

A Stochastic Lyapunov Feedback Technique for Propagator Generation of Quantum Systems on $U(n)$ *

H. B. Silveira¹, P. S. Pereira da Silva² and P. Rouchon³

Abstract—This work treats the problem of generating any desired goal propagator for a driftless quantum system that evolves on the unitary group $U(n)$. The physical relevance of such control problem is the realization of arbitrary quantum gates in quantum computers. Assuming only the controllability of the system, the paper constructs explicit stochastic control laws that assure global asymptotic convergence of the propagator of the system towards the goal propagator. The purpose of introducing a stochastic behaviour in the controls is to speed up convergence. The control strategy can be rigorously proved based on Lyapunov feedback and stochastic techniques. The controls laws rely on a reference trajectory that crosses the desired goal propagator in a time-periodic fashion and such that its corresponding linearised system generates the Lie algebra $\mathfrak{u}(n)$. Their existence is ensured by the Return Method of Coron, and standard Fourier series results allows them to be explicitly constructed.

I. INTRODUCTION

This paper summarizes some results of the authors in [14]. Consider a controllable driftless quantum system of the form

$$\dot{X}(t) = -\iota \sum_{k=1}^m u_k(t) S_k X(t) = \sum_{k=1}^m u_k(t) H_k X(t), \quad X(0) = I, \quad (1)$$

where $X(t) \in U(n)$ is the state (the propagator), $U(n)$ is the unitary group of n -square complex matrices, I is the identity matrix, $\iota \in \mathbb{C}$ is the imaginary unit, $u_k \in \mathbb{R}$ are the controls, $H_k = -\iota S_k \in \mathfrak{u}(n)$, and $\mathfrak{u}(n)$ is the Lie algebra associated to the Lie group $U(n)$. Recall that $\mathfrak{u}(n)$ corresponds to the real vector space of all skew-Hermitian n -square complex matrices, and that $U(n)$ is an embedded manifold in the real Banach space M^n of all n -square complex matrices endowed with the Frobenius (Euclidean) norm. The controllability assumption means that the Lie algebra generated by the H_k 's coincides with $\mathfrak{u}(n)$ [8], [1]. Choose any desired goal propagator $X_{goal} \in U(n)$. The control problem is to find bounded continuous open-loop controls $t \geq 0 \mapsto u_k(t) \in \mathbb{R}$ steering $X(t)$ from initial condition $X(0) = I$ to an arbitrary small neighbourhood of X_{goal} in some finite time $\bar{t} > 0$.

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¹H. B. Silveira is with Department of Automation and Systems (DAS), Federal University of Santa Catarina (UFSC), Brazil hector@das.ufsc.br

²P. S. Pereira da Silva is with Laboratory of Automation and Control (LAC), Department of Telecommunications and Control Engineering (PTC), University of São Paulo (USP), Brazil paulo@lac.usp.br

³P. Rouchon is with Mines ParisTech, Centre Automatique et Systèmes, Mathématiques et Systèmes (CAS), France pierre.rouchon@mines-paristech.fr

This work provides a solution merging Lyapunov feedback and stochastic methods (see Theorem 1 in Section II).

In quantum control theory, systems of the form (1) with a nonzero drift term $H_0 X$ correspond to the propagator associated to a bilinear control system that evolves on the unit sphere \mathbb{S}^{n-1} in \mathbb{C}^n (see e.g. [1, p. 1], [7, p. 26]). Under certain conditions, a standard technique to eliminate the drift term is to apply a global time-varying change of coordinates (the interaction picture) followed by the rotating wave approximation (RWA). The resulting system is driftless and has the form (1).

It follows from the principles of quantum mechanics that the Schrödinger equation determines the quantum dynamics as long as the system remains isolated from external measurements. When a measurement is made on the quantum system, its state collapses. Therefore, feedback control techniques cannot be directly applied. Although the proposed solution of the steering problem here treated determines feedback control laws $u_k = u_k(X, t)$ that depend on X and on time t , in practice its control values $u_k(t) = u_k(X(t), t)$ should be computed *off-line* by numerical integration of (1).

In the case of quantum systems consisting of n -qubits, the control problem considered in this paper for the associated propagator (after the RWA in the interaction picture) allows one to generate in an approximate manner arbitrary quantum logic gates that operate on n -qubits. The qubit (quantum bit) is the quantum analog of the usual bit in classical computation theory [12]. Thus the physical relevance of the present work lies in the realization of quantum gates for quantum computers.

There are known negative results in the literature with respect to the stabilization of driftless nonholonomic systems of the form $\dot{x} = \sum_{k=1}^m u_k f_k(x)$, where $x \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}$ are the controls and f_k are smooth mappings. If $m < n$ and $f_1(0), \dots, f_m(0)$ are linearly independent, then the origin $x = 0$ cannot be stabilized by continuous time-invariant feedbacks $u_k = u_k(x)$ [16], [2]. However, one may stabilize such systems by means of smooth time-varying feedbacks $u_k = u_k(t, x)$ that are periodic in t [3], [16] (see also [5], [6]).

Loosely speaking, this work constructs explicit control laws $u_k = u_k(X, t)$ that are T -periodic in t in order to solve the steering problem of system (1), where $T > 0$ is arbitrary. One points out that only the controllability of (1) is assumed; no additional hypothesis are required. The periodicity in the controls appears because one takes a T -periodic reference trajectory $(\bar{X}(t), \bar{u}_1(t), \dots, \bar{u}_m(t), t \geq 0)$ of system (1) with initial condition $\bar{X}(0) = X_{goal}$ such that its associated

linearised system generates the Lie algebra $\mathfrak{u}(n)$. The control strategy relies on Lyapunov feedback to force the samples $\tilde{X}(jT)$ of the tracking error¹ $\tilde{X}(t) = \bar{X}^\dagger(t)X(t)$ to asymptotically converge towards the identity matrix I as $j \rightarrow \infty$. This is a global result with no singularities. Although the existence of such reference trajectory is ensured by the Return Method of Coron in [4] (see also [5], [6]), they are in general not explicitly known. In order to overcome this drawback, one makes use of standard Fourier series results to explicitly construct the desired reference trajectory.

Although not here exhibited, computer simulations indicated that the convergence of speed of $\tilde{X}(jT)$ towards X_{goal} could be improved if one allowed the reference trajectory to vary stochastically while still crossing X_{goal} in a periodic fashion. With the aim of formalizing such heuristic result, the control strategy here proposed merges stochastic techniques into the Lyapunov feedback mentioned above. The development relies on a well-known result that can be considered as a stochastic version of Lasalle's invariance principle (see Theorem 4 in Appendix B). To sum up, this paper completely solves the steering problem for system (1) based on a new stochastic Lyapunov feedback technique. The solution is established in Theorem 1, which is the main result of this work.

The control problem approached in the present paper can be seen as a motion planning problem. The latter was considered in [20] based on optimal control theory, solved in an approximate manner in [11] with the use of averaging techniques (see also [17] for the case of quantum systems), treated in [13] for a single qubit quantum system on $SU(2)$ by means of a flatness approach, and studied in [18], [19] for driftless controllable systems that evolve on $SU(n)$. Based on decompositions of the Lie group $SU(n)$, [7] (see also the references therein) considers the problem of finding piecewise-constant inputs $u_k: [0, t_f] \rightarrow \mathbb{R}$ that steer the state of systems of the form (1) with a drift term H_0X to an arbitrary final state $X_f \in SU(n)$ in some finite instant of time. Furthermore, [7] also treats the problem of reaching a given final state in minimum time. Lyapunov stabilization of quantum systems have been analysed in [7] (see also the references therein), [9] and [22], for instance. The solution proposed in this work provides constructive explicit controls with no singularities and no further assumptions besides the controllability of (1).

The layout of the paper is as follows. Section II develops the proposed stochastic Lyapunov technique for the steering problem of system (1), and Section III presents the conclusions and discusses future work.

II. A STOCHASTIC LYAPUNOV FEEDBACK

Consider system (1). Take $T > 0$ and set $\omega = 2\pi/T$. Fix an integer $M > 0$ and let $a_{k,\ell} \in \mathbb{R}$ for $k = 1, \dots, m$, $\ell = 1, \dots, M$. Consider the T -periodic continuous reference

controls

$$\bar{u}_k(t) = \sum_{\ell=1}^M a_{k,\ell} \sin(\ell\omega t), \quad t \in \mathbb{R}, \quad k = 1, \dots, m, \quad (2)$$

and the associated reference trajectory $\bar{X}(t) \in U(n)$, $t \in \mathbb{R}$, solution of

$$\dot{\bar{X}}(t) = \sum_{k=1}^m \bar{u}_k(t) H_k \bar{X}(t), \quad \bar{X}(0) = X_{goal} \in U(n). \quad (3)$$

This means that $\bar{X}(t)$ is the solution of (1) with $u_k = \bar{u}_k$ and initial condition X_{goal} at $t = 0$. Since $\bar{u}_k(T-t) = -\bar{u}_k(t)$ for $t \in \mathbb{R}$, one has $\bar{X}(t) = \bar{X}(T-t)$ for $t \in \mathbb{R}$, and thus $\bar{X}(jT) = X_{goal}$ for all $j \in \mathbb{N}$ (see [6] for details). Therefore, $\bar{X}(t)$ is T -periodic.

The tracking error $\tilde{X}(t) = \bar{X}^\dagger(t)X(t)$ obeys

$$\dot{\tilde{X}}(t) = \sum_{k=1}^m \tilde{u}_k(t) \tilde{H}_k(t) \tilde{X}(t), \quad \tilde{X}(0) = X_{goal}, \quad (4)$$

where $\tilde{u}_k = u_k - \bar{u}_k$, and $\tilde{H}_k(t) = \bar{X}^\dagger(t)H_k\bar{X}(t) \in \mathfrak{u}(n)$ depends on $t \in \mathbb{R}$ and is T -periodic. The goal is to globally stabilize \tilde{X} towards the identity I . In order to accomplish this, one will choose a suitable Lyapunov function $\mathcal{V}(\tilde{X})$ to measure the distance from \tilde{X} to the identity matrix I .

Let $\mathcal{W} \subset U(n)$ be the set of $\tilde{X} \in U(n)$ which have all eigenvalues different from -1 . It follows from standard results on continuity of eigenvalues that \mathcal{W} is open in $U(n)$. Denote by \mathbf{H} the (real) vector space of all Hermitian n -square complex matrices and consider the map²

$$\mathcal{W} \ni \tilde{X} \mapsto \Upsilon(\tilde{X}) = \iota \frac{\tilde{X} - I}{\tilde{X} + I} \in \mathbf{H}.$$

One has that $\Upsilon: \mathcal{W} \rightarrow \mathbf{H}$ is a well-defined smooth map on the open submanifold \mathcal{W} of $U(n)$. Indeed, writing

$$\Upsilon(\tilde{X}) = \iota \frac{\tilde{X} - \tilde{X}^\dagger \tilde{X}}{\tilde{X} + \tilde{X}^\dagger \tilde{X}},$$

one gets

$$\begin{aligned} \Upsilon(\tilde{X})^\dagger &= -\iota \frac{\tilde{X}^\dagger - \tilde{X} \tilde{X}^\dagger}{\tilde{X}^\dagger + \tilde{X} \tilde{X}^\dagger} = -\iota (I - \tilde{X}) \tilde{X}^\dagger ((I + \tilde{X}) \tilde{X}^\dagger)^{-1} \\ &= \iota (\tilde{X} - I) (I + \tilde{X})^{-1} = \Upsilon(\tilde{X}). \end{aligned}$$

For $\tilde{X} \in \mathcal{W}$, the distance to I is measured by the Frobenius norm of $\Upsilon(\tilde{X})$:

$$\mathcal{V}(\tilde{X}) \triangleq \text{Tr} \left(\Upsilon(\tilde{X}) \Upsilon(\tilde{X})^\dagger \right) = -\text{Tr} \left(\frac{(\tilde{X} - I)^2}{(\tilde{X} + I)^2} \right) \geq 0. \quad (5)$$

Note that $\mathcal{V}: \mathcal{W} \rightarrow \mathbb{R}$ is smooth and non-negative. Moreover, $\mathcal{V}(\tilde{X}) = 0$ implies $\tilde{X} = I$. One will impose that $\dot{\mathcal{V}} \leq 0$.

²Given $A, B \in M^n$ with B invertible and $AB^{-1} = B^{-1}A$ (i.e. A and B^{-1} commute), one defines $A/B \triangleq AB^{-1} = B^{-1}A$. It is easy to see that A and B^{-1} commute whenever A and B commute.

¹The symbol \dagger denotes the conjugate transpose.

Using $\text{Tr}(AB) = \text{Tr}(BA)$, standard computations yield, for $(\tilde{X}, t) \in \mathcal{W} \times \mathbb{R}$,

$$\begin{aligned} \dot{\mathcal{V}}(\tilde{X}, t) &\triangleq \frac{d}{dt} \mathcal{V}(\tilde{X}) \\ &= -4 \sum_{k=1}^m \tilde{u}_k \text{Tr} \left(\tilde{X}(\tilde{X} - I)(\tilde{X} + I)^{-3} \tilde{H}_k(t) \right). \end{aligned}$$

Choose any (constant) feedback gains $f_k > 0$. Then the feedbacks

$$\tilde{u}_k(\tilde{X}, t) = f_k \text{Tr} \left(\tilde{X}(\tilde{X} - I)(\tilde{X} + I)^{-3} \tilde{H}_k(t) \right) \quad (6)$$

make

$$\dot{\mathcal{V}}(\tilde{X}, t) = -4 \sum_{k=1}^m f_k \text{Tr} \left(\tilde{X}(\tilde{X} - I)(\tilde{X} + I)^{-3} \tilde{H}_k(t) \right)^2 \leq 0, \quad (7)$$

$(\tilde{X}, t) \in \mathcal{W} \times \mathbb{R}$, and hence ensure that $\mathcal{V}(\tilde{X}(t))$ is non-increasing. The fact that the closed-loop system (4),(6) is only defined on $\mathcal{W} \subset U(n)$ is not a problem because the goal propagator X_{goal} is defined up to a global phase: there always exists $\theta \in [0, 2\pi)$ such that $e^{i\theta} X_{goal} \in \mathcal{W}$. Furthermore, \mathcal{W} is a positively invariant set, i.e. for every initial condition $\tilde{X}(t_0) = \tilde{X}_{t_0} \in \mathcal{W}$ at $t = t_0 \in \mathbb{R}$, the corresponding solution $\tilde{X}(t)$ evolves on \mathcal{W} and is defined for every $t \geq t_0$. Indeed, $\tilde{X}(t)$ remains in the compact set $K = \{\tilde{X} \in \mathcal{W} \mid 0 \leq \mathcal{V}(\tilde{X}) \leq \mathcal{V}(\tilde{X}_{t_0})\}$ (see Appendix A for details).

Therefore, up to a global phase change on X_{goal} , one can always assume that $X_{goal} \in \mathcal{W}$, and the closed-loop system (4),(6) is smooth on $\mathcal{W} \times \mathbb{R}$. The controls $\tilde{u}_k(t) = \tilde{u}_k(\tilde{X}(t), t)$ are defined for all $t \geq 0$, and they are also uniformly bounded on \mathbb{R}_+ since $\tilde{X}(t)$ evolves on the compact set $K = \{\tilde{X} \in \mathcal{W} \mid 0 \leq \mathcal{V}(\tilde{X}) \leq \mathcal{V}(\tilde{X}_{goal})\}$ in \mathcal{W} , the map $\mathcal{W} \ni \tilde{X} \mapsto \tilde{X}(\tilde{X} - I)(\tilde{X} + I)^{-3} \in M^n$ is smooth, and $\tilde{H}_k(t)$ is T -periodic (and thus is bounded).

In the sequel, the tracking error $\tilde{X}(t)$ is sampled with sampling period T in order to apply stochastic Lyapunov stability results that will assure that $\lim_{j \rightarrow \infty} \tilde{X}(jT) = I$. For each sampling interval $[jT, (j+1)T]$, $j \in \mathbb{N}$:

- At the time instant $t = jT$, one chooses the amplitudes $a_{k,\ell}^j$ of the reference controls $\bar{u}_k(t)$ in (2) for $t \in [jT, (j+1)T]$ in an independent stochastic manner following a uniform distribution on the interval $[-\frac{a_{max}}{2}, \frac{a_{max}}{2}]$, where $a_{max} > 0$ will be chosen afterwards. More precisely, assume the Lebesgue probability measure on $\mathbf{A} = [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]^{mM}$ (with the Borel algebra). An element $\mathbf{a}_j = (a_{k,\ell}^j) = (a_{1,1}^j, \dots, a_{1,M}^j, \dots, a_{m,1}^j, \dots, a_{m,M}^j) \in \mathbf{A}$ gathers the random amplitudes $a_{k,\ell}^j \in [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]$ of the reference controls $\bar{u}_k(t)$ in (2) for $t \in [jT, (j+1)T]$. Fix any i.i.d. (independent and identically distributed) random vectors $\mathbf{a}_j = (a_{k,\ell}^j): \mathbb{N} \rightarrow \mathbf{A} \subset \mathbb{R}^{mM}$ having a uniform distribution on \mathbf{A} . The resulting bounded continuous reference controls are

$$\bar{u}_k(t) = \sum_{\ell=1}^M a_{k,\ell}^j \sin(\ell \omega t), \quad (8)$$

for $t \in [jT, (j+1)T]$, $j \in \mathbb{N}$, $k = 1, \dots, m$.

- One defines similarly the reference trajectory $\bar{X}(t)$, the tracking error $\tilde{X}(t)$ and the feedbacks $\tilde{u}_k(t)$ by taking $\bar{u}_k(t)$ as in (8).
- One lets $\tilde{X}_j = \tilde{X}(jT) \in \mathcal{W}$ for $j \in \mathbb{N}$.

The vector field of the closed-loop system (4),(6) with (2) depends smoothly on the reference controls parameters $a_{k,\ell}$. Recall that $\tilde{H}_k(t) = \bar{X}^\dagger(t) H_k \bar{X}(t)$, where the reference trajectory $\bar{X}(t)$ is the solution of (3) with the reference controls $\bar{u}_k(t)$ in (2). Let $\Lambda: \mathcal{D} \times \mathbb{R} \times \mathcal{W} \times \mathbb{R}^{mM} \rightarrow \mathcal{W}$ be the (parameter dependent) smooth global flow of the closed-loop system (4),(6),(2), where $\mathcal{D} \subset \mathbb{R} \times \mathbb{R} \times \mathcal{W} \times \mathbb{R}^{mM}$ is open. This means that

$$\tilde{X}(t) = \Lambda(t, t_0, \hat{X}, \mathbf{a}) \in \mathcal{W},$$

for $t \in \mathcal{D}(t_0, \hat{X}, \mathbf{a}) \triangleq \{t \in \mathbb{R} \mid (t, t_0, \hat{X}, \mathbf{a}) \in \mathcal{D}\} \supset [t_0, \infty)$, is the maximal solution of system (4),(6),(2) with initial condition $\tilde{X}(t_0) = \hat{X} \in \mathcal{W}$ at $t = t_0$ and reference controls parameters $\mathbf{a} = (a_{k,\ell}) \in \mathbb{R}^{mM}$. In particular, the map $\mathcal{W} \times \mathbf{A} \ni (\tilde{X}, \mathbf{a}) \mapsto \Lambda(T, 0, \tilde{X}, \mathbf{a}) \in \mathcal{W}$ is continuous. Since system (4),(6),(2) is T -periodic in t , one has that $\Lambda(t + jT, jT, \tilde{X}, \mathbf{a}) = \Lambda(t, 0, \tilde{X}, \mathbf{a})$ for every $t \geq 0$, $j \in \mathbb{N}$, $(\tilde{X}, \mathbf{a}) \in \mathcal{W} \times \mathbf{A}$ [21, p. 143].

The reasoning above implies that

$$\tilde{X}_{j+1} = \Lambda(T, 0, \tilde{X}_j, \mathbf{a}_j), \quad \text{for } j \in \mathbb{N},$$

where $\tilde{X}_0 = X_{goal} \in \mathcal{W}$. Consequently, $\tilde{X}_j: \mathbf{A} \rightarrow \mathcal{W}$, $j \in \mathbb{N}$, is a Markov chain (with respect to the natural filtration and the Borel algebra on \mathcal{W}) because \mathbf{a}_j , $j \in \mathbb{N}$, are independent random vectors. Note that (7) assures that $\mathcal{V}(\tilde{X}_j)$, $j \in \mathbb{N}$, is a supermartingale. Define the continuous function $Q: \mathcal{W} \rightarrow \mathbb{R}$ as

$$\begin{aligned} Q(\hat{X}) &\triangleq \frac{-1}{a_{max}^{mM}} \int_{\mathbf{A}} \left(\int_0^T \dot{\mathcal{V}}(\Lambda(t, 0, \hat{X}, \mathbf{a}), t) dt \right) d\mathbf{a} \\ &= \frac{-1}{a_{max}^{mM}} \int_{\mathbf{A}} \left(\int_0^T \dot{\mathcal{V}}(\tilde{X}(t), t) dt \right) d\mathbf{a} = \frac{4}{a_{max}^{mM}} \\ &\times \int_{\mathbf{A}} \left(\int_0^T \text{Tr} \left(\tilde{X}(t)(\tilde{X}(t) - I)(\tilde{X}(t) + I)^{-3} \tilde{H}_k(t) \right)^2 dt \right) d\mathbf{a}. \end{aligned}$$

By (7), Q is non-negative, and (6) gives that $Q(I) = 0$. For each $j \in \mathbb{N}$, the conditional expectation of $\mathcal{V}(\tilde{X}_{j+1})$ knowing \tilde{X}_j is denoted by $\mathbb{E}(\mathcal{V}(\tilde{X}_{j+1})/\tilde{X}_j)$. Since \tilde{X}_j is independent of \mathbf{a}_j and $\dot{\mathcal{V}}(\tilde{X}, t)$ is T -periodic in t , one gets

$$\begin{aligned} &\mathbb{E}(\mathcal{V}(\tilde{X}_{j+1})/\tilde{X}_j) - \mathcal{V}(\tilde{X}_j) = \\ &= \frac{1}{a_{max}^{mM}} \left[\int_{\mathbf{A}} \left(\int_{jT}^{(j+1)T} \dot{\mathcal{V}}(\Lambda(t, jT, \hat{X}, \mathbf{a}), t) dt \right) d\mathbf{a} \right] \Big|_{\hat{X}=\tilde{X}_j} \\ &= -Q(\tilde{X}_j). \end{aligned}$$

Standard results on stochastic Lyapunov stability imply that $\lim_{j \rightarrow \infty} Q(\tilde{X}_j) = 0$ almost surely (see Theorem 4 in Appendix B). One will show that the only solution to $Q(\hat{X}) = 0$ is $\hat{X} = I$. This will prove almost sure convergence of $\tilde{X}_j =$

$\tilde{X}(jT)$ towards I because Q is continuous and \tilde{X}_j evolves on the compact set $K = \{\tilde{X} \in \mathcal{W} \mid 0 \leq \mathcal{V}(\tilde{X}) \leq \mathcal{V}(X_{goal})\}$ in \mathcal{W} .

Assume that $\hat{X} \in \mathcal{W}$ is such that $Q(\hat{X}) = 0$. Then,

$$\dot{\mathcal{V}}\left(\Lambda(t, 0, \hat{X}, \mathbf{a}), t\right) = 0$$

for all $t \in [0, T]$, $\mathbf{a} = (a_{k,\ell}) \in \mathbf{A}$. Hence, $\tilde{u}_k(t) = 0$ for $t \in [0, T]$, $a_{k,\ell} \in [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]$ in (2). Consequently, $\Lambda(t, 0, \hat{X}, \mathbf{a}) = \hat{X}$ for $t \in [0, T]$, $\mathbf{a} = (a_{k,\ell}) \in \mathbf{A}$. Therefore, (7) gives

$$\text{Tr}\left(\hat{X}(\hat{X} - I)(\hat{X} + I)^{-3}\bar{X}^\dagger(t)H_k\bar{X}(t)\right) = 0, \quad (9)$$

for all $t \in [0, T]$, $a_{k,\ell} \in [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]$, $k = 1, \dots, m$, where $\bar{X}(t)$ is the solution of system (3),(2). It is reasonable to expect that the controllability of system (1) ensures that by taking M and $a_{max} > 0$ sufficiently large, there exist $a_{k,\ell} \in [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]$ in (2) such that (9) implies

$$\text{Tr}\left(\hat{X}(\hat{X} - I)(\hat{X} + I)^{-3}Z\right) = 0, \quad \text{for } Z \in \mathfrak{u}(n). \quad (10)$$

Since $\hat{X} \in \mathcal{W} \subset U(n)$ is diagonalizable, so is

$$\hat{Y} = \hat{X}(\hat{X} - I)(\hat{X} + I)^{-3}.$$

Hence (10) provides that $\hat{Y} = 0$, and thus $\hat{X} = I$. Indeed, recall that $\text{Tr}(AB) = \text{Tr}(BA)$ and write $\hat{Y} = M^\dagger DM$, where $M \in U(n)$ and D is a diagonal matrix. Therefore, in order to show that $\hat{Y} = 0$, it suffices to take the Z 's in $\mathfrak{u}(n)$ such that $MZM^\dagger = \text{diag}(-\iota, 0, \dots, 0), \text{diag}(0, -\iota, 0, \dots, 0), \dots, \text{diag}(0, 0, \dots, 0, -\iota)$. The preceding argument will then prove that $Q(\hat{X}) = 0$ is achieved only for $\hat{X} = I$.

The main result of this paper is now presented:

Theorem 1: Consider a controllable quantum system of the form (1). Take $T > 0$ and specify any (constant) feedback gains $f_k > 0$. Choose any $X_{goal} \in U(n)$ such that³ all its eigenvalues are different than -1 . Then, there exist $\bar{M} > 0$ and $\bar{a}_{max} > 0$ such that, for all $M \geq \bar{M}$ and $a_{max} > \bar{a}_{max}$, the bounded continuous controls, for $t \geq 0$,

$$\begin{aligned} u_k(t) &= \bar{u}_k(t) + \tilde{u}_k(t) = \bar{u}_k(t) + \\ f_k \text{Tr}\left(X(t)(\bar{X}^\dagger(t)X(t) - I)(\bar{X}^\dagger(t)X(t) + I)^{-3}\bar{X}^\dagger(t)H_k\right), \end{aligned} \quad (11)$$

ensure that

$$\lim_{j \rightarrow \infty} X(jT) = X_{goal} \quad \text{almost surely}, \quad (12)$$

where $\bar{X}(t)$ is the corresponding trajectory of (3), and $\bar{u}_k(t)$ in (8) is specified by choosing any i.i.d. random vectors $\mathbf{a}_j = (a_{k,\ell}^j): \mathbb{N} \rightarrow \mathbf{A} \subset \mathbb{R}^{mM}$, $j \in \mathbb{N}$, having a uniform distribution on $\mathbf{A} = [-\frac{a_{max}}{2}, \frac{a_{max}}{2}]^{mM}$.

Proof: The proof relies on the Return Method of Coron ([4],[6]) and on standard Fourier series results. The details are essentially contained in [14]. ■

Remark 2: Simulation results have suggested that the speed of convergence of the state $X(jT)$ towards X_{goal} is

³This assumption always holds after a global phase change on X_{goal} .

considerably improved when one chooses the feedback gains as $f_k \cong Ma_{max}$.

Remark 3: The authors have also established that one can always take $\bar{a}_{max} = 0$ in Theorem 1. This means that as long as $M > 0$ is sufficiently large, one can then choose any $a_{max} > 0$. The proof of such stronger result is given in [14].

III. CONCLUSIONS

This paper has completely solved the problem of generating any desired goal propagator for a controllable driftless quantum system on $U(n)$. No additional assumptions are required and explicit stochastic control laws have been constructed. The proof of Remark 3 is given in [14]. In that work, the authors also establish that the stochastic techniques here developed can also be applied to systems of the form (1) that evolve on the special unitary group $SU(n)$. Furthermore, the control strategy proposed in the present paper can be given a deterministic version for controllable systems that evolve on $SU(n)$ and on $U(n)$. This is the subject of [15]. In particular, the controls laws used in the generation of the C-NOT gate for the quantum system in [18, Section III] can be given rigorous deterministic as well as stochastic versions. A comparison between both is exhibited in the simulation results of [14].

APPENDIX

A. Proof of the positive invariance property of \mathcal{W}

This will be proved by contradiction using (7), continuity arguments and the following results: (i) every $\tilde{X} \in U(n)$ can be decomposed as $\tilde{X} = M^\dagger DM$ with $M \in U(n)$, $D = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, $|\lambda_j| = 1$ for $j = 1, \dots, n$; (ii) for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, one has that⁴ $-(\lambda - 1)^2/(\lambda + 1)^2 \geq 0$ and $\lim_{\lambda \rightarrow -1} -(\lambda - 1)^2/(\lambda + 1)^2 = +\infty$. Hence, $\mathcal{V}(\tilde{X}) = \sum_{j=1}^n -(\lambda_j - 1)^2/(\lambda_j + 1)^2$ by (i).

Now, if $\tilde{X}(t_0) = \tilde{X}_{t_0} \in \mathcal{W}$ and \mathcal{W} is not a positively invariant set, then there exists $\bar{t} \geq t_0$ such that $\tilde{X}(\bar{t}) \notin \mathcal{W}$. Take $\bar{t} = \sup\{t_0 \leq t < \bar{t} \mid \tilde{X}(t) \in \mathcal{W}\}$. Then $t_0 < \bar{t} \leq \bar{t}$, $\tilde{X}(t) \in \mathcal{W}$ for $t_0 \leq t < \bar{t}$, $\tilde{X}(\bar{t}) \notin \mathcal{W}$ and $\lim_{t \rightarrow \bar{t}} \tilde{X}(t) = \tilde{X}(\bar{t})$. Hence, using well-known eigenvalue continuity results, one has that $\lim_{t \rightarrow \bar{t}} \mathcal{V}(\tilde{X}(t)) = +\infty$, which contradicts $\mathcal{V}(\tilde{X}(t)) \leq \mathcal{V}(\tilde{X}_{t_0})$ for $t_0 \leq t < \bar{t}$.

B. Stochastic Lyapunov stability result

Theorem 4: [10, Theorem 1, Chapter 8] Let Ω be a probability space and let \mathcal{W} be a measurable space. Consider that $X_j: \Omega \rightarrow \mathcal{W}$, $j \in \mathbb{N}$, is a Markov chain with respect to the natural filtration. Let $Q: \mathcal{W} \rightarrow \mathbb{R}$ and $\mathcal{V}: \mathcal{W} \rightarrow \mathbb{R}$ be measurable non-negative functions with $\mathcal{V}(X_j)$ integrable for all $j \in \mathbb{N}$. If

$$\mathbb{E}(\mathcal{V}(X_{j+1})/X_j) - \mathcal{V}(X_j) = -Q(X_j), \quad \text{for } j \in \mathbb{N},$$

then $\lim_{j \rightarrow \infty} Q(X_j) = 0$ almost surely.

Proof: Taking the expectation value on both sides of the equality above and reversing the signs, one gets

$$\mathbb{E}(\mathcal{V}(X_j)) - \mathbb{E}(\mathcal{V}(X_{j+1})) = \mathbb{E}(Q(X_j)) \geq 0, \quad j \in \mathbb{N}.$$

⁴In fact, $\iota(\lambda - 1)/(\lambda + 1)$ is real and use $2 \leq |1 - \lambda| + |\lambda + 1|$.

Iterating, one obtains, for $\ell \geq 1$,

$$\mathbb{E}(\mathcal{V}(X_0)) \geq \mathbb{E}(\mathcal{V}(X_0)) - \mathbb{E}(\mathcal{V}(X_\ell)) = \sum_{j=0}^{\ell-1} \mathbb{E}(Q(X_j)).$$

Therefore, $\sum_{j=0}^{\infty} \mathbb{E}(Q(X_j)) \leq \mathbb{E}(\mathcal{V}(X_0)) < \infty$, and the result then follows from [10, Lemma 1, Chapter 8] (Borel-Cantelli Lemma). ■

REFERENCES

- [1] F. Albertini and D. D'Alessandro. Notions of controllability for bilinear multilevel quantum systems. *IEEE Trans. Automat. Control*, 48(8):1399–1403, 2003.
- [2] R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. J. Sussmann, editors, *Differential Geometric Control Theory*, volume 27 of *Progr. Math.*, pages 181–191. Birkhäuser, Basel, 1983.
- [3] J.-M. Coron. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems*, 5(3):295–312, 1992.
- [4] J.-M. Coron. Linearized control systems and applications to smooth stabilization. *SIAM J. Control Optim.*, 32(2):358–386, 1994.
- [5] J.-M. Coron. On the stabilization of some nonlinear control systems: Results, tools, and applications. In F. H. Clarke and R. J. Stern, editors, *NATO Advanced Study Institute, Nonlinear analysis, differential equations, and control*, pages 307–367. Kluwer Academic Publishers, Holland, 1999.
- [6] J.-M. Coron. *Control and Nonlinearity*. American Mathematical Society, 2007.
- [7] D. D'Alessandro. *Introduction to Quantum Control and Dynamics*. Chapman & Hall/CRC, Boca Raton, 2008.
- [8] V. Jurdjevic and H. J. Sussmann. Control systems on Lie groups. *J. Diff. Equations*, 12:313–329, 1972.
- [9] S. Kuang and Cong. S. Lyapunov control methods of closed quantum systems. *Automatica*, 44(1):98–108, 2008.
- [10] H. Kushner. *Introduction to Stochastic Control*. Holt, Rinehart and Winston, Inc., New York, 1971.
- [11] N. E. Leonard and P. S. Krishnaprasad. Motion control of drift-free, left-invariant systems on Lie groups. *IEEE Trans. Automat. Control*, 40(9):1539–1554, 1995.
- [12] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, 2000.
- [13] P. S. Pereira da Silva and P. Rouchon. Flatness-based control of a single qubit gate. *IEEE Trans. Automat. Control*, 53(3):775–779, 2008.
- [14] P. S. Pereira da Silva, H. B. Silveira, and P. Rouchon. Approximate motion planning of driftless systems evolving on $U(n)$ and on $SU(n)$ based on stochastic Lyapunov feedback. 2013. Submitted to *Automatica*.
- [15] P. S. Pereira da Silva, H. B. Silveira, and P. Rouchon. Approximate motion planning of driftless systems on $SU(n)$ and on $U(n)$ based on a time-periodic Lyapunov technique. 2013. Submitted to *Int. J. Control*.
- [16] J.-B. Pomet. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems Control Lett.*, 18(2):147–158, 1992.
- [17] D. R. Sahoo and M. V. Salapaka. Constructive control of quantum-mechanical systems. In *Proc. of the 40th IEEE Conf. on Decision and Control – CDC*, pages 1601–1606, Orlando, United States, 2001.
- [18] H. B. Silveira, P. S. Pereira da Silva, and P. Rouchon. A time-periodic Lyapunov approach for motion planning of controllable driftless systems on $SU(n)$. In *Proc. of the 48th IEEE Conf. on Decision and Control – CDC*, pages 363–368, Shanghai, China, 2009.
- [19] H. B. Silveira, P. S. Pereira da Silva, and P. Rouchon. Explicit controls laws for the periodic motion planning of controllable driftless systems on $SU(n)$. In *Proc. of the 51st IEEE Conf. on Decision and Control – CDC*, pages 3640–3645, Maui, United States, 2012.
- [20] K. Spindler. Motion planning via optimal control theory. In *Proc. American Control Conference*, pages 1972–1977, Anchorage, United States, 2002.
- [21] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, New Jersey, 2nd edition, 1993.
- [22] X. Wang and S. G. Schirmer. Analysis of Lyapunov method for control of quantum states. *IEEE Trans. Automat. Control*, 55(10):2259–2270, 2010.