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Stackelberg Strategies for Singularly Perturbed Stochastic Systems

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***Abstract*— In this paper, a linear closed-loop Stackelberg strategy for a class of singularly perturbed stochastic systems (SPSS) governed by Itô differential equations is considered. Necessary conditions for the solution are established via a set of cross-coupled algebraic Lyapunov and Riccati equations (CALREs). After studying the asymptotic behavior of the solution for these stochastic equations, two new numerical algorithms based on Newton's method and semidefinite programming (SDP) for solving CALREs are given. A numerical example is solved to demonstrate the efficiency of the proposed algorithm.**

I. Introduction

Nash strategy and Stackelberg strategy are two important strategies in dynamic game. In the last decades, there are some theoretical studies on these strategies in optimal and robust control problems. In these problems, the solution of Nash strategy involves some parallel optimization problems. On the other hand, the solution of Stackelberg strategy involves a hierarchical combination of some optimization problems. Both strategies are appropriate when cooperation is not possible or when cooperation cannot be guaranteed (see e.g., [1], [2], [3] and reference therein). The difference lies in that Nash strategy is used in a parallel decision staructure while Stackelberg strategy is used in a hierarchical decision structure. Compared to Nash strategy, Stackelberg strategy seems intrinsically more difficult to analyze. In fact, in order to find Stackelberg strategy, a set of cross-coupled algebraic Lyapunov and Riccati equations (CALREs) has to be solved. There are various reliable approaches for solving CALREs numerically. In [1], the iterative algorithm has been introduced. However, there is no proof on the convergence of the algorithm. Furthermore, this algorithm is very slow in terms of convergence speed. In order to improve these drawbacks, Newton's method has been applied to this problem [10]. Although the Newton's method achieves the fast convergence and the local uniqueness, only the deterministic systems have been studied.

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Nash strategy for a class of stochastic controlled linear systems modeled by multiparameter singularly perturbed Itô differential equations has been initiated in [11], [12], [13]. Meanwhile, the linear Stackelberg strategies for a class of singularly perturbed systems (SPS) [8] have been investigated by using two-time scale decomposition technique [4]. After that, in order to obtain an exact strategy set for SPS, the numerical algorithms were discussed [5], [10]. Taking into consideration the fact that recent advances in numerical approaches have allowed the revision of existing algorithms, the new numerical approach for solving a set of cross-coupled stochastic algebraic Lyapunov and Riccati equations (CSALREs) corresponding to linear closed-loop Stackelberg strategy will be studied in this paper.

In this paper, the linear closed-loop Stackelberg strategy of the singularly perturbed stochastic systems (SPSS) with state dependent noise is investigated. For the stochastic case, it is shown that the existence condition of Stackelberg strategy set is given by using the CSALREs as necessary conditions for the first time. The rigorous proof is basically demonstrated along the lines of the proofs of the deterministic case [1] by adapting those proofs for the extended stochastic model. After defining a set of CSALREs, the uniqueness and boundedness of solution and their asymptotic structure are studied. In order to obtain a set of solution of CSALREs, Newton's method is applied. Furthermore, in order to reduce the computational difficulty, a new algorithm based on semidefinite programming (SDP) is also given. Finally, in order to show the validity and reliability of the developed theory, a simple numerical example and simulation results are demonstrated.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. $\mathrm{det} M$ denotes the determinant of M. $\mathrm{vec} M$ denotes an ordered stack of the columns of M [15]. $\otimes$ denotes Kronecker product. U_{lm} denotes a permutation matrix in Kronecker matrix sense [15] such that $U_{lm}\mathrm{vec} M = \mathrm{vec} M^T$, $(M \in \Re^{l \times m})$. $\mathbf{E}[\cdot]$ denotes the expectation. The trace of M is denoted by $\mathrm{Trace}\ M$.

II. Preliminary Results

Throughout this paper, let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t \geq 0},\ \boldsymbol{P})$ be a given filtered probability space where there exists a standard one-dimensional Wiener process $w(t)$, $t \geq 0$. The following two lemmas play a key technical role in this paper [7], [16].

Lemma 1: Assume that stochastic system is asymptotically mean-square stable.

$$dx(t) = Ax(t)dt + A_p x(t)dw(t). \tag{1}$$

Let us define

$$J = \mathbf{E}\left[\int_0^\infty x^T(t)C^T Cx(t)dt\right]. \tag{2}$$

If $(A,\ A_p \mid C)$ is exactly observable, then $(A,\ A_p)$ is stable iff the following stochastic algebraic Lyapunov equation (SALE):

$$A^T P + PA + A_p^T PA_p + C^T C = 0 \tag{3}$$

has a unique positive definite solution $P = P^T$. Moreover, $J = \mathbf{E}[x^T(0)Px(0)]$.

Lemma 2: Let us consider the following stochastic LQ control problem:

$$\min_u J(u) := \mathbf{E}\left[\int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt\right], \tag{4a}$$

$$\mathbf{s.t.}\ dx(t) = [Ax(t) + Bu(t)]dt + A_p x(t)dw(t), \tag{4b}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$.

It may be noted that the notation and definition of asymptotically mean-square stability and exactly observable are given in [7], [16].

Assume that there exists arbitrary $u(t)$ such that the closed-loop system is asymptotically mean-square stable. Suppose that the following stochastic algebraic Riccati equation (SARE) has a solution X.

$$A^T X + XA + A_p^T XA_p - XSX + Q = 0, \tag{5}$$

where $S = BR^{-1}B^T$.

Then, an optimal feedback control is given by

$$u(t) = Kx(t) = -R^{-1}B^T Xx(t). \tag{6}$$

Moreover, $J(u^*) = \mathbf{E}[x^T(0)Xx(0)]$.

On the other hand, the feedback gains K can be obtained by solving the following SDP. Moreover, X is a maximal solution X^*, which is the unique optimal solution.

$$\text{maximize}\ \ \mathbf{Tr}\ [X], \tag{7a}$$

$$\text{subject to}$$
$$\begin{bmatrix} A^T X + XA + A_p^T XA_p + Q & XB \\ B^T X & R \end{bmatrix} \geq 0. \tag{7b}$$

III. Problem Statement

Consider a linear time-invariant SPSS [9]

$$\begin{aligned} dx(t) = {} & [A_\varepsilon x(t) + B_{1\varepsilon}u_1(t) + B_{2\varepsilon}u_2(t)]dt \\ & + A_{p\varepsilon}x(t)dw(t), \end{aligned} \tag{8}$$

where

$$A_\varepsilon := \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{bmatrix},\ B_{i\varepsilon} := \begin{bmatrix} B_{1i} \\ \varepsilon^{-1}B_{2i} \end{bmatrix},$$
$$A_{p\varepsilon} := \begin{bmatrix} A_{p11} & \mu A_{p12} \\ \varepsilon^{\delta-1}A_{p21} & \varepsilon^{\delta-1}A_{p22} \end{bmatrix},\ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

$x_i(t) \in \Re^{n_i}$, $i = 1,\ 2$ are the state vectors, $u_i(t) \in \Re^{m_i}$ is the i-th control input. $\varepsilon > 0$ and $\mu > 0$ are small parameters. In order to simplify discussions, suppose that $\delta > 1/2$ is independent of ε. It should be noted that the parameters μ and δ have been introduced in [9]. $w(t) \in \Re$ is a one-dimensional standard Wiener process defined in the filtered probability space [6], [7]. It should be noted that the fast state matrix A_{22} may be singular.

The cost function for each strategy subset is defined by

$$\begin{aligned} J_i = \mathbf{E}\Bigg[\frac{1}{2}\int_0^\infty & [x^T(t)Q_i x(t) \\ & + u_i^T(t)R_{ii}u_i(t) + u_j^T(t)R_{ij}u_j(t)dt\Bigg], \end{aligned} \tag{9}$$

where

$$R_{ii} > 0,\ R_{ij} \geq 0,\ i,\ j = 1,\ 2,\ i \neq j,$$
$$Q_i = \begin{bmatrix} C_{i1}^T C_{i1} & C_{i1}^T C_{i2} \\ C_{i2}^T C_{i1} & C_{i2}^T C_{i2} \end{bmatrix} = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix}.$$

It is assumed that the decision-maker denoted by Player 2 is the leader, and Player 1 is the follower. Under the assumption that both players employ closed-loop strategies $u_i := u_i(x,\ t)$, a strategy set $(u_1^*,\ u_2^*)$ is called a Stackelberg strategy if for any admissible strategy set $(u_1,\ u_2)$, the following conditions hold.

$$J_2(u_1^*,\ u_2^*) \leq J_2(u_1^0(u_2),\ u_2),\ \forall u_2 \in \Re^{m_2}, \tag{10}$$

where

$$J_1(u_1^0(u_2),\ u_2) = \min_{u_1}\ J_1(u_1,\ u_2), \tag{11}$$

and

$$u_1^* = u_1^0(u_2^*). \tag{12}$$

Closed-loop Stackelberg strategies of the linear quadratic problems for the deterministic SPS have been studied in [4], [5], [10]. According to these studies, it is well-known that the closed-loop Stackelberg strategies have the following form.

$$u_i(x,\ t) = F_i x(t). \tag{13}$$

It is shown that the gain F_i is dependent on the initial state of the systems $x(0)$. To eliminate this dependence on $x(0)$, it is assumed that $\mathbf{E}[x(0)] = 0$, $\mathbf{E}[x(0)x^T(0)] = I_n$ under an uncorrelated stochastic process, where $n := n_1 + n_2$. On the other hand, without loss of generality, it is assumed that there exists an appropriate mapping from the admissible control inputs of the leader to the admissible control inputs of the follower [1].

The following theorem yields the extension of the existing results of [1], [4] for a stochastic case.

Theorem 1: Suppose that the following CSALREs has solutions $M_{i\varepsilon} \geq 0$, N_i and F_2.

$$
\begin{aligned}
&A_{F\varepsilon}^T M_{1\varepsilon} + M_{1\varepsilon} A_{F\varepsilon} + A_{p\varepsilon}^T M_{1\varepsilon} A_{p\varepsilon} \\
&\qquad + M_{1\varepsilon} S_{1\varepsilon} M_{1\varepsilon} + F_2^T R_{12} F_2 + Q_1 = 0, &\text{(14a)}\\
&A_{F\varepsilon}^T M_{2\varepsilon} + M_{2\varepsilon} A_{F\varepsilon} + A_{p\varepsilon}^T M_{2\varepsilon} A_{p\varepsilon} \\
&\qquad + M_{1\varepsilon} S_{2\varepsilon} M_{1\varepsilon} + F_2^T R_{22} F_2 + Q_2 = 0, &\text{(14b)}\\
&N_1 A_{F\varepsilon}^T + A_{F\varepsilon} N_1 - S_{1\varepsilon} M_{2\varepsilon} N_2 - N_2 M_{2\varepsilon} S_{1\varepsilon} \\
&\qquad + S_{2\varepsilon} M_{1\varepsilon} N_2 + N_2 M_{1\varepsilon} S_{2\varepsilon} + A_{p\varepsilon} N_1 A_{p\varepsilon}^T = 0, &\text{(14c)}\\
&N_2 A_{F\varepsilon}^T + A_{F\varepsilon} N_2 + A_{p\varepsilon} N_2 A_{p\varepsilon}^T + I_n = 0, &\text{(14d)}\\
&R_{12} F_2 N_1 + R_{22} F_2 N_2 + B_{2\varepsilon}^T (M_{1\varepsilon} N_1 + M_{2\varepsilon} N_2) = 0, &\text{(14e)}
\end{aligned}
$$

where $F_1 := -R_{11}^{-1} B_{1\varepsilon}^T M_{1\varepsilon}$, $S_{1\varepsilon} := B_{1\varepsilon} R_{11}^{-1} B_{1\varepsilon}^T$, $S_{2\varepsilon} := B_{1\varepsilon} R_{11}^{-1} R_{21} R_{11}^{-1} B_{1\varepsilon}^T$, $A_{F\varepsilon} := A_\varepsilon + B_{1\varepsilon} F_1 + B_{2\varepsilon} F_2$.

Then, this strategy set denotes the Stackelberg strategy.

Proof: Given arbitrary u_2, the corresponding u_1 is obtained by minimizing J_1 with respect to u_1. Let us consider the minimizing problem for the closed-loop stochastic system with arbitrary strategies $u_2(t) = F_2 x(t)$.

$$
\min_{u_1} J_1 = \mathbf{E}\left[\frac{1}{2}\int_0^\infty [x^T(t) Q_{F_1} x(t) + u_1^T(t) R_{11} u_1(t) dt\right], \quad \text{(15a)}
$$

$$
\mathbf{s.t.}\ dx(t) = [A_{F_2\varepsilon} x(t) + B_{1\varepsilon} u_1(t)]dt + A_{p\varepsilon} x(t) dw(t), \quad \text{(15b)}
$$

where $A_{F_2\varepsilon} := A_\varepsilon + B_{2\varepsilon} F_2$ and $Q_{F_1} := Q_1 + F_2^T R_{12} F_2$.

By using Lemma 2, the optimal state feedback controller $u_1^0(t)$ is given by

$$
u_1^0(t) = F_1 x(t) = -R_{11}^{-1} B_{1\varepsilon}^T M_{1\varepsilon} x(t), \quad \text{(16)}
$$

where

$$
\begin{aligned}
\boldsymbol{F}_1(M_{1\varepsilon},\ F_2) &= A_{F_2\varepsilon}^T M_{1\varepsilon} + M_{1\varepsilon} A_{F_2\varepsilon} + A_{p\varepsilon}^T M_{1\varepsilon} A_{p\varepsilon} \\
&\quad - M_{1\varepsilon} S_{1\varepsilon} M_{1\varepsilon} + Q_{F_1} = 0. \qquad \text{(17)}
\end{aligned}
$$

Then, it is easy to show that SARE (17) is equivalent to (14a). On the other hand, if $A_{F\varepsilon}$ is asymptotically mean-square stable, then the cost J_2 can be obtained by using Lemma 1 and $\mathbf{E}[x(0)x^T(0)] = I_n$.

$$
J_2(u_1^0,\ u_2) = \mathbf{Tr}\ [M_{2\varepsilon}], \quad \text{(18)}
$$

where $M_{2\varepsilon}$ is the solution of the following SALE (19).

$$
\begin{aligned}
\boldsymbol{F}_2(M_{1\varepsilon},\ M_{2\varepsilon},\ F_2) &= A_{F\varepsilon}^T M_{2\varepsilon} + M_{2\varepsilon} A_{F\varepsilon} + A_{p\varepsilon}^T M_{2\varepsilon} A_{p\varepsilon} \\
&\quad + M_{1\varepsilon} S_{2\varepsilon} M_{1\varepsilon} + Q_{F_2} = 0, \qquad \text{(19)}
\end{aligned}
$$

where $Q_{F_2} := Q_2 + F_2^T R_{22} F_2$.

Therefore, SARE (14b) holds. Let us consider the Lagrangian $\boldsymbol{H}$

$$
\begin{aligned}
\boldsymbol{H}(M_{1\varepsilon},\ M_{2\varepsilon},\ F_2) &= \mathbf{Tr}\ [M_{2\varepsilon}] + \mathbf{Tr}\ [N_1 \boldsymbol{F}_1(M_{1\varepsilon},\ F_2)] \\
&\quad + \mathbf{Tr}\ [N_2 \boldsymbol{F}_2(M_{1\varepsilon},\ M_{2\varepsilon},\ F_2)], \qquad \text{(20)}
\end{aligned}
$$

where N_i, $i = 1,\ 2$ are symmetric matrix of Lagrange multipliers.

As necessary condition, in order to minimize for $M_{2\varepsilon}$, the Lagrange multiplier technique results in the following equations.

$$
\begin{aligned}
\frac{\partial \boldsymbol{H}}{\partial M_{1\varepsilon}} &= N_1 A_{F\varepsilon}^T + A_{F\varepsilon} N_1 - S_{1\varepsilon} M_{2\varepsilon} N_2 - N_2 M_{2\varepsilon} S_{1\varepsilon} \\
&\quad + S_{2\varepsilon} M_{1\varepsilon} N_2 + N_2 M_{1\varepsilon} S_{2\varepsilon} + A_{p\varepsilon} N_1 A_{p\varepsilon}^T = 0, \quad \text{(21a)}\\
\frac{\partial \boldsymbol{H}}{\partial M_{2\varepsilon}} &= N_2 A_{F\varepsilon}^T + A_{F\varepsilon} N_2 + A_{p\varepsilon} N_2 A_{p\varepsilon}^T + I_n = 0, \quad \text{(21b)}\\
\frac{1}{2}\frac{\partial \boldsymbol{H}}{\partial F_2} &= R_{12} F_2 N_1 + R_{22} F_2 N_2 + B_{2\varepsilon}^T (M_{1\varepsilon} N_1 + M_{2\varepsilon} N_2) \\
&= 0. \quad \text{(21c)}
\end{aligned}
$$

Hence, (14c), (14d) and (14e) can be derived, respectively. This is the desired result. ■

Since A_ε, $B_{i\varepsilon}$ and $A_{p\varepsilon}$ have the term of ε^{-1}, the solution $M_{i\varepsilon}$ of CSALREs (14), if it exists, must contain terms of ε. Hence, in order to investigate the asymptotic structure of CSALREs (14), the following partitioned matrices are introduced.

$$
M_{i\varepsilon} := \begin{bmatrix} M_{i1} & \varepsilon M_{i2} \\ \varepsilon M_{i2}^T & \varepsilon M_{i3} \end{bmatrix},\ N_i := \begin{bmatrix} N_{i1} & N_{i2} \\ N_{i2}^T & N_{i3} \end{bmatrix},
$$

$$
F_i := \begin{bmatrix} F_{i1} & F_{i2} \end{bmatrix},\ i = 1,\ 2,
$$

where

$$
\begin{aligned}
&M_{i1} := M_{i1}(\varepsilon),\ M_{i2} := M_{i2}(\varepsilon),\ M_{i3} := M_{i3}(\varepsilon),\\
&N_{i1} := N_{i1}(\varepsilon),\ N_{i2} := N_{i2}(\varepsilon),\ N_{i3} := N_{i3}(\varepsilon),\\
&F_{i1} := F_{i1}(\varepsilon),\ F_{i2} := F_{i2}(\varepsilon).
\end{aligned}
$$

Let us define the following coefficient matrices.

$$
\begin{aligned}
A_{F\varepsilon} &= A_\varepsilon + B_{1\varepsilon} F_1 + B_{2\varepsilon} F_2 = \begin{bmatrix} A_{F11} & A_{F12} \\ \varepsilon^{-1} A_{F21} & \varepsilon^{-1} A_{F22} \end{bmatrix},\\
S_{1\varepsilon} &= B_{1\varepsilon} R_{11}^{-1} B_{1\varepsilon}^T = \begin{bmatrix} S_{11} & \varepsilon^{-1} S_{12} \\ \varepsilon^{-1} S_{12}^T & \varepsilon^{-2} S_{13} \end{bmatrix},\\
S_{2\varepsilon} &= B_{1\varepsilon} R_{11}^{-1} R_{21} R_{11}^{-1} B_{1\varepsilon}^T = \begin{bmatrix} S_{21} & \varepsilon^{-1} S_{22} \\ \varepsilon^{-1} S_{22}^T & \varepsilon^{-2} S_{23} \end{bmatrix}.
\end{aligned}
$$

By substituting matrices $A_{F\varepsilon}$, $A_{p\varepsilon}$, $S_{i\varepsilon}$, $M_{i\varepsilon}$ and N_i into CSALREs (14), and letting $\varepsilon = \mu = 0$, the parameter independent CSALREs (14) are also obtained. Then the asymptotic structure of CSALREs (14) is established in the next theorem.

Theorem 2: Assume that the parameter independent CSALREs (A.1)–(A.5) have solutions such that

$$
\begin{aligned}
&\det \boldsymbol{J}(\bar{M}_1,\ \bar{M}_2,\ \bar{N}_1,\ \bar{N}_2,\ \bar{F}_2) \\
&= \begin{bmatrix}
\Xi_{11} & 0 & \Xi_{13} & 0 & 0 \\
\Xi_{21} & \Xi_{11} & \Xi_{23} & 0 & 0 \\
\Xi_{31} & \Xi_{32} & \Xi_{33} & \Xi_{34} & \Xi_{35} \\
-S_1 \otimes \bar{N}_2 - \bar{N}_2 \otimes S_1 & 0 & \Xi_{43} & 0 & \Xi_{34} \\
\bar{N}_1 \otimes B_2^T & \bar{N}_2 \otimes B_2^T & \bar{N}_2 \otimes R_{22} + \bar{N}_1 \otimes R_{12} & \Xi_{54} & \Xi_{55}
\end{bmatrix} \neq 0, \quad \text{(22)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi_{11} &:= I_n \otimes A_F^T + A_F^T \otimes I_n + A_p^T \bar{M}_1 A_p, \\
\Xi_{13} &:= I_n \otimes (B_2^T \bar{M}_1 + R_{12} \bar{F}_2)^T \\
&\quad +[(B_2^T \bar{M}_1 + R_{12} \bar{F}_2)^T \otimes I_n] U_{nm_i}, \\
\Xi_{21} &:= I_n \otimes (S_2 \bar{M}_1 - S_1 \bar{M}_2)^T + (S_2 \bar{M}_1 - S_1 \bar{M}_2)^T \otimes I_n, \\
\Xi_{23} &:= I_n \otimes (R_{22} \bar{F}_2 + B_2^T \bar{M}_2)^T \\
&\quad +[(R_{22} \bar{F}_2 + B_2^T \bar{M}_2)^T \otimes I_n] U_{nm_i}, \\
\Xi_{31} &:= -S_1 \otimes (\Phi \bar{N}_1) - (\Phi \bar{N}_1) \otimes S_1 \\
&\quad +S_2 \otimes (\Phi \bar{N}_2) + (\Phi \bar{N}_2) \otimes S_2, \\
\Xi_{32} &:= -S_1 \otimes (\Phi \bar{N}_2) - (\Phi \bar{N}_2) \otimes S_1, \\
\Xi_{33} &:= B_2 \otimes (\Phi \bar{N}_1) + (\Phi \bar{N}_1) \otimes B_2, \\
\Xi_{34} &:= I_n \otimes A_F + A_F \otimes I_n + A_p \bar{N}_1 A_p^T, \\
\Xi_{35} &:= I_n \otimes (S_2 \bar{M}_1 - S_1 \bar{M}_2) + (S_2 \bar{M}_1 - S_1 \bar{M}_2) \otimes I_n, \\
\Xi_{43} &:= -B_2 \otimes (\Phi \bar{N}_2) - (\Phi \bar{N}_2) \otimes B_2, \\
\Xi_{54} &:= I_n \otimes (R_{12} \bar{F}_2 + B_2^T \bar{M}_1), \\
\Xi_{55} &:= I_n \otimes (R_{22} \bar{F}_2 + B_2^T \bar{M}_2), \\
A_F &:= A - S_1 \bar{M}_1 - B_2 \bar{F}_2, \ \Phi := \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \\
A &:= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ S_i := \begin{bmatrix} S_{1i} & S_{i2} \\ S_{i2}^T & S_{i3} \end{bmatrix}, \\
A_p &:= \begin{bmatrix} A_{p11} & 0 \\ 0 & 0 \end{bmatrix}, \ B_i := \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}, \\
\bar{M}_i &= \begin{bmatrix} \bar{M}_{i1} & 0 \\ \bar{M}_{i2} & \bar{M}_{i3} \end{bmatrix}, \ \bar{N}_i = \begin{bmatrix} \bar{N}_{i1} & \bar{N}_{i2}^T \\ \bar{N}_{i2} & \bar{N}_{i3} \end{bmatrix}, \\
\bar{F}_2 &= \begin{bmatrix} \bar{F}_{i1} & \bar{F}_{i2} \end{bmatrix}.
\end{aligned}
$$

Then there exists small $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \ \bar{\varepsilon})$, CSALREs (14) admits solutions $M_{i\varepsilon} \geq 0$, F_i and N_i, which can be written as

$$
M_{i\varepsilon} = \begin{bmatrix} \bar{M}_{i1} + O(\varepsilon) & \varepsilon \bar{M}_{i2}^T + O(\varepsilon^2) \\ \varepsilon \bar{M}_{i2} + O(\varepsilon^2) & \varepsilon \bar{M}_{i3} + O(\varepsilon^2) \end{bmatrix}, \tag{23a}
$$

$$
N_i = \begin{bmatrix} \bar{N}_{i1} + O(\varepsilon) & \bar{N}_{i2}^T + O(\varepsilon) \\ \bar{N}_{i2} + O(\varepsilon) & \bar{N}_{i3} + O(\varepsilon) \end{bmatrix}, \tag{23b}
$$

$$
F_2 = \begin{bmatrix} \bar{F}_{21} + O(\varepsilon) & \bar{F}_{22} + O(\varepsilon) \end{bmatrix}. \tag{23c}
$$

Proof: It can be done by applying the implicit function theorem to CSALREs (14). Since the used technique is similar to the reference [10], it is omitted. ■

In order to obtain the solutions of CSALREs (14), the following new numerical computation based on the Newton's method is given:

$$
\begin{aligned}
& A_{F\varepsilon}^{(n)T} M_{1\varepsilon}^{(n+1)} + M_{1\varepsilon}^{(n+1)} A_{F\varepsilon}^{(n)} + A_{p\varepsilon}^T M_{1\varepsilon}^{(n+1)} A_{p\varepsilon} \\
& \quad +(B_{2\varepsilon}^T M_{1\varepsilon}^{(n)} + R_{12} F_2^{(n)})^T F_2^{(n+1)} \\
& \quad +F_2^{(n+1)} (B_{2\varepsilon}^T M_{1\varepsilon}^{(n)} + R_{12} F_2^{(n)}) + \boldsymbol{L}_1^{(n)} = 0,
\end{aligned} \tag{24a}
$$

$$
\begin{aligned}
& (S_{2\varepsilon} M_{1\varepsilon}^{(n)} - S_{1\varepsilon} M_{2\varepsilon}^{(n)})^T M_{1\varepsilon}^{(n+1)} \\
& \quad +M_{1\varepsilon}^{(n+1)} (S_{2\varepsilon} M_{1\varepsilon}^{(n)} - S_{1\varepsilon} M_{2\varepsilon}^{(n)}) \\
& \quad +A_{F\varepsilon}^{(n)T} M_{2\varepsilon}^{(n+1)} + M_{2\varepsilon}^{(n+1)} A_{F\varepsilon}^{(n)} + A_{p\varepsilon}^T M_{2\varepsilon}^{(n+1)} A_{p\varepsilon} \\
& \quad +(B_{2\varepsilon}^T M_{2\varepsilon}^{(n)} + R_{22} F_2^{(n)})^T F_2^{(n+1)} \\
& \quad +F_2^{(n+1)T} (B_{2\varepsilon}^T M_{2\varepsilon}^{(n)} + R_{22} F_2^{(n)}) + \boldsymbol{L}_2^{(n)} = 0,
\end{aligned} \tag{24b}
$$

$$
\begin{aligned}
& -N_1^{(n)} M_{1\varepsilon}^{(n+1)} S_{1\varepsilon} - S_{1\varepsilon} M_{1\varepsilon}^{(n+1)} N_1^{(n)} \\
& \quad +N_2^{(n)} M_{1\varepsilon}^{(n+1)} S_{2\varepsilon} + S_{2\varepsilon} M_{1\varepsilon}^{(n+1)} N_2^{(n)} \\
& \quad -N_2^{(n)} M_{2\varepsilon}^{(n+1)} S_{1\varepsilon} - S_{1\varepsilon} M_{2\varepsilon}^{(n+1)} N_2^{(n)} \\
& \quad +N_1^{(n)} F_2^{(n+1)T} B_{2\varepsilon}^T + B_{2\varepsilon} F_2^{(n+1)} N_1^{(n)} \\
& \quad +A_{F\varepsilon}^{(n)} N_1^{(n+1)} + N_1^{(n+1)} A_{F\varepsilon}^{(n)T} + A_{p\varepsilon} N_1^{(n+1)} A_{p\varepsilon}^T \\
& \quad +(S_{2\varepsilon} M_{1\varepsilon}^{(n)} - S_{1\varepsilon} M_{2\varepsilon}^{(n)}) N_2^{(n+1)} \\
& \quad +N_2^{(n+1)} (S_{2\varepsilon} M_{1\varepsilon}^{(n)} - S_{1\varepsilon} M_{2\varepsilon}^{(n)})^T + \boldsymbol{L}_3^{(n)} = 0,
\end{aligned} \tag{24c}
$$

$$
\begin{aligned}
& -N_2^{(n)} M_{1\varepsilon}^{(n+1)} S_{1\varepsilon} - S_{1\varepsilon} M_{1\varepsilon}^{(n+1)} N_2^{(n)} \\
& \quad +N_2^{(n)} F_2^{(n+1)T} B_{2\varepsilon}^T + B_{2\varepsilon} F_2^{(n+1)} N_2^{(n)} \\
& \quad +A_{F\varepsilon}^{(n)} N_2^{(n+1)} + N_2^{(n+1)} A_{F\varepsilon}^{(n)T} \\
& \quad +A_{p\varepsilon} N_2^{(n+1)} A_{p\varepsilon}^T + \boldsymbol{L}_4^{(n)} = 0,
\end{aligned} \tag{24d}
$$

$$
\begin{aligned}
& B_{2\varepsilon}^T M_{1\varepsilon}^{(n+1)} N_1^{(n)} + B_{2\varepsilon}^T M_{2\varepsilon}^{(n+1)} N_2^{(n)} \\
& \quad +R_{22} F_2^{(n+1)} N_2^{(n)} + R_{12} F_2^{(n+1)} N_1^{(n)} \\
& \quad +(R_{12} F_2^{(n)} + B_{2\varepsilon}^T M_{1\varepsilon}^{(n)}) N_1^{(n+1)} \\
& \quad +(R_{22} F_2^{(n)} + B_{2\varepsilon}^T M_{2\varepsilon}^{(n)}) N_2^{(n+1)} + \boldsymbol{L}_5^{(n)} = 0,
\end{aligned} \tag{24e}
$$

where

$$
\begin{aligned}
M_{i\varepsilon}^{(0)} &= \begin{bmatrix} \bar{M}_{i1} & \varepsilon \bar{M}_{i2}^T \\ \varepsilon \bar{M}_{i2} & \varepsilon \bar{M}_{i3} \end{bmatrix}, \ N_i^{(0)} = \begin{bmatrix} \bar{N}_{i1} & \bar{N}_{i2}^T \\ \bar{N}_{i2} & \bar{N}_{i3} \end{bmatrix}, \\
F_2^{(0)} &= \begin{bmatrix} \bar{F}_{i1} & \bar{F}_{i2} \end{bmatrix}, \ A_{F\varepsilon}^{(n)} = A_\varepsilon + B_{1\varepsilon} F_1^{(n)} + B_{2\varepsilon} F_2^{(n)}, \\
\boldsymbol{L}_1^{(n)} &= M_{1\varepsilon}^{(n)} S_{1\varepsilon} M_{1\varepsilon}^{(n)} - F_2^{(n)T} B_{2\varepsilon}^T M_{1\varepsilon}^{(n)} - M_{1\varepsilon}^{(n)} B_{2\varepsilon} F_2^{(n)} \\
&\quad -F_2^{(n)T} R_{12} F_2^{(n)} + Q_1, \\
\boldsymbol{L}_2^{(n)} &= M_{1\varepsilon}^{(n)} S_{1\varepsilon} M_{2\varepsilon}^{(n)} + M_{2\varepsilon}^{(n)} S_{1\varepsilon} M_{1\varepsilon}^{(n)} - M_{1\varepsilon}^{(n)} S_{2\varepsilon} M_{1\varepsilon}^{(n)} \\
&\quad -F_2^{(n)T} B_{2\varepsilon}^T M_{2\varepsilon}^{(n)} - M_{2\varepsilon}^{(n)} B_{2\varepsilon} F_2^{(n)} \\
&\quad -F_2^{(n)T} R_{22} F_2^{(n)} + Q_2, \\
\boldsymbol{L}_3^{(n)} &= N_1^{(n)} M_{1\varepsilon}^{(n)} S_{1\varepsilon} + S_{1\varepsilon} M_{1\varepsilon}^{(n)} N_1^{(n)} \\
&\quad +N_2^{(n)} M_{2\varepsilon}^{(n)} S_{1\varepsilon} + S_{1\varepsilon} M_{2\varepsilon}^{(n)} N_2^{(n)} \\
&\quad -N_2^{(n)} M_{1\varepsilon}^{(n)} S_{2\varepsilon} - S_{2\varepsilon} M_{1\varepsilon}^{(n)} N_2^{(n)} \\
&\quad -N_1^{(n)} F_2^{(n)T} B_{2\varepsilon}^T - B_{2\varepsilon} F_2^{(n)} N_1^{(n)}, \\
\boldsymbol{L}_4^{(n)} &= N_2^{(n)} M_{1\varepsilon}^{(n)} S_{1\varepsilon} + S_{1\varepsilon} M_{1\varepsilon}^{(n)} N_2^{(n)} \\
&\quad -N_2^{(n)} F_2^{(n)T} B_{2\varepsilon}^T - B_{2\varepsilon} F_2^{(n)} N_2^{(n)} + I_n, \\
\boldsymbol{L}_5^{(n)} &= -R_{12} F_2^{(n)} N_1^{(n)} - R_{22} F_2^{(n)} N_2^{(n)} \\
&\quad -B_{2\varepsilon}^T (M_{1\varepsilon}^{(n)} N_1^{(n)} + M_{2\varepsilon}^{(n)} N_2^{(n)}).
\end{aligned}
$$

The following corollary indicates that the algorithm attains the quadratic convergence under the above initial conditions.

Corollary 1: Assume that the conditions of Theorem 2 hold. Then, there exists a small ε^* such that for all $\varepsilon \in (0, \ \varepsilon^*)$, Newton's method (24) converges to the exact solution of $M_{i\varepsilon}^*$, N_i^* and F_2^* with the rate of the quadratic convergence. Moreover, the convergence solution $M_{i\varepsilon}^*$, F_2^* and N_i^* is unique solutions of CSALREs (14) in the neighborhood of the initial condition (24). That is, the following relations are satisfied.

$$
\| M_{i\varepsilon}^{(n)} - M_{i\varepsilon}^* \| \leq O(\varepsilon^{2^n}), \ n = 0, \ 1, \ \ldots \ , \tag{25a}
$$

$$
\| N_i^{(n)} - N_i^* \| \leq O(\varepsilon^{2^n}), \ n = 0, \ 1, \ \ldots \ , \tag{25b}
$$

$$
\| F_2^{(n)} - F_2^* \| \leq O(\varepsilon^{2^n}), \ n = 0, \ 1, \ \ldots \ . \tag{25c}
$$

Proof: Since the proof of this theorem can be done by using Newton-Kantorovich theorem [14], it is omitted. ■

The leader's strategy F_2 of (14e) can be obtained by solving CSALREs (14). Newton's method involves computing the Jacobian matrix at every iteration step. Thus, one uses Newton's method (24), due to the large dimension and singularity of Jacobian matrix it may be hard to obtain the solutions. In order to avoid such disadvantages, a new algorithm based on SDP with linear matrix inequality (LMI) is given.

Let us consider the following new algorithm which is based on LMI.

Step 1. *Initialization: Set* $F_2^{(0)} = -R_{22}^{-1}B_{2\varepsilon}^T Z_\varepsilon$, *where* Z_ε *is the positive definite stabilizing solution of* $Z_\varepsilon A_\varepsilon + A_\varepsilon^T Z_\varepsilon - Z_\varepsilon S_{2\varepsilon} Z_\varepsilon + Q_2 = 0$.

Step 2. *Solve the SDP problem as formulated by* (7)*, with respect to* $M_{1\varepsilon}^{(n+1)}$*, subject to* (26).

$$\text{maximize } \ \mathbf{Tr}\ [M_{1\varepsilon}^{(n+1)}], \tag{26a}$$

subject to

$$\begin{bmatrix} \Phi_1(M_{1\varepsilon}^{(n+1)}) & M_{1\varepsilon}^{(n+1)}B_{1\varepsilon} \\ B_{1\varepsilon}^T M_{1\varepsilon}^{(n+1)} & R_{11} \end{bmatrix} \geq 0, \tag{26b}$$

$\Phi_1(M_{1\varepsilon}^{(n+1)}) := A_{F\varepsilon}^T M_{1\varepsilon}^{(n+1)} + M_{1\varepsilon}^{(n+1)} A_{F\varepsilon} + A_{p\varepsilon}^T M_{1\varepsilon}^{(n+1)} A_{p\varepsilon} + F_2^{(n)T} R_{12} F_2^{(n)} + Q_1$.

Step 3. *Solve the following SDP problem, with respect to* $M_{2\varepsilon}^{(n+1)}$*, subject to* (27).

$$\text{maximize } \ \mathbf{Tr}\ [M_{2\varepsilon}^{(n+1)}], \tag{27a}$$

subject to

$$\begin{aligned} &A_{F\varepsilon}^{(n)T} M_{2\varepsilon}^{(n+1)} + M_{2\varepsilon}^{(n+1)} A_{F\varepsilon}^{(n)} + A_{p\varepsilon}^T M_{2\varepsilon}^{(n+1)} A_{p\varepsilon} \\ &\quad + M_{1\varepsilon}^{(n+1)} S_{2\varepsilon} M_{1\varepsilon}^{(n+1)} \\ &\quad + F_2^{(n)T} R_{22} F_2^{(n)} + Q_2 \geq 0, \end{aligned} \tag{27b}$$

where $A_{F\varepsilon}^{(n)} := A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon}^{(n+1)} + B_{2\varepsilon} F_2^{(n)}$.

Step 4. *Solve the following SDP problem, with respect to* $N_2^{(n+1)}$*, subject to* (28).

$$\text{maximize } \ \mathbf{Tr}\ [N_2^{(n+1)}], \tag{28a}$$

subject to

$$\begin{aligned} &N_2^{(n+1)} A_{F\varepsilon}^{(n)T} + A_{F\varepsilon}^{(n)} N_2^{(n+1)} + A_{p\varepsilon} N_2^{(n+1)} A_{p\varepsilon}^T \\ &\quad + I_n \geq 0. \end{aligned} \tag{28b}$$

Step 5. *Solve the following SDP problem, with respect to* $N_1^{(n+1)}$*, subject to* (29).

$$\text{maximize } \ \mathbf{Tr}\ [N_1^{(n+1)}], \tag{29a}$$

subject to

$$\begin{aligned} &N_1^{(n+1)} A_{F\varepsilon}^{(n)T} + A_{F\varepsilon}^{(n)} N_1^{(n+1)} + A_{p\varepsilon} N_1^{(n+1)} A_{p\varepsilon}^T \\ &\quad - S_{1\varepsilon} M_{2\varepsilon}^{(n+1)} N_2^{(n+1)} - N_2^{(n+1)} M_{2\varepsilon}^{(n+1)} S_{1\varepsilon} \\ &\quad + S_{2\varepsilon} M_{1\varepsilon}^{(n+1)} N_2^{(n+1)} + N_2^{(n+1)} M_{1\varepsilon}^{(n+1)} S_{2\varepsilon} \\ &\quad \geq 0. \end{aligned} \tag{29b}$$

Step 5. *Compute* $F_2^{(n+1)}$.

$$\begin{aligned} &\text{vec } F_2^{(n+1)} \\ &= [N_1^{(n+1)} \otimes R_{12} + N_2^{(n+1)} \otimes R_{22}]^{-1} \\ &\qquad \text{vec}[B_{2\varepsilon}^T (M_{1\varepsilon}^{(n+1)} N_1^{(n+1)} + M_{2\varepsilon}^{(n+1)} N_2^{(n+1)})] \end{aligned} \tag{30}$$

Step 6. *If the algorithm converges, then* $F_2^{(n+1)}$ *is the solution, STOP. Otherwise, increment* $n \to n+1$ *and go to Step 2, until all LMIs* (26), (27), (28) *and* (29) *are simultaneously satisfied.*

It should be noted that convergence of the above algorithm cannot be guaranteed. However, it is observed that the proposed algorithm can be worked well in practice. Furthermore, it is worth pointing out that each iteration can be done by applying the LMI Control Toolbox with Matlab directly.

IV. Numerical Example

In order to demonstrate the efficiency of the proposed scheme, a simple example is given. The system matrices are given as follows.

$$A_\varepsilon = \begin{bmatrix} 0 & 1 \\ -\varepsilon^{-1} & -\varepsilon^{-1} \end{bmatrix},\ A_{p\varepsilon} = \begin{bmatrix} 0.1 & 0.5\varepsilon \\ \varepsilon & \varepsilon \end{bmatrix},$$

$$\delta = 2,\ \mu = 0.5\varepsilon,\ B_{1\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{-1} \end{bmatrix},\ B_{2\varepsilon} = \begin{bmatrix} 0.5 \\ 2\varepsilon^{-1} \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R_{11} = 2,\ R_{12} = 0.5,\ R_{21} = 0.1,\ R_{22} = 2.$$

The small parameter is chosen as $\varepsilon = 0.1$. It should be noted that the Newton's method (24) converges to the exact solution with a computational error of the order of $\mathrm{e}{-11}$ after four iterations. Moreover, for the same example, it can be verified that another algorithm that is based on SDP converges to same solutions. In this case, for the computational error of the order of $\mathrm{e}{-9}$, 38 iterations are needed.

In this case, the exact strategies F_i, $i = 1,\ 2$ and the solutions of CSALREs (14) are given below.

$$\begin{aligned} F_1 &= \begin{bmatrix} -8.7843\mathrm{e}{-2} & -2.4015\mathrm{e}{-1} \end{bmatrix}, \\ F_2 &= \begin{bmatrix} -3.0797\mathrm{e}{-1} & -5.8549\mathrm{e}{-1} \end{bmatrix}, \\ M_{1\varepsilon} &= \begin{bmatrix} 1.5606 & 1.7569\mathrm{e}{-2} \\ 1.7569\mathrm{e}{-2} & 4.8030\mathrm{e}{-2} \end{bmatrix}, \\ M_{2\varepsilon} &= \begin{bmatrix} 4.0230\mathrm{e}{-1} & 2.0831\mathrm{e}{-3} \\ 2.0831\mathrm{e}{-3} & 3.5164\mathrm{e}{-2} \end{bmatrix}, \\ N_1 &= \begin{bmatrix} 3.6721\mathrm{e}{-2} & 7.7362\mathrm{e}{-3} \\ 7.7362\mathrm{e}{-3} & -3.1389\mathrm{e}{-2} \end{bmatrix}, \\ N_2 &= \begin{bmatrix} 7.8400\mathrm{e}{-1} & -5.3871\mathrm{e}{-1} \\ -5.3871\mathrm{e}{-1} & 4.0143\mathrm{e}{-1} \end{bmatrix}. \end{aligned}$$

It is worth pointing out that the proposed strategies F_i can be computed for the small parameter ε.

V. Conclusion

The linear closed-loop Stackelberg strategies of the SPSS have been investigated. First, necessary conditions for the existence of the strategy set has been developed via a set of CALREs. After establishing the asymptotic structure of CALREs for the SPSS, Newton's method for solving the CSALRs has been introduced. As a result, the local quadratic convergence of the proposed algorithm was attained. Furthermore, in order to reduce the computational difficulty, SDP based algorithm is also given. The numerical example shows the effectiveness and usefulness of the proposed scheme. Finally, it is believed that our work will serve in reviving interest in the more general question of equilibrium refinement in the theory of stochastic Stackelberg game strategy.

VI. APPENDIX

Let $\bar{M}_{i1}$, $\bar{M}_{i2}$, $\bar{M}_{i3}$, $\bar{N}_{i1}$, $\bar{N}_{i2}$, $\bar{N}_{i3}$, $\bar{F}_{21}$ and $\bar{F}_{22}$ be the limiting solutions of CSALREs (14) as $\mu \to +0$, $\varepsilon \to +0$. Then we have

$$\begin{aligned}&\bar{M}_{11}\bar{A}_{F11} + \bar{A}_{F11}^T\bar{M}_{11} + \bar{M}_{12}^T\bar{A}_{F21} + \bar{A}_{F21}^T\bar{M}_{12}\\&\quad + A_{p11}^T\bar{M}_{11}A_{p11} + \bar{M}_{11}S_{11}\bar{M}_{11} + \bar{M}_{12}^TS_{13}\bar{M}_{12}\\&\quad + \bar{M}_{11}S_{12}\bar{M}_{12} + \bar{M}_{12}^TS_{12}^T\bar{M}_{11} + \bar{Q}_{F11} = 0, \qquad \text{(A.1a)}\end{aligned}$$

$$\begin{aligned}&\bar{A}_{F21}^T\bar{M}_{13} + \bar{M}_{11}\bar{A}_{F12} + \bar{M}_{12}^T\bar{A}_{F22}\\&\quad + \bar{M}_{11}S_{12}\bar{M}_{13} + \bar{M}_{12}^TS_{13}\bar{M}_{13} + \bar{Q}_{F12} = 0, \qquad \text{(A.1b)}\end{aligned}$$

$$\bar{M}_{13}\bar{A}_{F22} + \bar{A}_{F22}^T\bar{M}_{13} + \bar{M}_{13}S_{13}\bar{M}_{13} + \bar{Q}_{F13} = 0, \quad \text{(A.1c)}$$

$$\begin{aligned}&\bar{M}_{21}\bar{A}_{F11} + \bar{A}_{F11}^T\bar{M}_{21} + \bar{M}_{22}^T\bar{A}_{F21} + \bar{A}_{F21}^T\bar{M}_{22}\\&\quad + A_{p11}^T\bar{M}_{21}A_{p11} + \bar{M}_{11}S_{21}\bar{M}_{11} + \bar{M}_{12}^TS_{23}\bar{M}_{12}\\&\quad + \bar{M}_{11}S_{22}\bar{M}_{12} + \bar{M}_{12}^TS_{22}^T\bar{M}_{11} + \bar{Q}_{F21} = 0, \qquad \text{(A.2a)}\end{aligned}$$

$$\begin{aligned}&\bar{A}_{F21}^T\bar{M}_{23} + \bar{M}_{21}\bar{A}_{F12} + \bar{M}_{22}^T\bar{A}_{F22}\\&\quad + \bar{M}_{11}S_{22}\bar{M}_{13} + \bar{M}_{12}^TS_{23}\bar{M}_{13} + \bar{Q}_{F22} = 0, \qquad \text{(A.2b)}\end{aligned}$$

$$\bar{M}_{23}\bar{A}_{F22} + \bar{A}_{F22}^T\bar{M}_{23} + \bar{M}_{13}S_{23}\bar{M}_{13} + \bar{Q}_{F23} = 0, \quad \text{(A.2c)}$$

$$\begin{aligned}&\bar{N}_{11}A_{F11}^T + A_{F11}\bar{N}_{11} + \bar{N}_{12}A_{F12}^T + A_{F12}\bar{N}_{12}\\&\quad + A_{p11}\bar{N}_{11}A_{p11}^T - S_{11}\bar{M}_{21}\bar{N}_{21} - \bar{N}_{21}\bar{M}_{21}S_{11}\\&\quad - S_{12}\bar{M}_{22}^T\bar{N}_{21} - \bar{N}_{21}\bar{M}_{22}S_{12}^T\\&\quad - S_{12}\bar{M}_{23}\bar{N}_{22}^T - \bar{N}_{22}\bar{M}_{23}S_{12}^T\\&\quad + S_{21}\bar{M}_{11}\bar{N}_{21} + \bar{N}_{21}\bar{M}_{11}S_{21}\\&\quad + S_{22}\bar{M}_{12}^T\bar{N}_{21} + \bar{N}_{21}\bar{M}_{12}S_{22}^T\\&\quad + S_{22}\bar{M}_{13}\bar{N}_{22}^T + \bar{N}_{22}\bar{M}_{13}S_{22}^T = 0, \qquad \text{(A.3a)}\end{aligned}$$

$$\begin{aligned}&\bar{N}_{11}A_{F21}^T + \bar{N}_{12}A_{F22}^T\\&\quad - (\bar{N}_{21}\bar{M}_{21}S_{12} + \bar{N}_{21}\bar{M}_{22}S_{13} + \bar{N}_{22}\bar{M}_{23}S_{13})\\&\quad - (\bar{N}_{21}\bar{M}_{11}S_{22} + \bar{N}_{21}\bar{M}_{12}S_{23} + \bar{N}_{22}\bar{M}_{13}S_{13}) = 0, \text{(A.3b)}\end{aligned}$$

$$\begin{aligned}&\bar{N}_{13}A_{F22}^T + A_{F22}\bar{N}_{13} + \bar{N}_{12}^TA_{F21}^T + A_{F21}\bar{N}_{12}\\&\quad - S_{12}^T\bar{M}_{21}\bar{N}_{22} - \bar{N}_{22}^T\bar{M}_{21}S_{12}\\&\quad - S_{13}^T\bar{M}_{22}^T\bar{N}_{22} - \bar{N}_{22}\bar{M}_{22}S_{13}\\&\quad - S_{13}\bar{M}_{23}\bar{N}_{23} - \bar{N}_{23}\bar{M}_{23}S_{13}\\&\quad - S_{22}^T\bar{M}_{11}\bar{N}_{22} - \bar{N}_{22}^T\bar{M}_{11}S_{22}\\&\quad - S_{23}^T\bar{M}_{12}^T\bar{N}_{22} - \bar{N}_{22}\bar{M}_{12}S_{23}\\&\quad - S_{23}\bar{M}_{13}\bar{N}_{23} - \bar{N}_{23}\bar{M}_{13}S_{23} = 0, \qquad \text{(A.3c)}\end{aligned}$$

$$\begin{aligned}&\bar{N}_{21}A_{F11}^T + A_{F11}\bar{N}_{21} + \bar{N}_{22}A_{F12}^T + A_{F12}\bar{N}_{22}\\&\quad + A_{p11}\bar{N}_{21}A_{p11}^T + I_{\bar{N}_1} = 0, \qquad \text{(A.4a)}\end{aligned}$$

$$\bar{N}_{21}A_{F21}^T + \bar{N}_{22}A_{F22}^T = 0, \qquad \text{(A.4b)}$$

$$\begin{aligned}&\bar{N}_{23}A_{F22}^T + A_{F22}\bar{N}_{23} + \bar{N}_{22}^TA_{F21}^T + A_{F21}\bar{N}_{22}\\&\quad + I_{\bar{N}_2} = 0, \qquad \text{(A.4c)}\end{aligned}$$

$$\begin{aligned}&R_{12}\bar{F}_{21}\bar{N}_{11} + R_{12}\bar{F}_{22}\bar{N}_{12}^T + R_{22}\bar{F}_{21}\bar{N}_{21} + R_{22}\bar{F}_{22}\bar{N}_{22}^T\\&\quad + B_{21}^T\bar{M}_{11}\bar{N}_{11} + B_{22}^T\bar{M}_{12}^T\bar{N}_{11} + B_{22}^T\bar{M}_{13}\bar{N}_{12}^T\\&\quad + B_{21}^T\bar{M}_{21}\bar{N}_{21} + B_{22}^T\bar{M}_{22}^T\bar{N}_{21} + B_{22}^T\bar{M}_{23}\bar{N}_{22}^T = 0, \ \text{(A.5a)}\end{aligned}$$

$$\begin{aligned}&R_{12}\bar{F}_{21}\bar{N}_{12} + R_{12}\bar{F}_{22}\bar{N}_{13} + R_{22}\bar{F}_{21}\bar{N}_{22} + R_{22}\bar{F}_{22}\bar{N}_{23}^T\\&\quad + B_{21}^T\bar{M}_{11}\bar{N}_{12} + B_{22}^T\bar{M}_{12}^T\bar{N}_{12} + B_{22}^T\bar{M}_{13}\bar{N}_{13}\\&\quad + B_{21}^T\bar{M}_{21}\bar{N}_{22} + B_{22}^T\bar{M}_{22}^T\bar{N}_{22} + B_{21}^T\bar{M}_{23}\bar{N}_{23} = 0, \ \text{(A.5b)}\end{aligned}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}\bar{F}_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}\bar{F}_2 := \begin{bmatrix} \bar{A}_{F11} & \bar{A}_{F12} \\ \bar{A}_{F21} & \bar{A}_{F22} \end{bmatrix},$$

$$\bar{Q}_{Fi} = Q_i + \bar{F}_2^TR_{i2}\bar{F}_2 := \begin{bmatrix} \bar{Q}_{Fi1} & \bar{Q}_{Fi2} \\ \bar{Q}_{Fi2}^T & \bar{Q}_{Fi3} \end{bmatrix}.$$

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