Similarly, the modes (4) of the switching compensator $\Sigma_{C,\sigma(t)}$ are

$$\Sigma_{C,i} \equiv \begin{cases} \dot{x}_C(t) = A'_{C,i} x_C(t) + B'_{C,i} h(t), \\ u(t) = C'_{C,i} x_C(t) + D'_{C,i} h(t), \end{cases} i \in \mathcal{I}, \quad (43)$$

where $A'_{C,i}$, $B'_{C,i}$, $C'_{C,i}$, $D'_{C,i}$ are given by (37)–(40). Let

$$x(t) = \left[x_1(t)^\top \ x_2(t)^\top \ x_3(t)^\top \ x_4(t)^\top \right]^\top, \quad (44)$$

$$x_C(t) = \left[x_{C,1}(t)^\top \ x_{C,2}(t)^\top \right]^\top, \quad (45)$$

where the partitions are consistent with those considered in (16)–(19) and (37)–(40), respectively. Let

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} x_{C,1}(t) \\ x_{C,2}(t) \end{bmatrix}, \quad (46)$$

with $t \in \mathbb{R}_0^+$, so that

$$\dot{\eta}_1(t) = A'_{11,i}\,\eta_1(t) + A'_{12,i}\,\eta_2(t) + A'_{13,i}\,x_3(t) + A'_{14,i}\,x_4(t), \quad (47)$$

$$\dot{\eta}_2(t) = A'_{22,i}\,\eta_2(t) + A'_{23,i}\,x_3(t) + A'_{14,i}\,x_4(t). \quad (48)$$

In the light of (47)–(48), Assumption $\mathcal{A}.2$, condition (41), and definition (46) imply $\eta_1(t) = 0$ and $\eta_2(t) = 0$ for all $t \in \mathbb{R}_0^+$, or, equivalently,

$$x_{C,1}(t) = x_1(t), \quad (49)$$

$$x_{C,2}(t) = x_2(t), \quad (50)$$

for all $t \in \mathbb{R}_0^+$. With the control input generated by (43) and in view of (49)–(50), (42) can also be written as

$$\Sigma_i \equiv \begin{cases} \dot{x}(t) = A'_{F,i}\,x(t) + W'_i\,\Lambda_i\,h(t), \\ e(t) = E'_i\,x(t), \end{cases} \quad i \in \mathcal{I}, \quad (51)$$

where $A'_{F,i}$ is defined by (22), with $F'_{3,i} = O$ and $F'_{4,i} = O$. Recall that Lemma 4 proves that $\mathcal{V}_R^*$ is an internally stable $A_{F,\sigma(t)}$-invariant subspace. Moreover, by virtue of Lemma 3, $\mathrm{im}\,(W'_i\Lambda_i) \subseteq \mathcal{V}_R^*$. In the light of these facts, (51) guarantees that any state trajectory $x(t)$, with $t \in \mathbb{R}_0^+$, of $\Sigma_{\sigma(t)}$, starting from the origin, belongs to $\mathcal{V}_R^*$ for all $t \in \mathbb{R}_0^+$, for any admissible measurable disturbance input signal $h(t)$, with $t \in \mathbb{R}_0^+$. Hence, the state trajectory generates zero output. Moreover, it is quadratically stable under arbitrary switching. Furthermore, the resulting cascade switching system $\hat{\Sigma}_{\sigma(t)}$, defined by (5), with the modes (6), is quadratically stable under arbitrary switching by virtue of Lemma 5. ■

V. CONCLUSIONS

The synthesis of a feedforward dynamic switching compensator for rejecting measurable disturbance input signals in continuous-time linear switching systems, while ensuring quadratic stability under arbitrary switching of the cascade system, has been presented for plants which are assumed to be quadratically stable under arbitrary switching. Linear switching systems satisfying the weaker assumptions of being quadratically stabilizable by linear state feedback and by linear output injection can also be considered, provided that a quadratically stabilizing output dynamic feedback is preliminarly devised. The proposed procedure is applicable to switching systems whose modes may be nonleft-invertible. The theoretical bases rely on both classical and novel concepts of the geometric approach, enhanced with stability notions specifically oriented to linear switching systems.

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