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Stabilization of uncertain systems with finite data rates and Markovian packet losses

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Abstract— **This research addresses stabilization of uncertain systems over data rate constrained and lossy channels. While many of the existing works assume that the packet loss process is independent and identically distributed, we model it as a two-state Markov chain, which can deal with more practical situations including bursty dropouts. For parametrically uncertain plants, a necessary condition and a sufficient condition for mean square stability are derived. These conditions are represented by the product of the eigenvalues of the nominal plants, the data rate, the transition probabilities of the channel states, and the upper bounds of uncertainties. In particular, for scalar plants, the conditions coincide with each other.**

I. Introduction

Due to the development in communication and information technologies, networked control systems have attracted significant research interests [1], [2]. In such systems, the existence of communication channels causes several undesirable phenomena, e.g., quantization errors, losses and delays of packets containing samples, and time-varying sampling periods. It is known that these properties could be factors of degradation in control performance.

In this paper, we consider the stabilization problem of a linear discrete-time system over a channel and study how much information should be available through the channel to achieve stability. In the channel, the number of bits communicated at each time (data rate) is finite and transmitted packets may be randomly lost.

In the pioneering work by Wong and Brockett [3], it has been shown that there exists a critical value on the data rate for stability and this limitation is described by the product of the unstable poles of the plant. Stimulated by this result, data rate limitations have been developed in various setups including general discrete-time linear systems [4], stochastic systems [5], and nonlinear systems [6].

The effect of packet losses has also been studied in both stabilization problems [7]–[9] and state estimation problems [10]–[12] (see also [13], [14] and the references therein). In these works, the packet reception/loss status is modeled as a random process, and bounds on the loss probabilities required for stability or convergence of estimation error have been presented. It is remarkable that the bounds are also characterized by the unstable poles of the plant. Moreover, recent works have studied the case that the data rate constraint and the packet losses simultaneously exist [15], [16].

The objective of this research is to generalize the existing results to be applicable to more realistic situations especially in the following two aspects: taking account of plant uncertainties and employing more sophisticated models for representing the packet losses. It should be noted that in the literature, it is commonly assumed that the exact plant model is known and only a few works are available for uncertain cases. In [17], [18], linear time-invariant systems with norm bounded uncertainties have been considered, while in [19] scalar nonlinear systems with stochastic uncertainties have been dealt with. These papers study stabilization problems via data-rate-limited and lossless channels and provide sufficient conditions, but do not specify the necessity and the minimum data rate. On the other hand, in our previous works [20], [21], we have shown a necessary condition and a sufficient condition under the presence of data rate limitations and packet losses. To overcome difficulties due to the uncertainty, we have considered parametric uncertainties and have introduced a certain structure in the controller side. These features have enabled us to follow an analytical approach. In these works, however, the packet loss process is assumed to be independent and identically distributed (i.i.d.); this assumption is commonly employed to simplify the problem but is somewhat restrictive for modeling practical channels.

Hence, in this paper, we consider packet losses governed by Markov chains, which are more general channel models. Markov chains have been proposed to express practical communication failures including bursty losses [22], [23], and have been employed in several researches in the field of networked control: In [10], the state estimation problem is studied, and the stabilization problem is tackled both in infinite [24] and finite [25] data rate cases. In general, Markovian losses cause difficulties in the stability analysis since the channel states are no longer independent over time. To deal with this, we follow the approach of [25] and consider time intervals between successful transmissions, which become an i.i.d. process.

The contribution of this paper is as follows: Under the presence of the data rate constraint and the Markovian packet losses, we derive a necessary condition and a sufficient condition for mean square stability. These conditions provide the limitations on the data rate and the state transition probabilities of the channel. In particular, the necessary limitations are expressed by the product of the poles and the uncertainty bounds of the plant. Moreover, for the case

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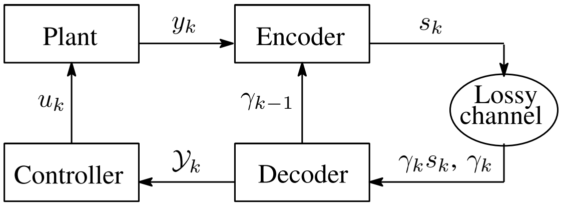

Fig. 1. Networked control system

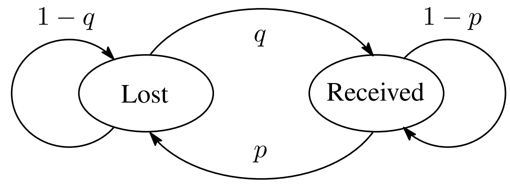

Fig. 2. State transition probabilities of a Markov channel

of scalar plants, the derived conditions are exact. Our results generalize those in [25] to the uncertain plants case and also those in [21] to a more practical channel case.

This paper is organized as follows. In the next section, we describe the system setup and formulate the problem considered here. We begin the analysis for scalar plants in Section III. The presented result will be extended to the general order plants case in Section IV. Finally in Section V, we provide the conclusion. Throughout this paper, we denote $\log_2(\cdot)$ simply as $\log(\cdot)$.

II. Problem formulation

We consider stabilization of a networked control system which has a communication channel at the side of the plant output (Fig. 1). At time $k \in \mathbb{Z}_+$, the encoder observes the plant output $y_k \in \mathbb{R}$ and quantizes it. The quantized signal $s_k \in \Sigma_N$ is transmitted to the decoder through the channel, where s_k may be randomly lost in the communication. Here, the set Σ_N represents all possible outputs of the encoder, and contains N symbols. The data rate is defined as $R := \log N$ [bits/sample]. Based on the transmitted signal, the decoder computes an interval $\mathcal{Y}_k \subset \mathbb{R}$, which is an estimate of y_k, and informs the encoder by the acknowledgement (ACK) signal that the packet has been received or lost. Finally, using the estimate, the controller provides a control input $u_k \in \mathbb{R}$.

In the following, we describe the details of each component in the system.

The plant is an n-dimensional autoregressive system[1] whose parameters are uncertain and may be time varying:

$$y_{k+1} = a_{1,k} y_k + a_{2,k} y_{k-1} + \cdots + a_{n,k} y_{k-n+1} + u_k. \quad (1)$$

Here, at the initial step, $y_0, y_{-1}, \ldots, y_{-n+1}$ are in known bounds $Y_0, Y_{-1}, \ldots, Y_{-n+1}$. Each uncertain parameter $a_{i,k}$ is represented by the nominal value a_i^* and the width $\epsilon_i \geq 0$ of the perturbation as

$$a_{i,k} \in \mathcal{A}_i := [a_i^* - \epsilon_i, a_i^* + \epsilon_i] \text{ for } i = 1, 2, \ldots, n. \quad (2)$$

The plant (1) can be expressed in the controllable canonical form:

$$x_{k+1} = A_k x_k + B u_k, \ y_k = C x_k, \quad (3)$$

[1]The results in this paper can easily be extended to the case where the plant is an ARX model as $y_{k+1} = \sum_{i=1}^{n_y} a_{i,k} y_{k-i+1} + \sum_{i=1}^{n_u} b_i u_{k-i+1}$, where $b_1, \ldots, b_{n_u}$ are known parameters and $b_1 \neq 0$.

where $x_k := [y_{k-n+1} \ y_{k-n+2} \ \cdots \ y_k]^T$ and

$$A_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_{n,k} & a_{n-1,k} & \cdots & a_{1,k} \end{bmatrix} \in \mathbb{R}^{n \times n},$$

$$B = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T \in \mathbb{R}^n, \ C = B^T \in \mathbb{R}^{1 \times n}.$$

Let A^* represent the nominal matrix, and let λ_{A^*} be the product of the eigenvalues of A^*:

$$A^* := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_n^* & a_{n-1}^* & \cdots & a_1^* \end{bmatrix}, \ \lambda_{A^*} := \prod_{i=1}^{n} \lambda_i(A^*) = a_n^*,$$

where $\lambda_i(\cdot)$ represents an eigenvalue of a matrix. Assume that for all $k \in \mathbb{Z}_+$ at least one eigenvalue of the matrix A_k is unstable and the product of the eigenvalues is grater than 1, i.e., $|a_n^*| - \epsilon_n > 1$.

The encoder quantizes the plant output y_k into the N-alphabet signal $s_k \in \Sigma_N$, where $\Sigma_N := \{1, 2, \ldots, N\}$. The input range of the encoder is centered at the origin and the width is defined by a scaling parameter $\sigma_k > 0$. In particular, the output s_k of the encoder is given as $s_k = \phi_N(y_k/\sigma_k)$, where $\phi_N(\cdot)$ is a quantizer which is a piece wise constant function and divides its input range $[-1/2, 1/2]$ into N cells. In the quantizer $\phi_N(\cdot)$, it is assumed that boundaries of the quantization cells are symmetric about the origin.

By its symmetry, the quantizer ϕ_N can be expressed by the set of boundary points $h_l \in \mathbb{R}$, $l = 0, 1, \ldots, \lceil N/2 \rceil$, of nonnegative quantization cells. These points must satisfy $h_0 = 0$, $h_{\lceil N/2 \rceil} = 1/2$, and $h_l < h_{l+1}$. The origin h_0 is a boundary only when the number N of quantization cells is even. However, for simplicity, we use the same notation above even if N is odd.

The transmitted signal s_k is randomly lost in the channel due to unreliability. We represent the state of the packet reception/loss at time k by the random variable $\gamma_k \in \{0, 1\}$: If $\gamma_k = 0$ then the packet is lost; otherwise, it arrives successfully. The channel state process $\{\gamma_k\}_k$ is modeled as a Markov chain which has two states: received and lost. We note that the Markov chain model contains i.i.d. processes studied in [21] as a special case and can deal with more practical behavior in communication. Fig. 2 shows the states and the transition probabilities. In the figure, p is the loss probability when the previous packet has arrived, and q

represents the recovery probability from the loss state:

$$\begin{aligned}
&\mathrm{Prob}(\gamma_k = 0|\gamma_{k-1} = 0) = 1 - q,\\
&\mathrm{Prob}(\gamma_k = 1|\gamma_{k-1} = 0) = q,\\
&\mathrm{Prob}(\gamma_k = 0|\gamma_{k-1} = 1) = p,\\
&\mathrm{Prob}(\gamma_k = 1|\gamma_{k-1} = 1) = 1 - p.
\end{aligned}$$

To make the process $\{\gamma_k\}_k$ ergodic, we consider the case of $p, q \in (0, 1)$. Moreover, without loss of generality, we assume that the transmitted signal at the initial time is successfully received, i.e., $\gamma_0 = 1$. In the case of loss, redefine the initial time as the the first step of succeeded communication. There exists such a finite time with probability 1 since $\{\gamma_k\}_k$ is ergodic.

The decoder converts the received signal $\gamma_k s_k$ to the interval $\mathcal{Y}_k \subset \mathbb{R}$. The interval $\mathcal{Y}_k$ provides an estimate of the set in which the plant output y_k should be included. If the packet arrives ($\gamma_k = 1$), then $\mathcal{Y}_k$ corresponds to the quantization cell that y_k fell in. Otherwise $\mathcal{Y}_k$ is equal to the entire input range $[-\sigma_k/2, \sigma_k/2]$ of the encoder.

The controller provides the control input u_k based on the estimates $\mathcal{Y}_{k-n+1}, \dots, \mathcal{Y}_k$ as

$$u_k = \sum_{i=1}^{n} f_{i,k}\left(\mathcal{Y}_{k-i+1}\right), \tag{4}$$

where $f_{i,k}(\cdot)$ is an arbitrary map from an interval on $\mathbb{R}$ to a real number.

We remark that the scaling parameter σ_k should be large enough to cover all possible inputs to the encoder. Otherwise, the quantizer may be saturated, in which case we lose track of the plant output y_k. On the other hand, if we take σ_k large, the quantization error also becomes large. Moreover, to achieve stabilization of the system, σ_k should decay to zero gradually.

We determine the scaling parameter σ_k as follows. At time k, the encoder and the decoder predict the next plant output y_{k+1} based on the estimates $\mathcal{Y}_0, \dots, \mathcal{Y}_k$. Let $\mathcal{Y}_{k+1}^- \subset \mathbb{R}$ be the set of all possible outputs y_{k+1} of the uncertain system (1). Then the scaling parameter σ_{k+1} is chosen such that

$$\sigma_{k+1} \geq \mu(\mathcal{Y}_{k+1}^-),$$

where $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$.

As the prediction set $\mathcal{Y}_{k+1}^-$, we employ the following one:

$$\begin{aligned}
\mathcal{Y}_{k+1}^- := \{&a_1' y_k' + \cdots + a_n' y_{k-n+1}' : a_1' \in \mathcal{A}_1, \dots, a_n' \in \mathcal{A}_n,\\
&y_k' \in \mathcal{Y}_k, \dots, y_{k-n+1}' \in \mathcal{Y}_{k-n+1}\}.
\end{aligned}$$

Under this definition, our prediction strategy is to use the information regarding $y_k, \dots, y_{k-n+1}$ independently such that $y_{k-i+1} \in \mathcal{Y}_{k-i+1}$ for each $i = 1, 2, \dots, n$, where $\mathcal{Y}_{k-i+1}$ is the interval received on the decoder side at time $k-i+1$. Then, clearly, $\mu(\mathcal{Y}_{k+1}^-)$ is large enough to include y_{k+1}, and it is computable on both sides of the channel. It should be noted that the encoder knows the previous loss state γ_{k-n+1} through the ACK signal from the decoder.

The control objective is to stabilize the system depicted in Fig. 1 in a stochastic sense as described below.

Definition 1: The system depicted in Fig. 1 is mean square stable (MSS) if the plant output y_k asymptotically goes to zero in the mean square sense for all possible uncertainties within the bounds in (2). That is, for all $a_{1,k} \in \mathcal{A}_1, a_{2,k} \in \mathcal{A}_2, \dots, a_{n,k} \in \mathcal{A}_n$, it holds that $\lim_{k\to\infty} \mathrm{E}[|y_k|^2] = 0$.

The problem of the paper is to find limitations on the data rate R and the channel state transition probabilities p, q for the overall system to be MSS.

III. Limitations for Scalar Plants

In this section, we consider the simple case where the plant is a scalar system as follows:

$$y_{k+1} = a_k y_k,\ a_k \in \mathcal{A} = [a^* - \epsilon, a^* + \epsilon],\ \epsilon \geq 0. \tag{5}$$

We derive a necessary and sufficient condition for the system to be MSS. The condition is characterized by the data rate R, the transition probabilities p, q, the uncertain bound ϵ, and the plant instability $|a^*|$.

To describe the limitations on R and q, we introduce the following notations:

$$\nu := \sqrt{1 + \frac{p((|a^*| + \epsilon)^2 - 1)}{1 - (1-q)(|a^*| + \epsilon)^2}},\ r := \frac{|a^*| - \epsilon}{|a^*| + \epsilon}.$$

The following theorem holds for the scalar plants.

Theorem 1: If the system depicted in Fig. 1 with the scalar plant (5) is MSS, then the following inequalities hold:

$$R > R_{\mathrm{nec}} := \begin{cases} \log \frac{\log(1-\epsilon\nu)^2}{\log r} & \text{if } \epsilon > 0,\\ \log\left(|a^*|\nu\right) & \text{if } \epsilon = 0, \end{cases} \tag{6}$$

$$q > q_{\mathrm{nec}} := 1 - \frac{1}{(|a^*| + \epsilon)^2}\left(1 - \frac{\epsilon^2 p\left((|a^*| + \epsilon)^2 - 1\right)}{1 - \epsilon^2}\right), \tag{7}$$

$$0 \leq \epsilon < 1. \tag{8}$$

Conversely, if these inequalities are satisfied for the data rate $R = \log N$ where N is an even number and the transition probabilities $p, q \in (0, 1)$, then there exists a control law such that the system is MSS.

For the case of $\epsilon = 0$, the above limitations R_{nec} and q_{nec} are equal to those presented in [25], where the plant is assumed to be known. In addition, if $R \to \infty$ then the condition (7) on the recovery probability coincides with that in [12]. Thus, this theorem generalizes these existing results to the uncertain plants case. Since the bounds R_{nec} and q_{nec} are increasing with respect to ϵ, as expected, plant uncertainty will result in higher requirements in communication with a large data rate and a high recovery probability. Furthermore, when the channel state process $\{\gamma_k\}_k$ is i.i.d., we have that $q = 1 - p$. In such a case, our problem can be reduced to that of [21], and the conditions (6)–(8) coincide with those given there.

To show this theorem, the approach developed in the analysis for the i.i.d. case [21] is not enough to deal with the Markov channel. It is because in the present case, the channel states are not independent of those at the previous or next step, and consequently, it is difficult to evaluate the mean square of the state estimation error. To overcome this difficulty, we consider the packet receptions as random measurements. Then, it has been shown that the process of the sampling intervals becomes i.i.d. [26]. We formally state this fact as a lemma below. Let t_j, $j \in \mathbb{Z}_+$, be the sampling times, i.e., the times satisfying $\gamma_{t_j} = 1$. From the assumption in the previous section, we have that $\gamma_0 = 1$. Thus, it follows that $0 = t_0 < t_1 < \cdots < t_j < \cdots$. In addition, let τ_j denote the sampling interval defined as follows:

$$\tau_j := t_j - t_{j-1},\ j \geq 1.$$

The following lemma holds [26].

Lemma 1: The process $\{\tau_j\}_j$ is i.i.d. and it holds that

$$\mathrm{Prob}(\tau_j = i) = \begin{cases} 1-p & \text{if } i = 1, \\ pq(1-q)^{i-2} & \text{if } i > 1, \end{cases}$$

for all $j \geq 1$.

The proof of Theorem 1 consists of three steps. We first provide a condition for stability under a given quantizer whose boundaries are $\{h_l\}_l$. To derive the condition, we will analyze how the plant state estimation error grows by the instability of the plant and how precise the quantization should be to achieve stability. We define the sequence of w_l, $l = 0, 1, \ldots, \lceil N/2 \rceil - 1$, as

$$w_l := \begin{cases} 2(|a^*| + \epsilon)h_{l+1} & \text{if } N \text{ is odd and } l = 0, \\ (|a^*| + \epsilon)h_{l+1} - (|a^*| - \epsilon)h_l & \text{else.} \end{cases}$$

This represents the rate of expansion of a quantization cell over time. Then, the following lemma holds.

Lemma 2: The system depicted in Fig. 1 is MSS if and only if

$$\max_{0 \leq l \leq \lceil N/2 \rceil - 1} w_l\, \nu < 1. \tag{9}$$

Lemma 2 provides a condition for stability for a given quantizer. The next step is to find the optimal quantizer, which minimizes the left-hand side of the inequality (9) under a fixed number N of quantization cells. Since ν does not depend on the structure of the quantizer, the problem is to find the quantizer minimizing $\max_l w_l$. We introduce the quantizer ϕ_N^* whose boundaries $\{h_l^*\}_l$ are given as follows:

(i) If $\epsilon > 0$, then

$$h_l^* = \begin{cases} \frac{1}{2}\frac{1-tr^l}{1-tr^{\lceil N/2 \rceil}} & \text{if } N \text{ is odd,} \\ \frac{1}{2}\frac{1-r^l}{1-r^{\lceil N/2 \rceil}} & \text{if } N \text{ is even,} \end{cases}$$

where $t := |a^*|/(|a^*| - \epsilon)$.

(ii) If $\epsilon = 0$, then

$$h_l^* = \begin{cases} \frac{1}{N}\left(l - \frac{1}{2}\right) & \text{if } N \text{ is odd,} \\ \frac{1}{N}l & \text{if } N \text{ is even.} \end{cases}$$

The following lemma shown in [21] provides the optimal quantizer.

Lemma 3: The quantizer q_N^* minimizes $\max_l w_l$.

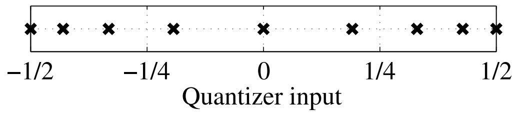

Fig. 3. Boundaries of the quantizer ϕ_N^* when $a^* = 3.0$, $\epsilon = 0.5$, and $N = 8$

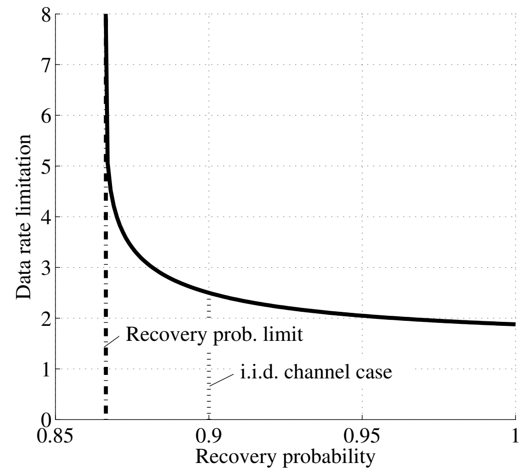

Fig. 4. Data rate limitation versus the recovery probability q

The optimal quantizer ϕ_N^* has a nonuniform structure when the plant is uncertain, i.e., $\epsilon > 0$ (see Fig. 3). In particular, it takes the width of the quantization cells smaller as its input becomes larger in magnitude. This structure helps to compensate the effect of the uncertainty in the expansion of estimation errors by plant instability; we note that in taking the product of intervals, the width of the resulting interval becomes large not only when the initial intervals are wide but also when they contain large values in magnitude. When there is no uncertainty in the plant, ϕ_N^* is the uniform quantizer, which is commonly employed in the literature.

It is remarkable that the structure of ϕ_N^* depends only on the level of instability $|a^*|$ and the uncertainty bound ϵ; the channel properties p and q do not affect it. Therefore, ϕ_N^* is equal to the optimal quantizer for the case of i.i.d. channels, which has been studied in [21].

The last step is deriving (6)–(8) from Lemmas 2 and 3: Assume that the quantizer is the optimal one. Then the lower bound on N satisfying (9) is the necessary data rate.

Example 1: We now illustrate the limitations on the data rate R_{nec} and the recovery probability q_{nec} by a numerical example. Consider a scalar plant of $a^* = 5$ and $\epsilon = 0.2$, and a channel where the loss probability is set as $p = 0.1$. In Fig. 4, the solid line shows the limitation R_{nec} versus the recovery probability q, and the vertical dash-dot line represents q_{nec}. The figure shows that larger data rate is required as the recovery probability becomes small toward q_{nec}, and if $q \leq q_{\text{nec}}$ then we can not stabilize the system for any data rate. The vertical dotted line corresponds to the probability when the channel state process is i.i.d., i.e.,

$q = 1 - p = 0.9$. From the figure, we observe that even if the recovery probability is less than 0.9, in which case more bursty packet losses may occur, we can make the system stable by selecting the data rate large enough.

IV. Limitations for multi-dimensional plants

In this section, we develop a necessary condition and a sufficient condition for stabilization of multi-dimensional plants in (1). The necessary condition is derived by generalizing the result shown in the previous section.

A. Necessary condition

To describe the necessary condition, we introduce the following notations corresponding to ν and r, respectively, in the previous section:

$$\nu_n := \sqrt{1 + \frac{p_n((|\lambda_{A^*}| + \epsilon_n)^2 - 1)}{1 - (1 - q_n)(|\lambda_{A^*}| + \epsilon_n)^2}},\ r_n := \frac{|\lambda_{A^*}| - \epsilon_n}{|\lambda_{A^*}| + \epsilon_n},$$

where p_n, q_n are defined as

$$p_n := \frac{1 - (1 - p - q)^n}{p + q} p,\ q_n := \frac{1 - (1 - p - q)^n}{p + q} q.$$

The following theorem provides the necessity result.

Theorem 2: If the system depicted in Fig. 1 is MSS, then the following inequalities hold:

$$R > R_{\mathrm{nec},n} := \begin{cases} \log \frac{\log(1-\epsilon_n\nu_n)^2}{\log r_n} & \text{if } \epsilon_n > 0, \\ \log(|\lambda_{A^*}|\nu_n) & \text{if } \epsilon_n = 0, \end{cases} \tag{10}$$

$$q_n > q_{\mathrm{nec},n} := 1 - \frac{1}{(|\lambda_{A^*}| + \epsilon_n)^2} \times \left(1 - \frac{\epsilon_n^2 p_n\left((|\lambda_{A^*}| + \epsilon_n)^2 - 1\right)}{1 - \epsilon_n^2}\right), \tag{11}$$

$$0 \le \epsilon_n < 1. \tag{12}$$

This theorem provides the limitations characterized by the product λ_{A^*} of the eigenvalues of the nominal plant. It can be viewed as an extension of the results in [25], where the known plants case has been studied. Let us compare the limitations $R_{\mathrm{nec},n}$ and $q_{\mathrm{nec},n}$ with those in [25] in the case of $\epsilon_n = 0$. For scalar plants case, i.e., $n = 1$, the inequalities (10)–(12) are equivalent to (6)–(8) in Theorem 1. Thus, $R_{\mathrm{nec},n}$ and $q_{\mathrm{nec},n}$ coincide with those in [25] as we mentioned in the previous section. However, when $n \ge 2$ and the channel state process is not i.i.d., i.e., $p + q \ne 1$, even if we assume that $\epsilon_n = 0$, the limitations $R_{\mathrm{nec},n}, q_{\mathrm{nec},n}$ may become larger than the bounds in [25]. In such a sense, Theorem 2 contains conservativeness. We would like to address this point in future research. On the other hand, when the channel state process is constrained to be i.i.d., the theorem is equivalent to the result in [21].

B. Sufficient condition

We next present a sufficient condition for the existence of a stabilizing tuple of an encoder, a decoder, and a controller. First, the update law of the scaling parameter and the control law are proposed. Furthermore, the stability analysis is carried out based on the theory of Markov jump linear systems.

Given a certain data rate R, or N, and a quantizer expressed by the boundaries $\{h_l\}_l$, we determine the scaling parameter as follows:

$$\sigma_k = \mu(\mathcal{Y}_k^-), \tag{13}$$

and in the controller (4), the input to the plant is given as

$$u_k = -\frac{1}{2}\Big(\sup_{y' \in \mathcal{Y}_{k+1}^-} y' + \inf_{y' \in \mathcal{Y}_{k+1}^-} y'\Big). \tag{14}$$

Next, we introduce some notations required for the analysis of the resulting system. For $i = 1, 2, \ldots, n$, let the random variables $\theta_{i,k}$ be given by

$$\theta_{i,k} := \begin{cases} |a_i^*| + \epsilon_i & \text{if } \gamma_{k-i+1} = 0, \\ \bar{w}_i & \text{if } \gamma_{k-i+1} = 1, \end{cases}$$

where $\bar{w}_i$ is defined as

$$\bar{w}_i := \begin{cases} \max\left\{\bar{w}_i^{(0)}, \bar{w}_i^{(1)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \not\ni 0, \\ \max\left\{\epsilon_i, \bar{w}_i^{(0)}\right\} & \text{if } N \text{ is odd and } \mathcal{A}_i \ni 0, \\ \bar{w}_i^{(1)} & \text{if } N \text{ is even and } \mathcal{A}_i \not\ni 0, \\ \epsilon_i & \text{if } N \text{ is even and } \mathcal{A}_i \ni 0, \end{cases}$$

$$\bar{w}_i^{(0)} := 2(|a_i^*| + \epsilon_i)h_1,$$
$$\bar{w}_i^{(1)} := \max_{l \in \{0, \ldots, \lceil N/2 \rceil - 1\}} \{(|a_i^*| + \epsilon_i)h_{l+1} - (|a_i^*| - \epsilon_i)h_l\}.$$

These are useful to bound the interval $\mathcal{A}_i\mathcal{Y}_{k-i+1}$ as $\mu(\mathcal{A}_i\mathcal{Y}_{k-i+1}) \le \theta_{i,k}\sigma_{k-i+1}$. Moreover, define the random variable matrix H_{Γ_k} containing $\theta_{1,k}, \ldots, \theta_{n,k}$ by

$$H_{\Gamma_k} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \theta_{n,k} & \theta_{n-1,k} & \cdots & \theta_{1,k} \end{bmatrix},$$

where $\Gamma_k := [\gamma_{k-n+1}\, \gamma_{k-n+2} \cdots \gamma_k]$. Here, this process Γ_k is a Markov chain, which has 2^n states given by $\Gamma^{(1)} := [0 \cdots 0\, 0]$, $\Gamma^{(2)} := [0 \cdots 0\, 1], \ldots, \Gamma^{(2^n)} := [1 \cdots 1\, 1]$ and the transition probability matrix $P \in \mathbb{R}^{2^n \times 2^n}$ is given by

$$P := \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \\ Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \in \mathbb{R}^{2^n \times 2^n},$$

$$Q := \begin{bmatrix} 1-q & q & 0 & 0 \\ 0 & 0 & p & 1-p \end{bmatrix} \in \mathbb{R}^{2 \times 4},$$

where the (i, j) element of P is equal to the transition probability from $\Gamma^{(i)}$ to $\Gamma^{(j)}$. When $n = 1$, the transition probability matrix takes a slightly different form as follows:

$$P = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}.$$

Note that H_{Γ_k} takes a controllable canonical form similar to A_k in (3). In fact, this matrix H_{Γ_k} corresponds to the A-matrix of a certain approximation of the overall uncertain system with the packet loss process. It is noted that this approximate system can be viewed as a Markov jump linear system. Finally, we define the matrix F using H_{Γ_k} and P by

$$F := F_1 F_2, \tag{15}$$

where

$$\begin{aligned} F_1 &:= P^T \otimes I_{n^2}, \\ F_2 &:= \operatorname{diag}\left(H_{\Gamma(1)} \otimes H_{\Gamma(1)}, \ldots, H_{\Gamma(2^n)} \otimes H_{\Gamma(2^n)}\right). \end{aligned}$$

Here, diag$(\cdot)$ denotes a block diagonal matrix and $\otimes$ is the Kronecker product. Also, let $\rho(\cdot)$ be the spectral radius of a matrix.

We are now ready to present the sufficient condition.

Theorem 3: Given the data rate $R = \log N$ and the quantizer whose boundaries are $\{h_l\}_l$, if the matrix F in (15) satisfies

$$\rho(F) < 1$$

then under the control law using (13) and (14), the system depicted in Fig. 1 is MSS.

From this theorem, we can find a sufficient data rate for stability by solving the stability test of the matrix F. This matrix is used to check mean square stability of Markov jump linear systems [27].

V. Conclusion

We have considered stabilization of uncertain systems over Markov channels. A necessary condition and a sufficient condition for stability has been derived where a non-uniform quantizer which is capable to reduce the required data rate is utilized. In particular, for scalar plants, the conditions are exact. The conditions provide the limitations on the data rate and the state transition probabilities, which are characterized by the product of eigenvalues of the plant and the bound of the uncertainty. In future research, we will consider a more general class of plants and encoders/decoders.

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