

# A shifting pole placement approach for the design of parameter-scheduled state-feedback controllers

Damiano Rotondo, Fatiha Nejari and Vicenç Puig

**Abstract**—In this paper, the problem of designing a parameter-scheduled state-feedback controller is investigated. The main novelty and contribution of this paper is the extension of the classical regional pole placement problem, that will be referred to as *shifting pole placement*, to the design of parameter scheduled controller taking advantage of polytopes and LMIs properties. By introducing some parameters, or using existing ones, the controller can be designed in such a way that different values of these parameters imply different regions where the closed-loop poles are situated.

The problem is analyzed in both linear time-invariant (LTI) and parameter-varying (LPV) cases, and some results obtained in simulation are shown so as to demonstrate the effectiveness of the proposed approach.

**Keywords:** Linear parameter-varying systems, regional pole placement, state-feedback control, LMIs

## I. INTRODUCTION

The design of control laws that place the closed-loop poles of the system to be controlled at some desired location of the complex plane is one of the most relevant problems in control theory. Historically, the problem of *exact* pole placement, where closed-loop poles are required to lie at some desired location, was studied at first [1]. However, in cases where exact pole placement is not required, *regional* pole placement, where closed-loop poles have to be placed within a prescribed region of the plane, is sufficient. Works about this problem have begun to appear in the 80s. [2] and [3] analyze eigenvalues placement in a vertical strip and a disk, respectively. Later, the works continue in an effort to establish a complete theory about this problem.

Remarkable results were obtained in [4], where the problem of designing state- or output- feedback  $H_\infty$  controllers that satisfy additional constraints on the closed-loop pole location was addressed. Sufficient conditions for feasibility, expressed in terms of linear matrix inequalities (LMIs) easily tractable from a numerical point of view, were derived for a general class of convex regions of the complex plane. [4] started from an extension of the Lyapunov characterization of stability made in [5], so as to obtain LMI-based conditions for pole clustering, obtained with the introduction of the so-called *LMI Regions*.

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In [6], a parameter-dependent Lyapunov approach has been used so as to deal with constant or time-varying uncertainty, obtaining less conservative results in the case of slow parametric variations. Further research has been aimed to achieve many objectives, among which to enforce robustness [7] or to solve the pole placement problem for filter design [8]. The big amount of articles appeared in the last decade, e.g. [9]–[16], demonstrates that the regional pole placement problem is still a hot topic of investigation.

Polytopic systems constitute an important field for regional pole placement application. The polytopic representation is one of the most used approaches for describing uncertainty or variability of the parameters in linear parameter-varying (LPV) systems. Looking at [4], [6], [8], [10], [14], [16], it can be seen that a lot of effort has been put in developing techniques for regional pole placement of polytopic systems, taking advantage of some useful properties of LMIs that allow to assure that the satisfaction of the LMIs at some points, i.e. the vertices of the polytope, implies their satisfaction in all points inside the polytope.

In this paper, the problem of designing a parameter-scheduled state-feedback controller is investigated. The main novelty and contribution of this paper is to take advantage of the properties of polytopes and LMIs to solve a new problem, an extension of the classical regional pole placement problem, that will be referred to as *shifting pole placement*. By introducing some parameters, or using the existing ones, the controller can be designed in such a way that different values of these parameters imply different regions where the closed-loop poles are situated.

From a practical point of view, reasons for which such a problem can be of interest include all situations where some performance degradation could be desirable, e.g. high-/low-gain control, control of systems with saturation nonlinearities, graceful performance degradation for active fault-tolerant control and actuator health degradation avoidance.

The problem is analyzed in both LTI and LPV cases, and results obtained in simulation will be shown so as to demonstrate the effectiveness of the proposed approach.

The rest of the paper is organized as follows. In Section II, notation used throughout the paper is explained, along with some definitions and known results that are a starting point for the rest of the paper. In Section III and Section IV, the shifting pole placement approach for the design of a parameter-scheduled state-feedback controller, that is the main topic of this paper, is outlined for both the LTI and the LPV cases. In Section V, an example is used to demonstrate the effectiveness of the proposed approach.

Finally, conclusions are outlined in Section VI.

## II. BACKGROUND

Following the notation used by [17],  $\sigma$  stands for the Laplace variable  $s$  in the continuous-time (C-T) case and for the Z-transform variable  $z$  in the discrete-time (D-T) case. Similarly,  $\tau$  will stand for the time  $t \in \mathbb{R}^+$  in the continuous-time case and for the time samples  $k \in \mathbb{Z}^+$  in the discrete-time case. The notation  $\sigma.x(\tau)$  stands for  $\dot{x}(t)$  for continuous-time signals and for  $x(k+1)$  for discrete-time signals. For real symmetric matrices  $M$ , the notation  $M > 0$  stands for *positive definite* and indicates that all the eigenvalues of  $M$  are positive. Similarly,  $M < 0$  means *negative definite*, that is, all the eigenvalues of  $M$  are negative. Finally, a matrix polytope is defined as the convex hull of a finite number of matrices  $H_i$  with the same dimensions:

$$\text{Co}\{H_i, i = 1, \dots, N\} := \left\{ \sum_{i=1}^N \alpha_i(\theta) H_i, \alpha_i(\theta) \geq 0, \sum_{i=1}^N \alpha_i(\theta) = 1 \right\} \quad (1)$$

*Definition 1: (LMI Regions)* A subset  $\mathcal{D}$  of the complex plane is called an LMI region if there exist a symmetric matrix  $\alpha = [\alpha_{kl}] \in \mathbb{R}^{m \times m}$  and a matrix  $\beta = [\beta_{kl}] \in \mathbb{R}^{m \times m}$  such that [4]:

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) < 0\} \quad (2)$$

with:

$$f_{\mathcal{D}}(z) := \alpha + z\beta + \bar{z}\beta^T = [\alpha_{kl} + \beta_{kl}z + \beta_{lk}\bar{z}]_{1 \leq k, l \leq m} \quad (3)$$

where  $\bar{z}$  is the complex conjugate of  $z$ .

In other words, an LMI region is a convex subset of the complex plane that is representable by an LMI in  $z$  and  $\bar{z}$ . In [4], it is shown that LMI regions not only include a wide variety of typical clustering regions, but also form a subset of the convex regions that are symmetric with respect to the real axis (e.g. disks, conic sectors, vertical half planes, vertical strips, horizontal strips, ellipses, parabolas, hyperbolic sectors and their intersections). Hence, LMI regions cover most practical needs for control purposes.

Pole location in a given LMI region can be characterized in terms of the  $m \times m$  block matrix:

$$M_{\mathcal{D}}(A, X) := \alpha \otimes X + \beta \otimes (AX) + \beta^T \otimes (AX)^T = [\alpha_{kl}X + \beta_{kl}AX + \beta_{lk}XA^T]_{1 \leq k, l \leq m} \quad (4)$$

through the following theorem [4].

*Theorem 1:* The matrix  $A$  is  $\mathcal{D}$ -stable if and only if there exists a symmetric matrix  $X$  such that:

$$\begin{cases} M_{\mathcal{D}}(A, X) < 0 \\ X > 0 \end{cases} \quad (5)$$

*Proof:* See [4]. ■

$M_{\mathcal{D}}$  in (4) and  $f_{\mathcal{D}}(z)$  in (3) are related by the substitution  $(X, AX, XA^T) \leftrightarrow (1, z, \bar{z})$ .

Following the terminology of [18], linear parameter-varying (LPV) systems are linear time-varying (LTV) plants whose state-space matrices are fixed functions of some vector of varying parameters  $\theta(\tau)$ . In the following, a system will

be assumed to be characterized only by its state equation, that in the LPV case takes the form:

$$\sigma.x(\tau) = A(\theta(\tau))x(\tau) + B(\theta(\tau))u(\tau) \quad (6)$$

where  $x(\tau) \in \mathbb{R}^{n_x}$ ,  $u(\tau) \in \mathbb{R}^{n_u}$  and  $y(\tau) \in \mathbb{R}^{n_y}$  denote the state, input and output vectors respectively, and  $\theta(\tau) \in \mathbb{R}^{n_\theta}$  is a time-varying vector of real parameters.

*Definition 2: (Polytopic LPV System)* An LPV system is called *polytopic* when it can be represented by state-space matrices whose dependence on the parameter vector  $\theta$ , that ranges over a polytope, is affine [17]:

$$\begin{pmatrix} A(\theta(\tau)) \\ B(\theta(\tau)) \end{pmatrix} \in \text{Co} \left\{ \begin{pmatrix} A_i \\ B_i \end{pmatrix}, i = 1, \dots, N \right\} \quad (7)$$

## III. SHIFTING POLE PLACEMENT APPROACH FOR LTI SYSTEMS

Consider the following LTI system:

$$\sigma.x(\tau) = Ax(\tau) + Bu(\tau) \quad (8)$$

under state-feedback control law  $u(\tau) = K(p(\tau))x(\tau)$ .

The problem considered in this section consists in designing the controller  $K(p(\tau))$  such that the closed-loop poles of (8) are placed in an LMI region  $\mathcal{D}(p(\tau))$ , whose characteristic function (3) depends on  $p(\tau)$ :

$$f_{\mathcal{D}}(z, p(\tau)) = \alpha(p(\tau)) + z\beta(p(\tau)) + \bar{z}\beta(p(\tau))^T = [\alpha_{kl}(p(\tau)) + \beta_{kl}(p(\tau))z + \beta_{lk}(p(\tau))\bar{z}]_{1 \leq k, l \leq m} \quad (9)$$

where  $p(\tau) \in \mathbb{R}^{n_p}$  is an exogenous parameter vector that can take values either in a finite set of possible values or in a continuous (hyper) interval  $\mathcal{P}$ .

**Remark:** Notice that, since the controller is LPV, also the closed-loop system is LPV. Hence, from a strict mathematical point of view, the idea of *poles* loses its meaning. However, following [19] and with a little abuse of language, the poles of an LPV system are defined as the set of all the poles of the LTI systems obtained by freezing  $p(\tau)$  to all its possible values  $p \in \mathcal{P}$ .

### A. Discrete-valued $p(\tau)$

In this case, for each value taken by  $p(\tau) \in \mathcal{P} = \{p_1, p_2, \dots, p_N\}$ , an LMI region  $\mathcal{D}_i, i = 1, \dots, N$  is defined. The corresponding block matrix  $M_{\mathcal{D}_i}(A_{cl,i}, X_i)$  is given by:

$$M_{\mathcal{D}_i}(A_{cl,i}, X_i) = \alpha(p_i) \otimes X_i + \beta(p_i) \otimes (A_{cl,i}X_i) + \beta(p_i)^T \otimes (A_{cl,i}X_i)^T = [\alpha_{kl}(p_i)X_i + \beta_{kl}(p_i)A_{cl,i}X_i + \beta_{lk}(p_i)X_iA_{cl,i}^T]_{1 \leq k, l \leq m} \quad (10)$$

where  $A_{cl,i} = A + BK_i$  with  $K_i \in \mathcal{K} = \{K_1, K_2, \dots, K_N\} \equiv \{K(p_1), \dots, K(p_N)\}$ .

Then, using Theorem 1, the shifting pole-placement problem is solved by finding symmetric matrices  $X_i$  and matrices  $K_i$  such that:

$$\begin{cases} M_{\mathcal{D}_i}(A + BK_i, X_i) < 0 \\ X_i > 0 \end{cases} \quad (11)$$

for  $i = 1, \dots, N$ . Notice that the problem is not convex because of the products  $K_i X_i$ . However, using the auxiliary variables  $L_i = K_i X_i$  [20], convexity is restored. Then, founding feasible solutions  $(X_i^*, L_i^*, i = 1, \dots, N)$ , the state-feedback gains can be obtained as  $K_i^* = L_i^* (X_i^*)^{-1}$ .

The stability is guaranteed as long as changes in the value of the parameter  $p$  are slow enough so as to allow considering the overall system as though as it were LTI and the desired LMI regions  $\mathcal{D}_i$  are inside the stability region, that is, the left-hand semi-plane in the continuous-time case and the unit circle in the discrete-time case. Cases of quick variations of  $p$  should be analyzed and treated using results from the hybrid system theory, that goes beyond the scope of this paper.

### B. Continuous-valued $p(\tau)$

In this case, the shifting pole placement problem is solved if and only if there exist a symmetric matrix  $X$  and a parameter-scheduled state-feedback gain  $K(p(\tau))$  such that:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \alpha_{kl}(p(\tau))X + \beta_{kl}(p(\tau))[A + BK(p(\tau))]X \\ + \beta_{lk}(p(\tau))X[A + BK(p(\tau))]^T \end{array} \right\}_{1 \leq k, l \leq m} < 0 \\ X > 0 \end{array} \right. \quad (12)$$

for each  $p \in \mathcal{P}$ . From a practical point of view, (12) is useless because it imposes an infinite number of constraints to be solved. This difficulty can be alleviated under the assumptions that:

(a)  $\mathcal{P}$  is a polytope:

$$\mathcal{P} = \text{Co}\{p_i, i = 1, \dots, N\} \quad (13)$$

(b)  $\beta(p(\tau))$  is a constant matrix, which implies that (9) reduces to:

$$f_D(z, p(\tau)) = \alpha(p(\tau)) + z\beta + \bar{z}\beta^T \quad (14)$$

(c) the dependence of the matrix  $\alpha(p(\tau))$  on  $p(\tau)$  is affine, so it ranges in a polytope of matrices whose vertices are the images of  $p_1, \dots, p_N$ :

$$\alpha(p(\tau)) \in \text{Co}\{\alpha_i = \alpha(p_i), i = 1, \dots, N\} \quad (15)$$

(d) the choice of the LPV state-feedback controller is polytopic, with the same coefficients as the polytopic decomposition of  $p(\tau)$  and  $\alpha(p(\tau))$ , that is, if:

$$p(\tau) = \sum_{j=1}^N \pi_j(p(\tau)) p_j \quad (16)$$

and:

$$\alpha(p(\tau)) = \sum_{j=1}^N \pi_j(p(\tau)) \alpha_j \quad (17)$$

then:

$$K(p(\tau)) = \sum_{j=1}^N \pi_j(p(\tau)) K_j \quad (18)$$

Under these assumptions, it is possible to reduce (12) to a system made up by a finite number of LMIs,  $N + 1$ , by rewriting the first condition in (12) at the polytope vertices.

Then, thanks to basic properties of matrices and LMIs [21], it can be shown that (12) results satisfied inside the polytope, thus satisfying the shifting pole placement problem. This can be stated in the following theorem:

*Theorem 2:* Let  $\mathcal{D}(p(\tau))$  be an LMI region whose characteristic function depends on the scheduling parameter  $p(\tau)$ , assume (a)-(d) and that a symmetric matrix  $X$  and  $N$  matrices  $K_1, \dots, K_N$  have been found such that:

$$\left\{ \begin{array}{l} \left\{ \alpha_{kl,i}X + \beta_{kl}[A + BK_i]X + \beta_{lk}X[A + BK_i]^T \right\}_{1 \leq k, l \leq m} < 0 \\ X > 0 \end{array} \right. \quad (19)$$

for each  $i = 1, \dots, N$ . Then, given a value  $p \in \mathcal{P}$  whose polytopic decomposition is (16), (12) results satisfied, that is, the controller (18) places the closed-loop poles of (8) in  $\mathcal{D}(p(\tau))$ .

*Proof:* Due to a basic property of matrices [21], any linear combination of (19) with non-negative coefficients is negative definite. Hence, using the linear combination brought by (16) leads to:

$$\sum_{i=1}^N \pi_i \left\{ \alpha_{kl,i}X + \beta_{kl}[A + BK_i]X + \beta_{lk}X[A + BK_i]^T \right\}_{1 \leq k, l \leq m} < 0 \quad (20)$$

Bringing inside the square brackets  $\sum_{i=1}^N \pi_i$  and taking into account that  $\sum_{i=1}^N \pi_i = 1$ , assumption (b), (17) and (18), the following result is obtained:

$$\left\{ \begin{array}{l} \alpha_{kl}(p(\tau))X + \beta_{kl}[A + BK(p(\tau))]X \\ + \beta_{lk}X[A + BK(p(\tau))]^T \end{array} \right\}_{1 \leq k, l \leq m} < 0 \quad (21)$$

that proves the theorem. ■

Also in this case, the problem is not convex because of the products  $K_i X$ . Convexity is restored by means of the auxiliary variables  $L_i = K_i X$ .

## IV. SHIFTING POLE PLACEMENT APPROACH FOR LPV SYSTEMS

Consider the following LPV system:

$$\sigma.x(\tau) = A(\theta(\tau))x(\tau) + B(\theta(\tau))u(\tau) \quad (22)$$

with  $\theta(\tau) = [\theta_s(\tau) \ \theta_r(\tau) \ \theta_p(\tau)]^T \in \Theta = \Theta_s \cup \Theta_r \cup \Theta_p^1$ . The problem considered in this section is that of designing the controller  $K(\theta_s(\tau), \theta_p(\tau))$  such that, for each  $\theta \in \Theta$ , the closed-loop poles of (22) are placed in an LMI region  $\mathcal{D}(\theta_p(\tau))$ , such that its characteristic function depends on  $\theta_p(\tau)$ :

$$f_{\mathcal{D}}(z, \theta_p(\tau)) = \alpha(\theta_p(\tau)) + z\beta(\theta_p(\tau)) + \bar{z}\beta(\theta_p(\tau))^T \\ = [\alpha_{kl}(\theta_p(\tau)) + \beta_{kl}(\theta_p(\tau))z + \beta_{lk}(\theta_p(\tau))\bar{z}]_{1 \leq k, l \leq m} \quad (23)$$

<sup>1</sup> $\theta_s(\tau)$  are varying parameters used to schedule the controller;  $\theta_r(\tau)$  are varying parameters that are not used to schedule the controller, and against whose variation robustness must be guaranteed;  $\theta_p(\tau)$  are varying parameters used to schedule the controller and the LMI region  $\mathcal{D}(\theta_p(\tau))$  too.

$\Theta_s$ ,  $\Theta_r$  and  $\Theta_p$  are assumed to be polytopes, such that:

$$\theta_s(\tau) = \sum_{i=1}^S s_i(\theta_s(\tau)) \theta_{s,i} \quad (24)$$

$$\theta_r(\tau) = \sum_{j=1}^R r_j(\theta_r(\tau)) \theta_{r,j} \quad (25)$$

$$\theta_p(\tau) = \sum_{h=1}^P \pi_h(\theta_p(\tau)) \theta_{p,h} \quad (26)$$

with  $S$ ,  $R$  and  $P$  number of vertices, denoted by  $\theta_{s,i}$ ,  $\theta_{r,j}$  and  $\theta_{p,h}$ , of  $\Theta_s$ ,  $\Theta_r$  and  $\Theta_p$ , respectively.

In this case, the shifting pole placement problem is solved if and only if there exist a symmetric positive definite matrix  $X > 0$  and a parameter-scheduled state-feedback gain  $K(p(\tau))$  such that:

$$\left\{ \begin{array}{l} \alpha_{kl}(\theta_p(\tau)) \\ +\beta_{kl}(\theta_p(\tau)) [A(\theta(\tau)) + B(\theta(\tau))K(\theta_s(\tau), \theta_p(\tau))]X \\ +\beta_{lk}(\theta_p(\tau))X [A(\theta(\tau)) + B(\theta(\tau))K(\theta_s(\tau), \theta_p(\tau))]^T \end{array} \right\}_{1 \leq k, l \leq m} < 0 \quad (27)$$

In this case, analogously to the previous section, (27) cannot be used for the controller design, as it imposes an infinite number of constraints that must be satisfied. Under the assumptions that  $\beta_{kl}(\theta_p(\tau))$  is a constant matrix, that  $B(\theta_s(\tau), \theta_r(\tau), \theta_p(\tau))$  only depends on  $\theta_r(\tau)$  and that the matrices  $\alpha_{kl}$ ,  $A$ ,  $B$  and  $K$  vary polytopically as follows:

$$\alpha_{kl}(\theta_p(\tau)) = \sum_{h=1}^P \pi_h(\theta_p(\tau)) \alpha_{kl,h} \quad (28)$$

$$A(\theta(\tau)) = \sum_{i=1}^S s_i(\theta_s(\tau)) \sum_{j=1}^R r_j(\theta_r(\tau)) \sum_{h=1}^P \pi_h(\theta_p(\tau)) A_{ijh} \quad (29)$$

$$B(\theta(\tau)) = B(\theta_r(\tau)) = \sum_{j=1}^R r_j(\theta_r(\tau)) B_j \quad (30)$$

$$K(\theta_s(\tau), \theta_p(\tau)) = \sum_{i=1}^S s_i(\theta_s(\tau)) \sum_{h=1}^P \pi_h(\theta_p(\tau)) K_{ih} \quad (31)$$

Notice that (27) can be reduced to a finite number of LMIs by rewriting it at the  $S \cdot R \cdot P$  vertices of the polytope  $\Theta$ , as stated in the following theorem, whose proof is omitted due to lack of space:

*Theorem 3:* Let  $\mathcal{D}(\theta_p(\tau))$  be an LMI region whose characteristic function depends on the scheduling parameter  $\theta_p(\tau)$  and assume that a symmetric positive definite matrix  $X > 0$  and  $S \cdot P$  matrices  $K_{11}, \dots, K_{ih}, \dots, K_{SP}$  have been found such that:

$$\left\{ \begin{array}{l} \alpha_{kl,h} + \beta_{kl} [A_{ijh} + B_j K_{ih}] X \\ +\beta_{lk} X [A_{ijh} + B_j K_{ih}]^T \end{array} \right\}_{1 \leq k, l \leq m} < 0 \quad (32)$$

TABLE I  
VARYING PARAMETERS AND POLYTOPIC VERTICES

$\theta_s$	$\theta_r$	$\theta_p$		
2	2	1	$A_1$	$K_1$
2	2	2	$A_2$	$K_2$
2	3	1	$A_3$	$K_1$
2	3	2	$A_4$	$K_2$
3	2	1	$A_5$	$K_3$
3	2	2	$A_6$	$K_4$
3	3	1	$A_7$	$K_3$
3	3	2	$A_8$	$K_4$

for each  $i = 1, \dots, S$ ,  $j = 1, \dots, R$  and  $h = 1, \dots, P$ . Then, given a triple  $\{\theta_s(\tau), \theta_r(\tau), \theta_p(\tau)\} \in \Theta$  which polytopic decompositions is given by (24)-(26), equation (27) results satisfied. Hence, the controller (31) places the closed-loop poles of (22) in  $\mathcal{D}(\theta_p(\tau))$ .

*Proof:* Similar to Theorem 2. ■

Also in this case, although the problem to be solved is not convex, convexity is restored using the auxiliary variables  $L_{ih} = K_{ih}X$ .

## V. APPLICATION EXAMPLE

The example proposed in this paper is an LPV mathematical system that illustrates some of the features of the proposed shifting pole placement approach for the design of parameter-scheduled state-feedback controllers.

Consider a continuous-time LPV system with a structure as in (22) ( $\tau = t$ ), a parameter-varying state matrix  $A(\theta(t))$  and a constant input matrix  $B$  defined as follows:

$$A(\theta(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\theta_p(t) & -\theta_s(t) & \theta_p(t) & 0 \\ 0 & 0 & 0 & 1 \\ \theta_p(t) & 0 & -\theta_p(t) & -\theta_r(t) \end{bmatrix} \quad (33)$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \quad (34)$$

where  $\theta_s \in [2, 3]$ ,  $\theta_r \in [2, 3]$  and  $\theta_p \in [1, 2]$  (in this example, the subscripts  $s, r, p$  are used following the notation used in Section IV and explained in footnote 1).

The controller  $u(t) = K(\theta_s(t), \theta_p(t))x(t)$  is designed such that the closed-loop poles of the LPV system are placed in a disk of radius  $r(\theta_p(t))$  and center  $(-q(\theta_p(t)), 0)$ , described by the characteristic function:

$$f_{\mathcal{D}}(z, \theta_p(t)) = \begin{pmatrix} -r(\theta_p(t)) & q(\theta_p(t)) + z \\ q(\theta_p(t)) + \bar{z} & -r(\theta_p(t)) \end{pmatrix} \quad (35)$$

with  $r(\theta_p(t))$  and  $q(\theta_p(t))$  defined as:

$$\begin{cases} r(\theta_p(t)) = 1 + \theta_p(t) \\ q(\theta_p(t)) = -1 + 3\theta_p(t) \end{cases} \quad (36)$$

The design is done using (32), that in this case is made up by nine LMIs in the variables  $X$ ,  $K_1$ ,  $K_2$ ,  $K_3$ ,  $K_4$  (see Table I).

$$\left\{ \begin{array}{l} \left( \begin{array}{cc} -2X & 2X + A_1X + BK_1X \\ 2X + XA_1^T + XK_1^T B^T & -2X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -3X & 5X + A_2X + BK_2X \\ 5X + XA_2^T + XK_2^T B^T & -3X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -2X & 2X + A_3X + BK_1X \\ 2X + XA_3^T + XK_1^T B^T & -2X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -3X & 5X + A_4X + BK_2X \\ 5X + XA_4^T + XK_2^T B^T & -3X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -2X & 2X + A_5X + BK_3X \\ 2X + XA_5^T + XK_3^T B^T & -2X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -3X & 5X + A_6X + BK_4X \\ 5X + XA_6^T + XK_4^T B^T & -3X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -2X & 2X + A_7X + BK_3X \\ 2X + XA_7^T + XK_3^T B^T & -2X \end{array} \right) < 0 \\ \left( \begin{array}{cc} -3X & 5X + A_8X + BK_4X \\ 5X + XA_8^T + XK_4^T B^T & -3X \end{array} \right) < 0 \end{array} \right. \quad (37)$$

$X > 0$

The resulting closed-loop poles for different values of the scheduling parameter  $\theta_p$  are shown in Fig. 1, proving that the shifting pole placement specification is correctly satisfied.

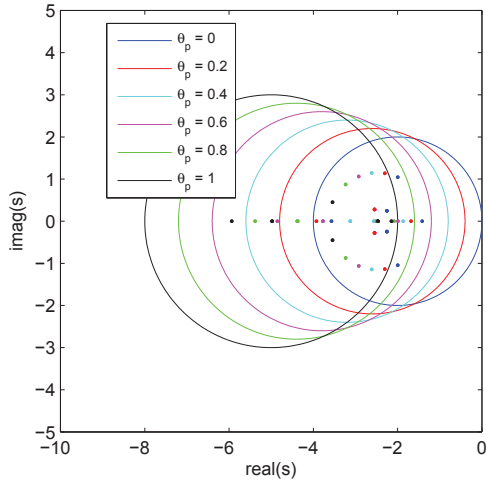


Fig. 1. Shifting pole placement for the LPV mathematical system.

The free responses of the state variables are shown in Fig. 2-5. These have been obtained starting from the initial state  $x(0) = [1 \ 0 \ 1 \ 0]^T$  in four different cases, three of which with constant values of the scheduling parameter  $\theta_p(t)$  ( $\theta_p = 0$ ,  $\theta_p = 0.5$  and  $\theta_p = 1$ , corresponding to blue, purple and red line, respectively), and one with a varying scheduling parameter  $\theta_p(t) = 0.5 + 0.5 \sin(\pi t/2)$  (corresponding to black line). The remaining scheduling parameters have been chosen as  $\theta_s(t) = 2.5 + 0.5 \cos t$  and  $\theta_r(t) = 2.5 + 0.5 \sin t$ .

It can be seen from the figures that the closed-loop system behaves as expected: a big value of  $\theta_p$  corresponds to faster dynamics of the closed-loop system. In the fourth case, that is, with a time-varying  $\theta_p$ , the dynamics of the closed-loop system around  $t = 0s$  is the same as that of the closed-loop system scheduled by the constant  $\theta_p = 0.5$ . As the time increases, so does the value of  $\theta_p$  and the system gets

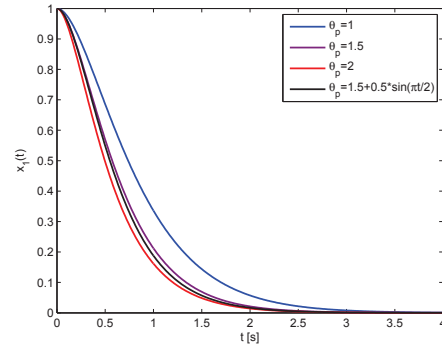


Fig. 2. Free response of  $x_1(t)$  (closed-loop).

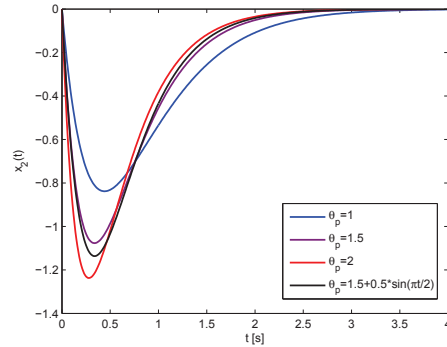


Fig. 3. Free response of  $x_2(t)$  (closed-loop).

faster until  $t = 1s$  when  $\theta_p$  begins to decrease and the trend reverses, with the system getting slower. However, this last effect and the increasing of speed from time  $t = 3s$  are not appreciable because the steady-state has almost been reached. The input signals  $u_1$  and  $u_2$  are shown in Fig. 6 and Fig. 7. It can be seen that the bigger is  $\theta_p$ , the bigger are the control signals, and vice versa. This is consistent with the fact that strong control actions are required to make the controlled system faster.

## VI. CONCLUSIONS AND FUTURE WORK

In this paper, the problem of designing a parameter-scheduled state-feedback controller that satisfies a new kind of specification, referred to as *shifting pole placement*, has been investigated. The problem has been analyzed in both the LTI and the LPV cases. The results obtained with an LPV mathematical example have demonstrated the effectiveness of the proposed approach.

Future work will focus on applying the proposed approach to all those situations where some performance degradation could be desirable, e.g. control of systems with saturation non-linearities and graceful performance degradation in fault-tolerant control.

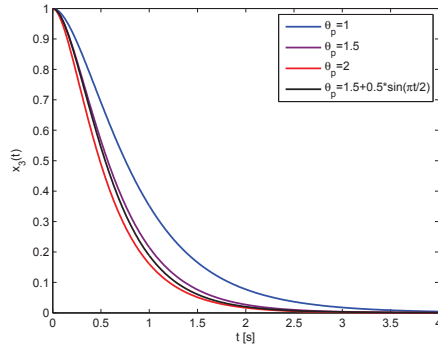


Fig. 4. Free response of  $x_3(t)$  (closed-loop).

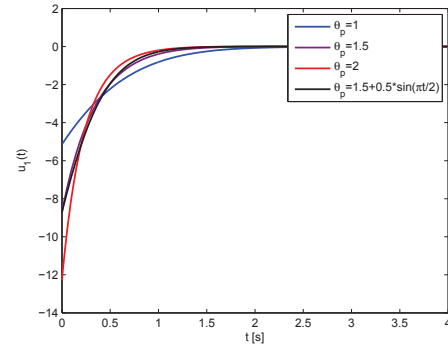


Fig. 6. Input signal  $u_1(t)$ .

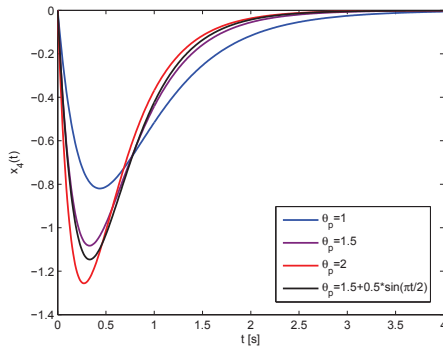


Fig. 5. Free response of  $x_4(t)$  (closed-loop).

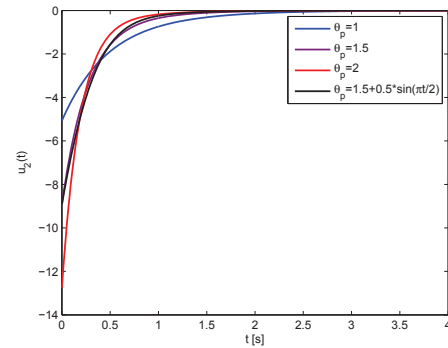


Fig. 7. Input signal  $u_2(t)$ .

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