

Stochastic Optimal Control in the Perspective of the Wiener Chaos

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Abstract—We propose a novel and generic methodology for solving continuous finite-horizon stochastic optimal control problems. We develop innovative ideas for approximating controlled stochastic differential equations within the Wiener chaos framework and expand them to reformulate stochastic optimal control problems directly into deterministic ones. Within our approach we present how to preserve the feedback character of the optimal Markov decision rules.

This transformation allows the use of state-of-the-art methods of solving deterministic optimal control problems in the broad context of stochastic differential equations.

We illustrate this new methodology by a numerical example and show the validity of the developed reformulations with huge computational savings compared to standard approaches to stochastic optimal control.

I. INTRODUCTION

In the course of this paper we want to solve *continuous finite-horizon stochastic optimal control* (SOC) problems of the form

$$\min_{u \in \mathcal{A}} \mathbb{E} \left[\int_0^T L(t, X_t, u_t) dt + G(T, X_T) \right] \quad (1a)$$

$$\text{s.t. } dX_t = b(t, X_t, u_t) dt + \sigma(t, X_t, u_t) dB_t, \quad (1b)$$

$$X_0 = x_0, \quad (1c)$$

where X_t is a n_X -dimensional stochastic process within the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and B_t a n_B -dimensional *Brownian motion*. The process X_t is determined by the controlled *stochastic differential equation* (SDE) (1b) with drift term b and diffusion σ . Furthermore, X_t is assumed to be stopped at the final time T . The control u is chosen over the set \mathcal{A} of *admissible* controls as to minimize the cost functional (1a). It therefore has to be a random process as well, being at least \mathcal{F}_t -measurable. The most common choices of admissible controls functions are [1]

- *deterministic* controls $u(t, \omega) = u(t)$,
- *open-loop controls* $u(t, \omega)$ which are non-anticipative with respect to the Brownian motion B_t , and
- *Markov controls* $u(t, \omega) = u_M(t, X_t(\omega))$ with a non-random and Lebesgue-measurable function u_M . Such a control makes the process X_t an Itô diffusion.

In the following we restrict ourselves to Markov controls, writing $u_t = u(t, X_t)$.

In economics and finance, starting with classical problems like Merton's portfolio management [2], this class of problems is usually used as the underlying modeling framework.

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In general, such problems can be solved by considering the corresponding *Hamilton-Jacobi-Bellman* (HJB) equation [1], [3]. But often this leads to an intricate second-order partial differential equation, which is why numerical—and especially *direct* numerical—ideas are often the method of choice to obtain optimal decision rules.

Popular methods of that kind (apart from discretizing the HJB equation) are based on *Markov chains*, i. e., discretizing the continuous process in space and time. Examples are the approach of Kushner [4], [5] using finite-difference and finite-element ideas to obtain the needed transition probabilities or the work of Krawczyk [6], [7] which builds on *weak approximation* schemes for SDEs to design a *Markov decision chain* from the original problem. In each case the resulting Bellman equation can be solved using value iteration. A third related idea introduced by Pagès is the *quantization* of stochastic processes [8], [9], [10], [11], which projects the original process onto a random vector taking values on a finite grid. The resulting problem can, again, be solved by a dynamic programming procedure.

In this paper we propose an entirely different methodology to solve continuous finite-horizon SOC problems. We utilize the *polynomial* or *Wiener chaos* framework to reformulate the original SOC problem as a deterministic optimal control problem that can be solved by existing sophisticated methods of deterministic control, e. g., Bock's *direct multiple shooting* approach [12]. Furthermore this transformation allows the utilization of state-of-the-art techniques of deterministic optimization in the context of random processes.

By means of the chaos expansion we express the considered stochastic processes in terms of deterministic coefficient functions and orthonormal basis polynomials spanning the underlying *Wiener chaos space*. Due to this construction the Brownian motion driving the stochastic processes is represented in terms of random variables closely related to its *Karhunen-Loève expansion*. Still, in order to reformulate the appearing SDE as a system of ordinary differential equations (ODEs), the stochastic integral characterizing the diffusion has to be treated cautiously, because one integrates with respect to a nowhere differentiable function. In the context of this paper we utilize *Malliavin calculus* [13], [14] to overcome this difficulty. The resulting *propagator*, appearing in a related fashion in [15], [16] for solving a class of partial differential equations with random forcing, implicitly includes all the randomness of the SDE. It is enhanced afterwards to controlled SDEs. Therefore it is crucial to implement feedback formulations of the control process to preserve the non-anticipativity of optimal decision rules.

The emphasis of this paper is on the general idea of transforming a SOC problem into a deterministic one. It is organized as follows. In Section II we introduce the general chaos expansion and give an overview on the pieces of Malliavin calculus needed for our approach. Thereafter we apply those ideas to stochastic differential equations and SOC problems resulting in the transformation to deterministic control problems. In that context we establish a method to preserve the feedback character of the control policy. In Section IV we give a short numerical example to illustrate this novel approach for solving continuous finite-horizon SOC problems.

II. NOTATIONS AND MATHEMATICAL PRELIMINARIES

A. The Wiener Chaos Expansion

The starting point of the presented methodology is the *Wiener chaos expansion*. Based on Wiener's *homogeneous chaos* [17], Cameron and Martin [18] generalized the first developments by constructing an orthogonal basis for non-linear functions in a Fourier-Hermite setting. Thus, Wiener's expansion of functionals depending on Gaussian random variables could be transferred to hold for arbitrary random processes [19], [20]. In the past years, the concept of polynomial chaos has attracted much attention again, e. g., in technical applications (including von Mises stress [21] or robustness in shape optimization [22]).

Theorem 1 (Cameron and Martin [18]) *Assume that the process X_t satisfies the integrability condition $X_t \in L^2(\Omega \times [0, T])$, i. e., $\mathbb{E} \left[\int_0^T |X_t|^2 dt \right] < \infty$. Then X_t can be expanded in $[0, T]$ as*

$$X_t = \sum_{\alpha} x_{\alpha}(t) \Psi^{\alpha}(\xi) \quad (2)$$

with (deterministic) coefficients $x_{\alpha}(t)$ and $\{\Psi^{\alpha}(\xi)\}_{\alpha}$ being an orthonormal basis of the Wiener chaos space $L^2(\Omega \times [0, T])$.

Within this theorem, α denotes a multi-index from the set

$$\mathcal{I} = \left\{ \alpha = (\alpha_i)_{i \geq 1} \mid \alpha_i \geq 0, |\alpha| = \sum_{i=1}^{\infty} \alpha_i < \infty \right\}. \quad (3)$$

For convenience, in the rest of the paper we will shortly write

$$\begin{aligned} \alpha &= \vec{0} && \text{if } \alpha_i = 0 \text{ for all } i, \\ \alpha &= \vec{e}_j && \text{if } \alpha_i = \delta_{ij}. \end{aligned}$$

To construct the basis polynomials $\Psi^{\alpha}(\xi)$ we first need

Lemma 1 *Let $\{m_i(t)\}$ be an orthonormal basis of the Hilbert space $L^2([0, T])$. Then the Itô integrals*

$$\xi_i = \int_0^T m_i(t) dB_t \quad (4)$$

define independent standard Gaussian random variables.

Definition 1 *With the one-dimensional Hermite polynomials*

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (5)$$

the basis polynomials of the Wiener chaos space are defined by

$$\Psi^{\alpha}(\xi) = \sqrt{\alpha!} \prod_i H_{\alpha_i}(\xi_i). \quad (6)$$

From that definition and the properties of the Hermite polynomials one directly shows the orthonormality of the basis polynomials $\Psi^{\alpha}(\xi)$.

Hence, the coefficient functions $x_{\alpha}(t)$ can be interpreted as projections of the process X_t onto the corresponding chaos basis as

$$x_{\alpha}(t) = \mathbb{E} [X_t \Psi^{\alpha}].$$

Particularly the zero-order coefficient has a special meaning as it coincides with the expectation of the process X_t .

$$x_{\vec{0}}(t) = \mathbb{E} [X_t \Psi^{\vec{0}}] = \mathbb{E} [X_t]. \quad (7)$$

In a similar fashion the variance of X_t and, consequently, all higher moments can be expressed in terms of the coefficient functions $x_{\alpha}(t)$ only [21].

Lemma 1 additionally reveals a Fourier expansion of the Brownian motion B_t . Due to the scaling property of this process we can restrict the considerations to the time interval $[0, 1]$. Keeping Equation (4) in mind, one can rewrite B_t as [16]

$$B(t) = \sum_{i=1}^{\infty} \xi_i \int_0^t m_i(s) ds. \quad (8)$$

This expansion converges in the mean square sense. Letting the orthonormal basis $\{m_i(t)\}$ of $L^2([0, 1])$ be given by

$$m_i(t) = \sqrt{2} \cos \left(\left(i - \frac{1}{2} \right) \pi t \right),$$

one obtains an expansion of B_t based on Gaussian random variables that coincides with its Karhunen-Loève expansion [23], [24]. On larger time horizons $[0, T]$ one has to modify the time variable appropriately [25].

B. Malliavin Calculus

The second major concept that is needed to transform the stochastic problem (1) into a deterministic one is the *Malliavin calculus*. Introductions to that topic can be found in, e. g., [13], [14]. Here we only want to summarize the basic ingredients needed for our method to solve problems of type (1). Again, we need the Hermite polynomials (5) and the chaos basis functions $\Psi^{\alpha}(\xi)$ (6).

Let $W = \{W(h) \mid h \in H\}$ denote an isonormal Gaussian process defined in $(\Omega, \mathcal{F}, \mathbb{P})$ and associated with the Hilbert space H (compare the definition of ξ_i depending on $m_i \in L^2$ in Lemma 1). Further on, assume F to be a smooth random variable of the form

$$F = f(W(h_1), \dots, W(h_n)) \quad (9)$$

with $f \in C_p^{\infty}(\mathbb{R}^n)$ and $h_i \in H$, $i = 1, \dots, n$.

Definition 2 (Derivative D [14]) The Malliavin derivative of a smooth random variable of the form (9) is the H -valued random variable

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) \cdot h_i. \quad (10)$$

Hence we can regard DF as a directional derivative. We denote the domain of D in $L^2(\Omega)$ by $\mathbb{D}^{1,2}$. This space is again a Hilbert space with the scalar product $\langle F, G \rangle_{1,2} = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG \rangle_H]$. Additionally, the derivative of a random variable $F \in \mathbb{D}^{1,2}$ will be a stochastic process denoted by $\{D_t F \mid t \in [0, T]\}$.

To give an illustrating example we calculate the Malliavin derivative of the basis polynomial $\Psi^\alpha(\xi)$, which is indeed a random variable in $\mathbb{D}^{1,2}$.

Example 1 Consider the basis polynomial $\Psi^\alpha(\xi)$ for fixed $\alpha \in \mathcal{I}$. We deduce (exploiting the rules for differentiating Hermite polynomials (5) and the definition of ξ_i)

$$\begin{aligned} D_s \Psi^\alpha(\xi) &= \sum_{j=1}^{\infty} \sqrt{\alpha_j} \prod_{\substack{i \\ i \neq j}} H_{\alpha_i}(\xi_i) H_{\alpha_j-1}(\xi_j) m_j(s) \\ &= \sum_{j=1}^{\infty} \sqrt{\alpha_j} m_j(s) \Psi^{\alpha^-(j)}(\xi) \end{aligned} \quad (11)$$

with the diminished multi-index $\alpha^-(j)$ defined as

$$\alpha_i^-(j) = \begin{cases} \alpha_i, & i \neq j \\ \alpha_i - 1, & i = j \end{cases}. \quad (12)$$

In the context of stochastic integrals of adapted processes Malliavin calculus provides an important *integration by parts formula* [14] on the time interval $[0, t]$ that is a key component in the following section.

Lemma 2 (Integration by parts) Let X_t be a square integrable and \mathcal{F}_t -measurable random variable for all $t \in [0, T]$. Then for all $F \in \mathbb{D}^{1,2}$ the formula

$$\mathbb{E} \left[F \cdot \int_0^t X_s dB_s \right] = \mathbb{E} \left[\int_0^t D_s F \cdot X_s ds \right]. \quad (13)$$

holds.

With all these ingredients together, we consider now a stochastic process X_t given by an Itô stochastic differential equation in its Wiener chaos expansion.

III. STOCHASTIC OPTIMAL CONTROL PROBLEMS AND THE WIENER CHAOS

A. The Uncontrolled Case

Let us start with an uncontrolled (one-dimensional) process X_t defined by the autonomous SDE on the time interval $[0, T]$

$$dX_t = f(X_t) dt + \phi(X_t) dB_t, \quad X_0 = x_0, \quad (14a)$$

or, conveniently, written in its integral form

$$X_t = x_0 + \int_0^t f(X_s) ds + \int_0^t \phi(X_s) dB_s. \quad (14b)$$

The generalization to multi-dimensional processes X_t and B_t is straightforward. We can apply the Wiener chaos expansion from Theorem 1 if the SDE has a square integrable solution on $[0, T]$.

Theorem 2 Let X_t be given by (14) and assume that $X_t \in L^2([0, T] \times \Omega)$. Then X_t can be written in its Wiener chaos expansion (2) with the coefficients $x_\alpha(t)$ determined by the propagator on $[0, T]$

$$\dot{x}_\alpha(t) = f(X_t)_\alpha + \sum_{j=1}^{\infty} \sqrt{\alpha_j} m_j(t) \phi(X_t)_{\alpha^-(j)}, \quad (15a)$$

$$x_\alpha(0) = \mathbf{1}_{\{\alpha=\bar{0}\}} \cdot x_0. \quad (15b)$$

Within this system of ordinary differential equations ϕ_α denotes again the α -coefficient of the chaos expansion of the function ϕ (depending on X_t) and $\alpha^-(j)$ the diminished multi-index as defined in (12).

Proof: Inserting the expansion (2) into (14b), multiplying with the basis polynomial $\Psi^\beta(\xi)$, $\beta \in \mathcal{I}$, and calculating expectations yields for all $\alpha \in \mathcal{I}$ and $t \in [0, T]$

$$\begin{aligned} x_\alpha(t) &= x_0 \cdot \mathbf{1}_{\{\alpha=\bar{0}\}} + \int_0^t \mathbb{E} [f(X_s) \Psi^\alpha(\xi)] ds \\ &\quad + \mathbb{E} \left[\Psi^\alpha(\xi) \cdot \int_0^t \phi(X_s) dB_s \right]. \end{aligned}$$

While the first appearing integral is deterministic and the expectation within can be represented by the corresponding coefficient function of the expansion of $f(X_t)$, the second integral has to be treated with the integration by parts formula (13). This results in

$$\begin{aligned} x_\alpha(t) &= \mathbf{1}_{\{\alpha=\bar{0}\}} x_0 + \int_0^t f(X_s)_\alpha ds \\ &\quad + \mathbb{E} \left[\int_0^t D_s \Psi^\alpha(\xi) \phi(X_s) ds \right] \\ &= \mathbf{1}_{\{\alpha=\bar{0}\}} x_0 + \int_0^t f(X_s)_\alpha ds \\ &\quad + \sum_{j=1}^{\infty} \int_0^t \sqrt{\alpha_j} m_j(s) \mathbb{E} \left[\phi(X_s) \Psi^{\alpha^-(j)}(\xi) \right] ds \\ &= \mathbf{1}_{\{\alpha=\bar{0}\}} x_0 + \int_0^t f(X_s)_\alpha ds \\ &\quad + \sum_{j=1}^{\infty} \int_0^t \sqrt{\alpha_j} m_j(s) \phi(X_s)_{\alpha^-(j)} ds. \end{aligned}$$

Remark 1 Keeping in mind the definition of the random variables ξ_i (4) and their significance in constructing the Brownian motion (8), we see that all information about the behavior of the stochastic process X_t is implicitly captured within the deterministic ODE system (15). ■

One major advantage of this approach of solving SDEs (apart from the fact that it is independent of any random number) is that the expectation of the process, $\mathbb{E}[X_t]$, is

directly given by the zero-order coefficient $x_{\bar{0}}(t)$. Hence, it need not be calculated by, e. g., *Monte Carlo* methods, where a huge amount of sample paths (computed by some standard stochastic integration method) is necessary. Similar to the expectation, the variance of the process and all higher moments are completely specified by the deterministic coefficient functions of the chaos expansion.

B. The Propagator of the Control Problem

Let us return to the SOC problem (1). Our novel methodology applies the ideas of the previous section directly to this control problem, without considering the value function and its corresponding Hamilton-Jacobi-Bellman equation.

We consider a controlled SDE (compare especially Equation (1b)), where we proceed in a similar way as above to obtain the propagator of the system. Besides the expansion of the state process X_t we have to include a second chaos expansion determining the control process u_t , i. e.,

$$u_t = \sum_{\alpha \in \mathcal{I}} u_\alpha(t) \Psi^\alpha(\xi). \quad (16)$$

Remark 2 With incorporating expansion (16) directly in the propagator obtained for a controlled SDE, we cannot guarantee the assumed feedback character of the Markov control $u_t = u(t, X_t)$ anymore.

The remedy to that problem lies in

Theorem 3 If the Markov control $u_t = u(t, X_t)$ is expanded in terms of X_t by the q -th order polynomial

$$u^q(t, X_t) = \sum_{i=0}^q \hat{u}_i(t) X_t^i, \quad (17)$$

the original control coefficients $u_\alpha(t)$ of (16) are characterized completely by the $q+1$ new control functions $\hat{u}_i(t)$, $i = 0, \dots, q$, and the state coefficients $x_\alpha(t)$. Furthermore, the resulting control process u_t^q is automatically non-anticipative and tends to u_t for $q \rightarrow \infty$.

Proof: In contrast to expanding $u(t, X_t)$ in t , for calculating the expansion in terms of X_t we do not need a stochastic Taylor expansion. Thus, a (infinite) Taylor expansion in $X_t = a$ yields

$$u_t = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial x} u^{(n)}(t, X_t) \Big|_{X_t=a} (X_t - a)^n.$$

which can always be rewritten in powers of X_t . Hence, with defining new control functions $\hat{u}_i(t)$ as the coefficient terms of these powers, one arrives at the infinite version of (17). Similarly, a finite version up to order q can be defined with the q -th term corresponding to the remaining error. The convergence to u_t follows directly and so does the non-anticipativity as we express u_t through the state process which fulfills the property by definition.

Now if we compare (16) and (17)

$$\sum_{\alpha \in \mathcal{I}} u_\alpha(t) \Psi^\alpha(\xi) = \sum_{i=0}^q \hat{u}_i(t) X_t^i \quad (18)$$

by inserting the chaos expansion (2) of X_t and projecting the resulting expression onto the chaos bases, we obtain a system describing the original control coefficients $u_\alpha(t)$ by the new control functions $\hat{u}_i(t)$ and the state coefficients $x_\alpha(t)$, while having the feedback character of the Markov control included implicitly. ■

Example 2 Assume $q = 2$. Then the quadratic and non-anticipative expansion of the control process u_t is given by (16), where the coefficients $u_\alpha(t)$ are defined by the system

$$u_\alpha(t) = \hat{u}_0(t) \cdot \mathbf{1}_{\{\alpha \equiv \bar{0}\}} + \hat{u}_1(t) \cdot x_\alpha(t) + \hat{u}_2(t) \cdot \sum_{\beta \in \mathcal{I}} \sum_{0 \leq \gamma \leq \alpha} C(\alpha, \gamma, \beta) x_{\alpha-\gamma+\beta}(t) x_{\gamma+\beta}(t) \quad (19)$$

for all $\alpha \in \mathcal{I}$ and $C(\alpha, \gamma, \beta)$ given by (compare [16])

$$C(\alpha, \gamma, \beta) = \sqrt{\binom{\alpha}{\gamma} \binom{\gamma+\beta}{\beta} \binom{\alpha-\gamma+\beta}{\beta}}. \quad (20)$$

All multi-index operations are defined component-wise, including the binomial coefficient that is calculated as the product of the component's binomial coefficients.

Remark 3 To reflect that optimal controls can be discontinuous, our preference for solving the resulting deterministic control problem after applying expansion (17) is Bock's direct multiple shooting approach [12]. This state of the art simultaneous method solves optimization and simulation tasks at the same time. Controls are identified on a discrete multiple shooting grid, allowing discontinuous control profiles.

Combining Theorems 2 and 3 we obtain a deterministic reformulation of the controlled SDE (1b). Hence, the only missing part of our transformation method is the objective function (1a) of the original SOC problem. But as this is already formulated as an expectation value, it can be rewritten directly in terms of the deterministic coefficients $x_\alpha(t)$ of the state process X_t and the (new) control functions $\hat{u}_i(t)$. We will give a detailed example in Section IV.

C. Truncating the Propagator

For numerical applications the propagator certainly has to be truncated. Basically, there are three major types of truncation: (A) The first one follows directly from the approximation (17) of the Markov control. (B) Due to their construction via the basis polynomials $m_i(t)$ of $L^2([0, T])$, the random variables ξ_i give less information for increasing index i . Hence, the second type of truncation is based on the number k of random variables ξ_i , $i = 1, \dots, k$, that are used within the construction of the basis polynomials Ψ^α . (C) By a similar argument we see that the information obtained by the coefficient functions with corresponding basis polynomial Ψ^α with $|\alpha| = p$ becomes less for increasing p , so that the third type is the maximum order p of the basis polynomials.

I. e., we have the (simply) truncated multi-index set

$$\mathcal{I}_{k,p} = \left\{ \alpha = (\alpha_1, \dots, \alpha_k) \mid \alpha_i \geq 0, \sum_{i=1}^k \alpha_i \leq p \right\}. \quad (21)$$

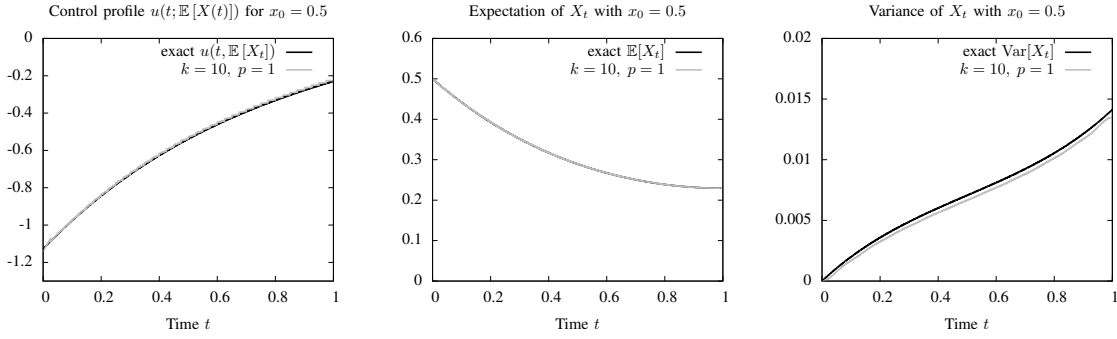


Fig. 1. Solution paths of the linear-quadratic stochastic regulator problem (22) with initial value $x_0 = 0.5$ (and the parameter $\sigma = 0.15$). All plots show a comparison of exact solutions and the corresponding path obtained by the introduced chaos approach, i. e., a solution to the transformed deterministic optimal control problem (24) with $k = 10$ included random variables and an approximation order $p = 1$ resulting in eleven basis polynomials that describe the stochastic system. Therefore, the emerging deterministic problem can be solved in only 2 seconds to optimality. The first plot depicts the control profile $u(t; \mathbb{E}[X_t])$, the second one the expectation of the state process $\mathbb{E}[X_t]$, and the third figure its variance $\text{Var}[X_t]$.

Based on these arguments of decaying importance, one can additionally define *sparse index sets* or even *adaptive ones* [16] (compare as well for exemplary error estimates in the context of stochastic partial differential equations), [26] to reduce the number of coefficients appearing within the chaos expansion and, therefore, computational effort without impairing the solution.

IV. NUMERICAL APPLICATION: THE STOCHASTIC REGULATOR

We illustrate our approach by considering the standard linear-quadratic stochastic regulator problem [1], [3]. The advantage of this academic example is that we can solve the corresponding HJB equation analytically and have an exact solution to compare our results with. This problem reads as

$$\min_{u \in \mathcal{A}} \mathbb{E} \left[\frac{1}{2} \int_0^1 (X_t^2 + u_t^2) dt + \frac{1}{2} X_1^2 \right] \quad (22a)$$

$$\text{s.t. } dX_t = (X_t + u_t) dt + \sigma dB_t, \quad X_0 = x_0 \quad (22b)$$

Then the optimal Markov control can be calculated as

$$u_t = u^*(t, X_t) = \left(\sqrt{2} \tanh(\sqrt{2}(t-1)) - 1 \right) \cdot X_t. \quad (23)$$

Note that the feedback rule at each time instant t depends on the actual state of the system, as each such pair of time and state can be interpreted as the initial point of a separate problem. The Markovian feedback control u_t depends linearly on the system's state X_t and explicitly on t . By the help of (23) we can deduce the optimal cost of the problem and the expectation and variance of the solution process analytically.

Applying the propagator of the previous section to the SDE (22b) in its integral form and inserting the chaos expansions of X_t and u_t (according to (17)) into the cost functional (22a) (making use of Itô's formula [3]) we arrive at the *deterministic* optimal control problem

$$\min_{\hat{u}_0(\cdot), \hat{u}_1(\cdot), \hat{u}_2(\cdot)} \left\{ \frac{1}{2} (x_0^2 + \sigma^2) + \frac{1}{2} \int_0^1 \sum_{\alpha \in \mathcal{I}} \left((x_\alpha(t) + u_\alpha(t))^2 + 2x_\alpha^2(t) \right) dt \right\} \quad (24a)$$

$$\text{s.t. } \dot{x}_\alpha(t) = x_\alpha(t) + u_\alpha(t) + \sigma m_j(t) \cdot \mathbf{1}_{\{\alpha = \bar{e}_j\}} \quad (24b)$$

$$x_\alpha(0) = x_0 \cdot \mathbf{1}_{\{\alpha = \bar{0}\}} \quad (24c)$$

with u_t given as in Theorem 3. After choosing k and p appropriately, the resulting problem (24) can be solved by sophisticated methods of deterministic optimal control as it does not explicitly involve random components anymore. Instead all stochastic information is included implicitly within the system (24b).

In the sequel we assume a quadratic approximation of the control rule, i. e., $q = 2$, compare (19) in Example 2. Remember that the exact control (23) is only linear in X_t . The solution shown in Figure 1 has been calculated by using $k = 10$ random variables and an approximation order $p = 1$, i. e., by a rather low (and only Gaussian) approximation of the propagator. Problem (24) includes then $|\mathcal{I}_{k,p}| = 11$ state functions corresponding to the coefficients $x_\alpha(t)$ of the chaos expansion and three control functions as we use a quadratic approximation of the feedback rule.

Figure 1 shows solution paths of the transformed deterministic optimal control problem (24) with the mentioned truncation scheme in comparison with the appropriate exact solutions of the original stochastic problem (22). Within this figure the first plot depicts the optimal control profile depending on the time $t \in [0, 1]$ and the expectation of the process at that time, i. e., $u(t; \mathbb{E}[X_t | X_0 = x_0])$. This unusual profile imparts an impression of the accuracy of the numerically obtained control at states where the process will be most likely at time t . The remaining two plots of Figure 1 show the corresponding expectation and variance of the processes.

From purely visual comparison of these figures we see how well the introduced chaos method works, even for very low approximations of the Wiener chaos space. This holds especially if we are interested in calculating the objective and expectations of the solution process for a given initial value as these items are fitted very well. The absolute errors of the control, expectation and variance processes decrease with the number of incorporated random variables k and, particularly, with the approximation order p . E. g., the absolute error of the expectation process $\mathbb{E}[X_t]$ in $[0, 1]$ is

at most $1 \cdot 10^{-4}$ for the low approximation $(k, p) = (10, 1)$ and decays to $2 \cdot 10^{-5}$ for $(k, p) = (40, 2)$, which is a very astonishing result. Additionally, the resulting deterministic control problem for $(k, p) = (10, 1)$ was solved to optimality in only 2 seconds. Computation times increase to minutes for better approximations using sparse and adaptive truncation techniques, but remain small compared to standard approaches based on dynamic programming and, especially, Monte Carlo methods.

Even as it is not visible from Figure 1, our approach yields $\hat{u}_2(t) \approx 0$ for approximation orders $p \geq 2$, professing the linearity of the optimal decision rule.

However, if instead we are interested in a certain robustness of control profiles against deviations in the initial values x_0 , we have to apply more accurate approximations of the chaos space including more random variables ξ_i and, especially, a higher approximation order p .

V. RESULTS

In this paper we developed a novel and generic methodology to solve finite horizon stochastic optimal control problems. By the help of the Wiener chaos expansion and Malliavin calculus we are able to transform the underlying stochastic differential equation driving the state process into a system of ordinary differential equations. Additionally, we ensure the feedback character of the Markov control process by expanding in a Taylor-like fashion. Hence, after reformulating the original objective function in terms of the chaos expansion, we obtain a deterministic optimal control problem that implicitly contains all the random information of the original stochastic problem. This resulting problem then can be solved by sophisticated methods of deterministic control.

By a numerical example we showed that our approach yields very promising results. If the optimizer is mainly interested in the optimal objective value, the expectation of the optimally controlled state process, and corresponding higher moments for only one or a small number of initial values, he can obtain fast and reliable results with quite low approximations of the chaos space. At that, making use of sparse and adaptive truncation schemes is beneficial to reduce the resulting computational effort. If the optimizer instead emphasizes the control profile calculated for a fixed initial value to be robust against deviations in the initial values to some extent, he has to apply more accurate approximations of the chaos space with more incorporated random variables and a higher order.

A very important feature of the introduced chaos approach is that we obtain expectations, variances, and all higher moments as a byproduct of solving the resulting deterministic control problem. Hence we do not need any simulation to deduce these quantities.

Therefore, we get an efficient alternative in solving this challenging class of problems, apart from the Hamilton-Jacobi-Bellman theory and dynamic programming techniques based on Markov chain approximations or quantization of the stochastic processes.

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