

Stability analysis for structured feedback interconnections of distributed-parameter systems and time-varying uncertainties

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Abstract—This paper considers stability analysis of systems with constantly-proper Callier-Desoer distributed-parameter transfer functions, interconnected via feedback channels with time-varying parameters and bandwidth-limited time-varying delays. A point-wise in frequency condition that implies stability is established by applying a robust stability result based on integral-quadratic-constraints and a linear time-varying generalization of the ν -gap metric. The stability condition is applied in an example to illustrate the proposed analysis method.

Index Terms—Distributed-parameter systems, feedback stability, IQCs, ν -gap metric, time-varying systems

I. INTRODUCTION

Considers the problem of certifying the stability of the interconnection shown in Figure 1, where M_a and M_b are potentially unstable distributed-parameter systems and each feedback channel, Δ_a and Δ_b , models a bandwidth-limited time-varying delay with a rate-limited varying channel gain. More precisely, M_a and M_b are the causal systems associated with multiplication by the two potentially unstable distributed-parameter transfer functions from the Callier-Desoer class [1], and Δ_a and Δ_b are taken to be bounded operators in the class $\mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$ defined as follows:

$$\begin{aligned} \mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3) &:= \\ &\{ \Delta = \mathbf{S} \circ \mathbf{D} : \mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3), \mathbf{D} \in \mathcal{U}_D(\rho, \alpha, \beta) \} \\ \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3) &:= \\ &\{ \mathbf{S} : (\mathbf{S}v)(t) = \delta(t)v(t) \forall t \in \mathbb{R}; \\ &\quad \delta : \mathbb{R} \rightarrow [\gamma_1, \gamma_2]; |\dot{\delta}(t)| \leq 2\gamma_3\delta(t), \forall t \in \mathbb{R}; \} \\ \mathcal{U}_D(\rho, \alpha, \beta) &:= \\ &\{ \mathbf{D} : (\mathbf{D}v)(t) = \bar{v}(t - d(t)) \forall t \in \mathbb{R}; \\ &\quad \bar{v}(t) = \int_{-\infty}^t (\exp(-(t-s)/\rho)/\rho)v(s)ds \forall t \in \mathbb{R}; \\ &\quad d : \mathbb{R} \rightarrow [0, \alpha]; |\dot{d}(t)| < \beta, \forall t \in \mathbb{R} \} \end{aligned} \quad (1)$$

where $\rho > 0$, $\alpha \geq 0$, $0 \leq \beta < 1$, $0 < \gamma_1 \leq 1 \leq \gamma_2$, and $\gamma_3 \geq 0$. Given $(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$ and nominal stability of the interconnection of M_a and M_b via zero-delay unit-gain channels $\Delta_{0a}, \Delta_{0b} \in \mathcal{U}(\rho, 0, 0, 1, 1, 0)$, the problem of interest is to certify stability of the feedback interconnection of M_a and M_b with arbitrary $\Delta_a, \Delta_b \in \mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$.

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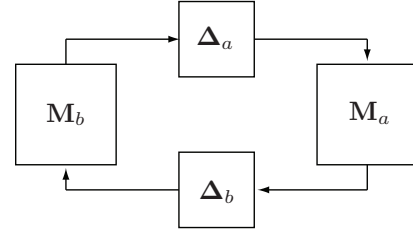


Fig. 1. Structured feedback interconnection with time-varying delays

$\mathcal{U}(T, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$. Such problem can be found in a wide range of engineering applications, such as control of flexible structures, heat distribution, wave propagation, etc. The problem is tackled in the subsequent sections via a recent mixed IQC and ν -gap metric robust stability result from [2], as in [3] where the case of constant gains is studied. Related work carried out in terms of Lyapunov stability analysis, or the notion of energy dissipation, can be found in [4], [5], [6] for constant delays and in [7], [8], [9], [10], [11] for time-varying delays.

To formally state the robust stability result from [2], required notation and concepts are developed in Section II. In part, this involves identifying the key assumptions that underpin the abstract result. Application of the robust stability result [2, Theorem 5.2] involves constructing a path from the nominal zero-delay instance $\mathbf{D}_0 \in \mathcal{U}_D(\rho, 0, 0)$ to an arbitrary bandwidth-limited time-varying delay $\mathbf{D} \in \mathcal{U}_D(\rho, \alpha, \beta)$, and a path from the normal unit gain instance $\mathbf{S}_0 \equiv \mathbf{I}$ to an arbitrary rate-limited time-varying scaling $\mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$. These paths must be continuous with respect to the linear time-varying (LTV) generalization of the ν -gap metric developed in [2]. Verification of Assumption 1 for systems in $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ and $\mathcal{U}_D(\rho, \alpha, \beta)$ and construction of the ν -gap continuous paths are the topics of Sections III and IV. To apply the robust stability result, one must also identify an integral-quadratic-constraint (IQC) that is satisfied by the finite-energy input-output pairs of each instance of $\mathcal{U}_D(\rho, \alpha, \beta)$ and $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ along the ν -gap continuous paths. This is achieved in Section V. Finally, the result is applied in Section VI to establish a point-wise in frequency stability certificate for feedback interconnection via arbitrary $\Delta_a, \Delta_b \in \mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$. An illustrative example is presented in Section VII.

II. PRELIMINARIES

A. Terminology

Given $\tau \in \mathbb{R}$, let the forward interval $\mathbb{R}_\tau^+ := (\tau, +\infty) \subset \mathbb{R}$ and the backward interval $\mathbb{R}_\tau^- := (-\infty, \tau) \subset \mathbb{R}$. Also let \mathcal{A}_ρ

denote the algebra of transfer functions obtained by taking the Laplace transform of impulse responses

$$\kappa(t) = \kappa_r(t) + \kappa_a \delta(t),$$

where $\kappa_r(t) \in \mathbf{L}_1(\mathbb{R}_0^+) := \{f : \mathbb{R} \rightarrow \mathbb{R} \mid \int_{-\infty}^{\infty} |f(t)| dt < \infty, f(t) = 0 \forall t \in \mathbb{R}_0^-\}$, $\kappa_a \in \mathbb{R}$ and $\delta(\cdot)$ is the Dirac distribution; see [1]. One may associate with each $M \in \mathcal{A}_{cp}$ a causal convolution $\mathbf{M} : \mathbf{L}_2(\mathbb{R}) \rightarrow \mathbf{L}_2(\mathbb{R})$ defined by $(\mathbf{M}u)(t) := \int_{-\infty}^t \kappa_r(t-\tau)u(\tau) d\tau + \kappa_a u(t) \forall t \in \mathbb{R}$, where for any interval $\mathbb{I} \subset \mathbb{R}$, $\mathbf{L}_2(\mathbb{I})$ denotes the Hilbert space of square Lebesgue integrable functions (i.e. finite-energy signals) $w : \mathbb{R} \rightarrow \mathbb{R}$ such that $w(t) = 0$ for $t \in \mathbb{R} \setminus \mathbb{I}$; in $\mathbf{L}_2(\mathbb{R})$ the inner-product is denoted $\langle w, v \rangle_{\mathbf{L}_2(\mathbb{R})}$ and the norm $\|\cdot\|_2$. Indeed, the convolution \mathbf{M} is equivalent to the multiplication operator $\hat{u} \in \mathbf{L}_2(j\mathbb{R}) \mapsto \hat{y} \in \mathbf{L}_2(j\mathbb{R})$ defined by $\hat{y}(j\omega) := M(j\omega)\hat{u}(j\omega) \forall \omega \in \mathbb{R}$, where \hat{u} denotes the Fourier transform of u and $\mathbf{L}_2(j\mathbb{R})$ denotes the frequency-domain signal space that is isometrically isomorphic to $\mathbf{L}_2(\mathbb{R})$ via the Fourier transform. Note that the elements of \mathcal{A}_{cp} are constantly proper in the sense that there is a constant limit as $|s| \rightarrow \infty$ in $\mathbb{C}_{0+} := \{s \in \mathbb{C} : \Re(s) > 0\}$. The Callier-Desoer class \mathcal{B}_{cp} consists of transfer functions in the quotient algebra $\mathcal{A}_{cp}[\mathcal{A}_{cp,\infty}]^{-1}$, where

$$\mathcal{A}_{cp,\infty} := \{M \in \mathcal{A}_{cp} : \inf_{s \in \mathbb{C}_{0+}; |s| > \varrho} |M(s)| > 0 \text{ as } \varrho \rightarrow \infty\}.$$

This subsumes the rationals, in that it includes irrational transfer functions; however, these can only have a finite number of unstable poles of finite multiplicity.

B. Basic operator theory

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Consider a linear operator $\mathbf{X} : \text{dom}(\mathbf{X}) \rightarrow \mathcal{H}_2$, where $\text{dom}(\mathbf{X})$ is a linear space contained in \mathcal{H}_1 . The graph of \mathbf{X} is denoted

$$\mathcal{G}(\mathbf{X}) := \left\{ \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{H}_1 \times \mathcal{H}_2 : v \in \text{dom}(\mathbf{X}); w = \mathbf{X}v \right\};$$

the inverse-graph is denoted $\mathcal{G}'(\mathbf{X}) := \{[\begin{smallmatrix} w \\ v \end{smallmatrix}] \in \mathcal{H}_2 \times \mathcal{H}_1 : [\begin{smallmatrix} v \\ w \end{smallmatrix}] \in \mathcal{G}(\mathbf{X})\}$. The image and kernel of \mathbf{X} are denoted $\text{img}(\mathbf{X}) := \{w \in \mathcal{H}_2 : w = \mathbf{X}v; v \in \text{dom}(\mathbf{X})\}$ and $\text{ker}(\mathbf{X}) := \{v \in \text{dom}(\mathbf{X}) : \mathbf{X}v = 0\}$, respectively. The operator is called bounded if there exists a $c > 0$ such that $\|\mathbf{X}v\|_{\mathcal{H}_2} \leq c\|v\|_{\mathcal{H}_1}$ for all $v \in \text{dom}(\mathbf{X})$.

Consider a bounded $\mathbf{X} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. Let

$$\gamma(\mathbf{X}) := \sup_{v \neq 0} \frac{\|\mathbf{X}v\|_{\mathcal{H}_2}}{\|v\|_{\mathcal{H}_1}} \text{ and } \mu(\mathbf{X}) := \inf_{v \neq 0} \frac{\|\mathbf{X}v\|_{\mathcal{H}_2}}{\|v\|_{\mathcal{H}_1}}.$$

Denote the Hilbert adjoint of \mathbf{X} as \mathbf{X}^* . It holds that $\gamma(\mathbf{X}) = \gamma(\mathbf{X}^*)$ and that $\text{img}(\mathbf{X})^\perp = \text{ker}(\mathbf{X}^*)$ and $\text{ker}(\mathbf{X}^*)^\perp = \text{cl img}(\mathbf{X})$ [12, Theorem 11.3], where cl denotes closure and the superscript \perp denotes the orthogonal complement of a linear subspace, which is closed. The restriction of \mathbf{X} to $\mathcal{V} \subset \mathcal{H}_1$ is denoted $\mathbf{X}|_{\mathcal{V}}$; this is clearly bounded. If for every bounded sequence $\{v_k\} \subset \mathcal{H}_1$ there exists a sub-sequence of $\{\mathbf{X}v_k\}$ that is convergent in \mathcal{H}_2 , then \mathbf{X} is said to be *compact*. For the properties regarding the compactness of an operator, readers are referred to [12, Chapter 13].

Definition 1: A bounded operator $\mathbf{X} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is said to be Fredholm if the dimensions of $\text{ker}(\mathbf{X})$ and $\text{coker}(\mathbf{X})$ are both finite, where coker denotes the quotient space $\mathcal{H}_2/\text{img}(\mathbf{X}) := \{[w] : w \in \mathcal{H}_2\}$ and $[w] := w + \text{img}(\mathbf{X})$ denotes the equivalence class of w defined by the equivalence relation $w_1 \sim w_2$ if $w_1 - w_2 \in \text{img}(\mathbf{X})$.

If $\mathbf{X} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is Fredholm, $\text{img}(\mathbf{X})$ is necessarily closed, since $\text{coker}(\mathbf{X})$ is finite dimensional [13, Corollary XI.2.3], whereby $\dim \text{coker}(\mathbf{X}) = \dim \text{img}(\mathbf{X})^\perp = \dim \text{ker}(\mathbf{X}^*)$ and the Fredholm index is defined by

$$\begin{aligned} \text{ind}(\mathbf{X}) &:= \dim \text{ker}(\mathbf{X}) - \dim \text{coker}(\mathbf{X}) \\ &= \dim \text{ker}(\mathbf{X}) - \dim \text{ker}(\mathbf{X}^*). \end{aligned}$$

Note that if \mathbf{X} is one-to-one and onto \mathcal{H}_2 , then it is Fredholm with $\text{ind}(\mathbf{X}) = 0$.

As noted above, in the subsequent stability analysis, finite-energy signals are of primary concern. In the sequel, open-loop systems are considered to be possibly unbounded operators that map from a domain in \mathbf{L}_{2+} into \mathbf{L}_{2+} , where $\mathbf{L}_{2+} := \bigcup_{\tau \in \mathbb{R}} \mathbf{L}_2(\mathbb{R}_\tau^+)$, which is dense in $\mathbf{L}_2(\mathbb{R})$. Note that $\mathbf{L}_2(\mathbb{R}) = \mathbf{L}_2(\mathbb{R}_\tau^-) \oplus \mathbf{L}_2(\mathbb{R}_\tau^+) \forall \tau \in \mathbb{R}$, where \oplus denotes orthogonal sum. The orthogonal projections from $\mathbf{L}_2(\mathbb{R})$ onto $\mathbf{L}_2(\mathbb{R}_\tau^+)$ (respectively $\mathbf{L}_2(\mathbb{R}_\tau^-)$) are denoted \mathbf{P}_τ (respectively $\mathbf{Q}_\tau = \mathbf{I} - \mathbf{P}_\tau$, where \mathbf{I} denotes the identity operator on $\mathbf{L}_2(\mathbb{R})$). Given a bounded $\mathbf{X} : \mathbf{L}_2(\mathbb{R}) \rightarrow \mathbf{L}_2(\mathbb{R})$ and a $\tau \in \mathbb{R}$, the corresponding Toeplitz operators

$$\mathbf{T}_{\mathbf{X},\tau} := \mathbf{P}_\tau \mathbf{X}|_{\mathbf{L}_2(\mathbb{R}_\tau^+)} \text{ and } \mathbf{L}_{\mathbf{X},\tau} := \mathbf{Q}_\tau \mathbf{X}|_{\mathbf{L}_2(\mathbb{R}_\tau^-)}$$

and Hankel operators

$$\mathbf{H}_{\mathbf{X},\tau}^{+-} := \mathbf{P}_\tau \mathbf{X}|_{\mathbf{L}_2(\mathbb{R}_\tau^-)} \text{ and } \mathbf{H}_{\mathbf{X},\tau}^{-+} := \mathbf{Q}_\tau \mathbf{X}|_{\mathbf{L}_2(\mathbb{R}_\tau^+)}$$

are also relevant in the subsequent analysis. Note that the projections \mathbf{P}_τ and \mathbf{Q}_τ give rise to the following.

Definition 2: A map $\mathbf{X} : \text{dom}(\mathbf{X}) \subset \mathbf{L}_2(\mathbb{R}) \rightarrow \mathbf{L}_2(\mathbb{R})$ is *causal* if given any $\tau \in \mathbb{R}$ and $[\begin{smallmatrix} v \\ w \end{smallmatrix}] \in \mathbf{Q}_\tau \mathcal{G}(\mathbf{X}) \subset \mathbf{L}_2(\mathbb{R}_\tau^-)$, $v = 0$ implies $w = 0$. When $\text{dom}(\mathbf{X}) = \mathbf{L}_2(\mathbb{R})$, this is equivalent to $\mathbf{Q}_\tau \mathbf{X} \mathbf{Q}_\tau = \mathbf{Q}_\tau \mathbf{X} \forall \tau \in \mathbb{R}$. If $\mathbf{P}_\tau \mathbf{X} \mathbf{P}_\tau = \mathbf{P}_\tau \mathbf{X} \forall \tau \in \mathbb{R}$ instead, then \mathbf{X} is *anti-causal*. If \mathbf{X} is both causal and anti-causal it is *memory-less*. Note that the composition of bounded causal (resp. anti-causal; memory-less) operators is causal (resp. anti-causal; memory-less).

C. Feedback interconnections and robust stability

The stability analysis problem described in the introduction is tackled in this paper via a recent robust stability result from [2]. The result involves IQCs and an LTV generalization of the ν -gap metric. In particular, it is formulated in an abstract setting that is underpinned by Assumption 1 below, regarding the existence of particular strong right and left representations of the open-loop system graphs. The remainder of this section establishes the notation and concepts required to formulate a precise statement of the robust stability result.

Given open-loop $\mathbf{M} : \text{dom}(\mathbf{M}) \subset \mathbf{L}_{2+} \rightarrow \mathbf{L}_{2+}$ and $\Delta : \text{dom}(\Delta) \subset \mathbf{L}_{2+} \rightarrow \mathbf{L}_{2+}$, both causal in the sense

of Definition 2, the following feedback interconnection is of interest:

$$[\mathbf{M}, \Delta] := \begin{cases} e_1 = -\Delta e_2 + r_1 \\ e_2 = -\mathbf{M}e_1 + r_2 \end{cases} \quad (2)$$

where r_1 and r_2 are exogenous inputs and the signals e_1 and e_2 are the internal signals at the input to \mathbf{M} and the input to Δ , respectively.

Assumption 1: There exist *bounded causal operators* $\mathbf{V}, \mathbf{U}, \tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}$ and $\Theta, \Phi, \tilde{\Theta}, \tilde{\Phi}, \Psi, \Xi, \tilde{\Psi}, \tilde{\Xi}$, all with domains equal to $L_2(\mathbb{R})$, for which:

$$(A1) \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\tilde{\mathbf{U}} & \tilde{\mathbf{V}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & -\tilde{\mathbf{Y}} \\ \mathbf{U} & \tilde{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} \mathbf{V} & -\tilde{\mathbf{Y}} \\ \mathbf{U} & \tilde{\mathbf{X}} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\tilde{\mathbf{U}} & \tilde{\mathbf{V}} \end{bmatrix} = \mathbf{I};$$

$$(A2) \begin{bmatrix} \Xi & \Psi \\ \tilde{\Theta} & -\tilde{\Phi} \end{bmatrix} \begin{bmatrix} \Phi & \tilde{\Psi} \\ \Theta & -\tilde{\Xi} \end{bmatrix} = \begin{bmatrix} \Phi & \tilde{\Psi} \\ \Theta & -\tilde{\Xi} \end{bmatrix} \begin{bmatrix} \Xi & \Psi \\ \tilde{\Theta} & -\tilde{\Phi} \end{bmatrix} = \mathbf{I};$$

(B1) $\mathbf{G}^* := [\mathbf{V}^* \ \mathbf{U}^*]$ and $\tilde{\mathbf{G}} := [-\tilde{\mathbf{U}} \ \tilde{\mathbf{V}}]$ are normalized in the sense $\mathbf{G}^* \mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}} \tilde{\mathbf{G}}^* = \mathbf{I}$;

(B2) $\mathbf{\Gamma}^* := [\Phi^* \ \Theta^*]$ and $\tilde{\mathbf{\Gamma}} := [\tilde{\Theta} \ -\tilde{\Phi}]$ are normalized in the sense $\mathbf{\Gamma}^* \mathbf{\Gamma} = \mathbf{I}$ and $\tilde{\mathbf{\Gamma}} \tilde{\mathbf{\Gamma}}^* = \mathbf{I}$;

(C1) $\mathcal{G}_\tau(\mathbf{M}) := \mathcal{G}(\mathbf{M}) \cap L_2(\mathbb{R}_\tau^+)$ = $\text{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau}) \forall \tau \in \mathbb{R}$;

(C2) $\mathcal{G}'_\tau(\Delta) := \mathcal{G}'(\Delta) \cap L_2(\mathbb{R}_\tau^+)$ = $\text{img}(\mathbf{T}_{\mathbf{\Gamma},\tau}) = \ker(\mathbf{T}_{\tilde{\mathbf{\Gamma}},\tau}) \forall \tau \in \mathbb{R}$;

(D1) $\mathbf{H}_{\mathbf{G},\tau}^{+-}$ and $\mathbf{H}_{\tilde{\mathbf{G}},\tau}^{+-}$ are compact for all $\tau \in \mathbb{R}$; and

(D2) $\mathbf{H}_{\mathbf{\Gamma},\tau}^{+-}$ and $\mathbf{H}_{\tilde{\mathbf{\Gamma}},\tau}^{+-}$ are compact for all $\tau \in \mathbb{R}$.

Remark 1: Assumption 1 holds for multiplication by constantly-proper Callier-Desoer transfer functions in the frequency domain, as shown in [2, Section 4.1]. The assumption is verified for $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ in Section III and for $\mathcal{U}_D(\rho, \alpha, \beta)$ in Section IV.

In the following, $\mathbf{\Gamma}_k$ and $\tilde{\mathbf{\Gamma}}_k$ denote normalized strong right and left graph representations of a causal $\Delta_k : \text{dom}(\Delta_k) \subset L_{2+} \rightarrow L_{2+}$ in accordance with Assumption 1, for $k \in \{1, 2\}$. Similarly, \mathbf{G} and $\tilde{\mathbf{G}}$ denote normalized strong right and left graph representations of a causal $\mathbf{M} : \text{dom}(\mathbf{M}) \subset L_{2+} \rightarrow L_{2+}$.

Definition 3: $[\mathbf{M}, \Delta]$ is said to be *stable* if

$$\mathbf{F}_\tau := \left[\begin{array}{c|c} \mathbf{I} & \Delta \\ \hline \mathbf{M} & \mathbf{I} \end{array} \right] \Big|_{(\text{dom}(\mathbf{M}) \times \text{dom}(\Delta)) \cap L_2^{m+p}(\mathbb{R}_\tau^+)} \quad (3)$$

is one-to-one and onto $L_2^{m+p}(\mathbb{R}_\tau^+)$ for all $\tau \in \mathbb{R}$ and $\sup_{\tau \in \mathbb{R}} \gamma(\mathbf{F}_\tau^{-1}) < \infty$.

It can be shown that this definition of feedback stability naturally imposes a positive arrow of time in closed-loop. Indeed, if $[\mathbf{M}, \Delta]$ is stable then $\mathbf{F}_\tau^{-1} : L_2(\mathbb{R}_\tau^+) \rightarrow (\text{dom}(\mathbf{M}) \times \text{dom}(\Delta)) \cap L_2(\mathbb{R}_\tau^+)$ is necessarily causal for all $\tau \in \mathbb{R}$ [2, Theorem 3.3]. Moreover, given strong right and left representations \mathbf{G} and $\mathbf{\Gamma}$ (respectively $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{\Gamma}}$) of $\mathcal{G}(\mathbf{M})$ (respectively $\mathcal{G}(\Delta)$), stability of $[\mathbf{M}, \Delta]$ is equivalent to causal invertibility of the bounded causal operators $\tilde{\mathbf{\Gamma}}\mathbf{G}$ and $\tilde{\mathbf{G}}\mathbf{\Gamma}$ [2, Theorem 3.5].

The robust stability result from [2] stated at the end of this subsection involves an LTV generalization of the ν -gap, which is known to specialize to the established ν -gap metric for transfer functions in the frequency-domain [14], [15].

Definition 4: The ν -gap between Δ_1 and Δ_2 is

$$\delta_\nu(\Delta_1, \Delta_2) := \begin{cases} \gamma(\tilde{\mathbf{\Gamma}}_2 \mathbf{\Gamma}_1) & \text{if } \mathbf{T}_{\mathbf{\Gamma}_2^* \mathbf{\Gamma}_1, \tau} \text{ is Fredholm with} \\ & \text{ind}(\mathbf{T}_{\mathbf{\Gamma}_2^* \mathbf{\Gamma}_1, \tau}) = 0 \forall \tau \in \mathbb{R} \\ 1 & \text{otherwise} \end{cases}$$

Note that $0 \leq \delta_\nu(\Delta_1, \Delta_2) \leq 1$. The result also involves IQCs on the input-output pairs of the open-loop systems concerned. These are typically defined in the frequency-domain via the Fourier transform isomorphism [16]. Let $L_\infty^{2 \times 2}(j\mathbb{R}) := \{\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{2 \times 2} : \|\Pi\|_\infty < \infty\}$, where $\|\Pi\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\Pi(j\omega))$. An IQC involves a quadratic form $\sigma_\Pi : L_2^2(j\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\sigma_\Pi(\hat{u}) := \langle \hat{u}, \Pi \hat{u} \rangle_{L_2(j\mathbb{R})} = \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \Pi(j\omega) \hat{u}(j\omega) d\omega$$

for a given $\Pi = \Pi_* \in L_\infty^{2 \times 2}(j\mathbb{R})$, where $\Pi_*(j\omega) := \Pi(j\omega)^* \forall \omega \in \mathbb{R}$. Since the bound $\langle \hat{u}, \Pi \hat{u} \rangle_{L_2(j\mathbb{R})} \leq \|\Pi\|_\infty \|\hat{u}\|_{L_2(j\mathbb{R})}^2$ holds, the quadratic form $\sigma_\Pi(\cdot)$ is continuous.

A causal $\mathbf{M} : \text{dom}(\mathbf{M}) \subset L_{2+} \rightarrow L_{2+}$ with normalized strong right graph representation \mathbf{G} is said to satisfy the *strict* IQC defined by $\Pi \in L_\infty^{2 \times 2}(j\mathbb{R})$ if there exists an $\epsilon > 0$ such that

$$\sigma_\Pi(\hat{v}) \geq \epsilon \|\hat{v}\|_2^2 \forall v \in \mathcal{G}(\mathbf{M}) = \text{img}(\mathbf{G}) \cap L_{2+}^2. \quad (4)$$

This is denoted $\mathbf{M} \in \text{SIQC}(\Pi)$. The IQC is said to be non-strict if (4) only holds with $\epsilon = 0$ and this is denoted $\mathbf{M} \in \text{IQC}(\Pi)$. Similarly, a causal $\Delta : \text{dom}(\Delta) \subset L_{2+} \rightarrow L_{2+}$, with normalized strong right (inverse) graph representation $\mathbf{\Gamma}$, is said to satisfy the *complementary* IQC if

$$\sigma_\Pi(\hat{w}) \leq 0 \forall w \in \mathcal{G}'(\Delta) = \text{img}(\mathbf{\Gamma}) \cap L_{2+}^2. \quad (5)$$

This is denoted $\Delta \in \text{IQC}^c(\Pi)$. Observe that (4) and (5) imply satisfaction of the quadratic constraints therein on the graph closures $\text{cl } \mathcal{G}(\mathbf{M}) = \text{img}(\mathbf{G})$ and $\text{cl } \mathcal{G}'(\Delta) = \text{img}(\mathbf{\Gamma})$, because L_{2+} is dense in $L_2(\mathbb{R})$ and $\sigma_\Pi(\cdot)$ is continuous.

Given causal Δ_0 and Δ_1 , as above, let the path

$$\lambda \in [0, 1] \mapsto \Delta_\lambda : \text{dom}(\Delta_\lambda) \subset L_{2+} \rightarrow L_{2+}, \quad (6)$$

be ν -gap continuous in the sense that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\delta_\nu(\Delta_\theta, \Delta_\varphi) < \epsilon$ if $|\theta - \varphi| < \delta$. Then under Assumption 1 an IQC robust stability result holds.

Theorem 2.1: [2, Theorem 5.2] Given $\mathbf{M} : \text{dom}(\mathbf{M}) \subset L_{2+} \rightarrow L_{2+}$ and the ν -gap continuous path (6), suppose the feedback interconnection $[\mathbf{M}, \Delta_0]$ is stable. If there exists a multiplier $\Pi = \Pi_* \in L_\infty^{2 \times 2}(j\mathbb{R})$ such that

(i) $\mathbf{M} \in \text{SIQC}(\Pi)$ and

(ii) $\Delta_\lambda \in \text{IQC}^c(\Pi) \forall \lambda \in [0, 1]$,

then $[\mathbf{M}, \Delta_\lambda]$ is stable for all $\lambda \in [0, 1]$.

In order to apply Theorem 2.1 within the context of the stability problem described in Section I, Assumption 1 must be verified for $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ and $\mathcal{U}_D(\rho, \alpha, \beta)$. Moreover, ν -gap continuous paths are required from the nominal zero-delay instance $\mathbf{D}_0 \in \mathcal{U}_D(\rho, 0, 0)$ to an arbitrary bandwidth-limited time-varying delay $\mathbf{D} \in \mathcal{U}_D(\rho, \alpha, \beta)$, and from the normal unit gain instance $\mathbf{S}_0 \equiv \mathbf{I}$ to an arbitrary rate-limited time-varying scaling $\mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$. These are the topics for the next two sections.

III. NORMALIZED STRONG REPRESENTATION AND ν -GAP CONTINUOUS PATHS FOR SYSTEMS IN $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$

In this section, Assumption 1 is verified for $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ and a ν -gap continuous path are established from $\mathbf{S}_0 \equiv \mathbf{I}$ to any $\mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$.

Given $0 < \gamma_1 \leq 1 \leq \gamma_2$, and $\gamma_3 > 0$, consider a rate-limited time-varying scaling operator $\mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$, $(\mathbf{S}v)(t) = \delta(t)v(t)$. Let \mathbf{R} be defined by $(\mathbf{R}v)(t) := (1 + \delta(t))^{-1/2}v(t)$, $\forall t \in \mathbb{R}$. Clearly, the operators

$$\mathbf{\Gamma} := \begin{bmatrix} \mathbf{S} \\ \mathbf{I} \end{bmatrix} \mathbf{R} \quad \text{and} \quad \tilde{\mathbf{\Gamma}} := \mathbf{R} \begin{bmatrix} \mathbf{I} & -\mathbf{S} \end{bmatrix} \quad (7)$$

are normalized strong right and left representation of $\mathcal{G}'(\mathbf{S})$. Moreover, since both $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are memoryless, the forward Hankel operators $\mathbf{H}_{\mathbf{\Gamma}, \tau}^+$ and $\mathbf{H}_{\tilde{\mathbf{\Gamma}}, \tau}^+$ are equal to zero and are thus compact.

For the path from \mathbf{S}_0 to \mathbf{S} , let us consider

$$\lambda \in [0, 1] \mapsto \mathbf{S}_\lambda \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3) \quad (8)$$

where $\mathbf{S}_\lambda := (1 - \lambda)\mathbf{I} + \lambda\mathbf{S}$. Let $\delta_\lambda(t) := (1 - \lambda) + \lambda\delta(t)$. The normalized strong right and left representation of $\mathcal{G}'(\mathbf{S}_\lambda)$ have the form (7) with the corresponding \mathbf{R}_λ defined as $(\mathbf{R}_\lambda v)(t) := (1 + \delta_\lambda(t))^{1/2}v(t)$. Now given $\varepsilon > 0$ and $|\vartheta - \varphi| < \varepsilon/(1 + \gamma_2)$, $\gamma(\tilde{\mathbf{\Gamma}}_\vartheta \mathbf{\Gamma}_\varphi) \leq \gamma(\mathbf{R}_\vartheta)\gamma(\mathbf{S}_\vartheta - \mathbf{S}_\varphi)\gamma(\mathbf{R}_\varphi) \leq \gamma(\mathbf{S}_\vartheta - \mathbf{S}_\varphi) \leq \varepsilon$, where the inequalities follow $\gamma(\mathbf{R}_\vartheta) \leq 1$, $\gamma(\mathbf{R}_\varphi) \leq 1$, and $|\vartheta - \varphi|(1 + \delta(t)) \leq \varepsilon$. Moreover, the operator $\mathbf{T}_{\mathbf{\Gamma}_\vartheta^* \mathbf{\Gamma}_\varphi, \tau}$ is Fredholm for all τ as $\mathbf{\Gamma}_\vartheta^* \mathbf{\Gamma}_\varphi = \mathbf{R}_\vartheta^*(\mathbf{I} + \mathbf{S}_\vartheta^* \mathbf{S}_\varphi)\mathbf{R}_\varphi$ is clearly memoryless and invertible. This implies $\text{ind}(\mathbf{T}_{\mathbf{\Gamma}_\vartheta^* \mathbf{\Gamma}_\varphi, \tau}) = 0$ and therefore $\delta_\nu(\mathbf{S}_\vartheta, \mathbf{S}_\varphi) < \varepsilon$, which in turn implies that the path $\lambda \in [0, 1] \mapsto \mathbf{S}_\lambda$ is continuous in the ν -gap metric.

IV. NORMALIZED STRONG REPRESENTATION AND ν -GAP CONTINUOUS PATHS FOR SYSTEMS IN $\mathcal{U}_D(\rho, \alpha, \beta)$

Given $\rho > 0$, $\alpha \geq 0$ and $0 \leq \beta < 1$, consider a bandwidth-limited time-varying delay $\mathbf{D} \in \mathcal{U}_D(\rho, \alpha, \beta)$. It can be shown that the operators

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{D} \\ \mathbf{I} \end{bmatrix} \mathbf{R}^{-1} \quad \text{and} \quad \tilde{\mathbf{\Gamma}} = \tilde{\mathbf{R}}^{-1} \begin{bmatrix} \mathbf{I} & -\mathbf{D} \end{bmatrix} \quad (9)$$

are normalized strong right and left representation for $\mathcal{G}'(\mathbf{D})$, where \mathbf{R} is taken to have state-space realization

$$\dot{z}(t) = -(1/\rho)z(t) + (1/\rho)v(t); \quad w(t) = -F(t)z(t) + v(t),$$

with $F(t) := -(1/\rho)X(t)$, $X(t) := \lim_{t_f \rightarrow \infty} X_b(t, t_f)$ and $X_b(t, t_f)$ satisfying the Riccati differential equation

$$\frac{\partial}{\partial t} X_b(t, t_f) = (2/\rho)X_b(t, t_f) + (1/\rho^2)X_b^2(t, t_f) - \xi(t)$$

for $t < t_f$, with $X_b(t_f, t_f) = 0$; $\tilde{\mathbf{R}}$ taken to have the state-space realization

$$\dot{\tilde{z}}(t) = -(1/\rho)\tilde{z}(t) - L(t)\tilde{v}(t); \quad \tilde{w}(t) = -(1/\rho)\tilde{z}(t) + \tilde{v}(t),$$

where $L(t) := -(1/\rho)Y(t)$, $Y(t) := \lim_{t_0 \rightarrow -\infty} Y_f(t_0, t)$ and $Y_f(t_0, t)$ satisfies the Riccati differential equation

$$-\frac{\partial}{\partial t} Y_f(t_0, t) = (2/\rho)Y_f(t_0, t) + (1/\rho^2)Y_f^2(t, t_f) - \chi(t)$$

for $t > t_0$, with $Y_f(t_0, t_0) = 0$. Moreover, it can also be shown that the forward Hankel operators associated with $\mathbf{\Gamma}$ and $\tilde{\mathbf{\Gamma}}$ are compact; see [3, Section III]. Now let $\mathbf{D}_0 \in \mathcal{U}_D(\rho, 0, 0)$ be defined as

$$(\mathbf{D}_0 v)(t) := \int_{-\infty}^t (\exp(-(t-s)/\rho)/\rho)v(s)ds \quad (10)$$

for all $t \in \mathbb{R}$. For the path from \mathbf{D}_0 to \mathbf{D} , consider

$$\lambda \in [0, 1] \mapsto \mathbf{D}_\lambda \in \mathcal{U}_D(\rho, \alpha, \beta) \quad (11)$$

where $(\mathbf{D}_\lambda v)(t) := \bar{v}(t - \lambda d(t))$ with $\bar{v} := \mathbf{D}_0 v \in L_2(\mathbb{R})$. Let $\mathbf{\Gamma}_\lambda$ and $\tilde{\mathbf{\Gamma}}_\lambda$ be the strong right and left graph representation of $\mathcal{G}'(\mathbf{D}_\lambda)$, constructed as shown in (9). One can show that, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\vartheta - \varphi| < \delta$ implies $\gamma(\tilde{\mathbf{\Gamma}}_\vartheta \mathbf{\Gamma}_\varphi) < \varepsilon$. Moreover, the Toeplitz operator $\mathbf{T}_{\mathbf{\Gamma}_\vartheta^* \mathbf{\Gamma}_\varphi, \tau}$ is itself with $\text{ind}(\mathbf{T}_{\mathbf{\Gamma}_\vartheta^* \mathbf{\Gamma}_\varphi, \tau}) = 0$ and this holds for all $\tau \in \mathbb{R}$. This implies the path defined in (11) is ν -gap continuous. For details, the readers are referred to [3].

V. IQC FOR SYSTEMS IN $\mathcal{U}_D(\rho, \alpha, \beta)$ AND $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$

To apply Theorem 2.1 to the stability problem described in Section I, IQC characterizations $\mathcal{U}_D(\rho, \alpha, \beta)$ and $\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ are also required, as provided below.

Lemma 5.1 ([3]): $\mathbf{D} \in \mathcal{U}_D(\rho, \alpha, \beta)$ satisfies $\mathbf{D} \in \text{IQC}^c(\Pi)$ in the sense of (5) for any multiplier $\Pi = \Pi_* \in L_\infty^{2 \times 2}(j\mathbb{R})$ such that

$$\Pi(j\omega) = \begin{bmatrix} x_1 + x_2 & -x_2 B(j\omega) \\ (\star) & (x_2 - \frac{x_1}{1-\beta})|B(j\omega)|^2 - x_2 \alpha^2 |\tilde{B}(j\omega)|^2 \end{bmatrix}, \quad (12)$$

where $B(j\omega) := \frac{1}{j\omega\rho+1}$ and $\tilde{B}(j\omega) := \frac{j\omega}{j\omega\rho+1}$ for all $\omega \in \mathbb{R}$, and x_1, x_2 are any positive real numbers.

Lemma 5.2 ([2]): $\mathbf{S} \in \mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ satisfies $\mathbf{S} \in \text{IQC}^c(\Pi)$ in the sense of (5) for any multiplier $\Pi = \Pi_* \in L_\infty^{2 \times 2}(j\mathbb{R})$ such that

$$\Pi(j\omega) = \begin{bmatrix} 2x_1 & -x_1(\gamma_1 + \gamma_2) - x_2(K_1(j\omega) + K_2(j\omega)^*) \\ (\star) & 2x_1\gamma_1\gamma_2 \end{bmatrix}, \quad (13)$$

where $K_1, K_2 \in \mathcal{A}_{\text{cp}}(-\gamma_3)$ are such that $K_i(j\omega - \gamma_3) + K_i(j\omega - \gamma_3)^* \geq 0 \forall \omega \in \mathbb{R}$, $i \in \{1, 2\}$, and x_1, x_2 are any positive real numbers.

VI. THE MAIN RESULT

Consider the closed-loop illustrated in Figure 1. Note that by expressing $\Delta_a(\Delta_b)$ as a composition of two operators $\mathbf{S}_a(\mathbf{S}_b)$ and $\mathbf{D}_a(\mathbf{D}_b)$, the system in Figure 1 can be equivalently viewed as feedback interconnection of $\tilde{\mathbf{M}}$ and $\tilde{\Delta}$, where $\tilde{\Delta} = \text{diag}(\mathbf{S}_a, \mathbf{S}_b, \mathbf{D}_a, \mathbf{D}_b)$ and

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \\ 0 & \mathbf{M}_b & 0 & 0 \\ \mathbf{M}_a & 0 & 0 & 0 \end{bmatrix}. \quad (14)$$

Stability of system described in Figure 1 is therefore equivalent to stability of the structured feedback interconnection $[\tilde{\mathbf{M}}, \tilde{\Delta}]$. For $\mathbf{D}_a, \mathbf{D}_b \in \mathcal{U}_D(\rho, \alpha, \beta)$, $\mathbf{S}_a, \mathbf{S}_b \in$

$\mathcal{U}_S(\gamma_1, \gamma_2, \gamma_3)$ Assumption 1 clearly holds for $\tilde{\Delta}$ and a ν -gap continuous path can be established from nominal $\tilde{\Delta}_0 := \text{diag}(\mathbf{I}, \mathbf{I}, \mathbf{B}, \mathbf{B})$ to $\tilde{\Delta}$. Moreover, by Lemmas 5.1 and 5.2, $\tilde{\Delta} \in \text{IQC}^c(\Psi)$ if $\Psi = \Psi_* \in \mathcal{L}_\infty^{8 \times 8}(j\mathbb{R})$ is chosen to have the form

$$\Psi = \begin{bmatrix} \text{diag}(\Pi_{sa}^{11}, \Pi_{sb}^{11}, \Pi_{da}^{11}, \Pi_{db}^{11}) & \text{diag}(\Pi_{sa}^{12}, \Pi_{sb}^{12}, \Pi_{da}^{12}, \Pi_{db}^{12}) \\ (\star) & \text{diag}(\Pi_{sa}^{22}, \Pi_{sb}^{22}, \Pi_{da}^{22}, \Pi_{db}^{22}) \end{bmatrix} \quad (15)$$

where Π_{dk}^{ij} and Π_{sk}^{ij} is the (i, j) entry of Π_{dk} in the form of (12) and Π_{sk} in the form of (13), respectively, for $k \in \{a, b\}$ and $i, j \in \{1, 2\}$. Note that form of Ψ is parametrised by a set of positive real numbers. Applying Theorem 2.1 yields the main result.

Theorem 6.1: Consider the structured feedback interconnection shown in Figure 1, where $\mathbf{M}_a, \mathbf{M}_b \in \mathcal{B}_{\text{cp}}$ and $\Delta_a, \Delta_b \in \mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$ for specified values of $\rho > 0$, $\alpha \geq 0$, $0 \leq \beta < 1$, $0 < \gamma_1 \leq 1 \leq \gamma_2$, and $\gamma_3 > 0$ as in (1). Given this, let $\tilde{\Delta} := \text{diag}(\mathbf{S}_a, \mathbf{S}_b, \mathbf{D}_a, \mathbf{D}_b)$, \mathbf{M} be as defined in (14) and \mathbf{D}_0 as defined in (10). If

- (i) the bandwidth limitation $1/\rho$ is such that nominal interconnection $[\tilde{\mathbf{M}}, \tilde{\Delta}_0]$ is stable, where $\tilde{\Delta}_0 = \text{diag}(\mathbf{I}, \mathbf{I}, \mathbf{D}_0, \mathbf{D}_0)$ corresponds to zero channel delay and unit gain, and
- (ii) there exists a multiplier Ψ in the form of (15) such that $\tilde{\mathbf{M}} \in \text{SIQC}(\Psi)$,

then $[\tilde{\mathbf{M}}, \tilde{\Delta}]$ is stable. In this case, the structured feedback interconnection in Figure 1 is also stable. Indeed, this is true for any $\Delta_a, \Delta_b \in \mathcal{U}(\rho, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$.

Remark 2: With M being the transfer matrix corresponding to \mathbf{M} and $Z := \begin{bmatrix} I \\ M \end{bmatrix}$, the condition $\mathbf{M} \in \text{SIQC}(\Psi)$ in Theorem 6.1 is equivalent to the existence of an $\epsilon > 0$ such that $Z(j\omega)^* \Psi(j\omega) Z(j\omega) \geq \epsilon I$ almost everywhere in frequency, except for at the potentially finite number of finite-order poles $j\omega_k$ of M , where the condition becomes $\lim_{\omega \rightarrow \omega_k} \dot{Z}(j\omega)^* \Psi(j\omega) \dot{Z}(j\omega) \geq \epsilon I$, with $\dot{Z}(j\omega) := Z(j\omega)(I + M(j\omega)^* M(j\omega))^{-1/2}$. Alternatively, given any (not necessarily normalized) coprime factorization $M = UV^{-1}$, let $Z := \begin{bmatrix} U \\ V \end{bmatrix} \in \mathcal{A}_{\text{cp}, -}^{8 \times 4}$, whereby $\mathbf{M} \in \text{SIQC}(\Psi)$ if, and only if, there exists an $\epsilon > 0$ such that $Z(j\omega)^* \Psi(j\omega) Z(j\omega) \geq \epsilon I \forall \omega \in \mathbb{R}$.

Remark 3: For a simplified scenario where one of the Δ 's (say Δ_b) is \mathbf{I} , the same stability conditions hold with $\tilde{\Delta}$ reducing to $\text{diag}(\mathbf{S}_a, \mathbf{D}_a)$, and

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{M}_a \mathbf{M}_b & 0 \end{bmatrix}, \Psi = \begin{bmatrix} \text{diag}(\Pi_{sa}^{11}, \Pi_{da}^{11}) & \text{diag}(\Pi_{sa}^{12}, \Pi_{da}^{12}) \\ (\star) & \text{diag}(\Pi_{sa}^{22}, \Pi_{da}^{22}) \end{bmatrix}$$

VII. EXAMPLE

Consider the following controlled heat equation

$$\begin{aligned} x_t(\xi, t) &= x_{\xi\xi}(\xi, t) + \frac{1}{2\nu_0} \mathbf{1}_{[\xi_0 - \nu_0, \xi_0 + \nu_0]} u(t) \\ x_\xi(0, t) &= x_\xi(1, t) = 0, \quad x(\xi, 0) = x_0(\xi) \\ y(t) &= \frac{1}{2\nu_1} \int_{\xi_1 - \nu_1}^{\xi_1 + \nu_1} x(\xi, t) d\xi. \end{aligned} \quad (16)$$

The equation arises as a reasonable mathematical model for temperature control of a unit length metal rod by the use of a

small heating element around the point ξ_0 and a temperature sensor around a small neighborhood of the point ξ_1 for feedback. Here $\mathbf{1}_{[\eta_1, \eta_2]}$ denotes the indicator function

$$\mathbf{1}_{[\eta_1, \eta_2]}(\xi) = \begin{cases} 1 & \xi \in [\eta_1, \eta_2] \\ 0 & \text{otherwise} \end{cases}$$

Assuming that the intersection of $[\xi_0 - \nu_0, \xi_0 + \nu_0]$ and $[\xi_1 - \nu_1, \xi_1 + \nu_1]$ is empty, it can be shown that system (16) has a transfer function representation

$$g(s) = \frac{\sinh(\nu_0 \sqrt{s}) \sinh(\nu_1 \sqrt{s}) \cosh(\underline{\xi} \sqrt{s}) \cosh((1 - \bar{\xi}) \sqrt{s})}{\nu_0 \nu_1 s \sqrt{s} \sinh(\sqrt{s})}$$

where $\underline{\xi} := \min\{\xi_0, \xi_1\}$ and $\bar{\xi} := \max\{\xi_0, \xi_1\}$. Furthermore, $g \in \mathcal{B}_{\text{cp}}$ and it has the coprime factorization $g_r(s)(s/(s+1))^{-1}$, where $g_r(s)$ is

$$\frac{\sinh(\nu_0 \sqrt{s}) \sinh(\nu_1 \sqrt{s}) \cosh(\underline{\xi} \sqrt{s}) \cosh((1 - \bar{\xi}) \sqrt{s})}{\nu_0 \nu_1 (s+1) \sqrt{s} \sinh(\sqrt{s})}$$

See [1] for the details.

To stabilize the system, a low-pass-filtered negative feedback $u = -(1/(0.1s + 1))y$ is applied. Moreover, an uncertain time-varying delay and a time-varying parameter are introduced to model the varying channel gain and the delay associated with heat exchange between the controlled heating/cooling device and the metal rod. The resulting closed-loop system can be formulated as the feedback interconnection shown in Figure 1, with $M_b(s) = 1$, $M_a(s) = -g(s)$, $\Delta_b = \mathbf{I}$, and $\Delta_a \in \mathcal{U}(0.1, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$. One can verify that the interconnection with unit gain and zero delay is stable by the Nyquist criterion. Applying Theorem 6.1, the interconnection with delay is stable if there exist $x_i > 0$, $i = 1, \dots, 4$, and $\epsilon > 0$, such that

$$\begin{aligned} & \begin{bmatrix} x_1 + x_2 & -x_2 B(j\omega) g_r(j\omega) \\ (\star) & ((x_2 - \frac{x_1}{1-\beta}) |B(j\omega)|^2 - x_2 \alpha^2 |\tilde{B}(j\omega)|^2) |g_r(j\omega)|^2 \end{bmatrix} \\ & + \begin{bmatrix} 2x_3 \gamma_1 \gamma_2 & (-x_3(\gamma_1 + \gamma_2) - x_4(K_1(j\omega)^* + K_2(j\omega))) \frac{j\omega}{j\omega+1} \\ (\star) & 2x_3 \left| \frac{j\omega}{j\omega+1} \right|^2 \end{bmatrix} \\ & \geq \epsilon I, \quad \forall \omega \in [0, \infty] \end{aligned} \quad (17)$$

where $B(j\omega) := \frac{1}{0.1j\omega+1}$, $\tilde{B}(j\omega) := \frac{j\omega}{0.1j\omega+1}$, and K_i , $i = 1, 2$, are such that $K_i(j\omega - \gamma_3) + K_i(j\omega - \gamma_3)^* \geq 0 \forall \omega \in \mathbb{R}$. For simplicity, let us choose $K_1(s) = K_2(s) = \frac{\gamma_3}{s + \gamma_3}$. This point-wise in frequency stability certificate can be approximately verified by solving a set of linear matrix inequalities resulting from evaluating ω on a dense subset of $[0, \infty]$.

If the channel gain is unity and stationary, by setting $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = 0$ one can simplify condition (17) to

$$x_1 s_1(\omega; \beta) + x_2 s_2(\omega; \alpha) > \epsilon \quad \forall \omega \in [0, \infty], \quad (18)$$

where

$$\begin{aligned} s_1(\omega; \beta) &:= \left| \frac{j\omega}{j\omega+1} \right|^2 - \frac{1}{1-\beta} \left| \frac{g_r(j\omega)}{(0.1j\omega+1)} \right|^2, \\ s_2(\omega; \alpha) &:= \left| \frac{j\omega}{j\omega+1} - \frac{g_r(j\omega)}{(0.1j\omega+1)} \right|^2 - \alpha^2 \left| \frac{j\omega g_r(j\omega)}{(0.1j\omega+1)} \right|^2. \end{aligned}$$

Given (α, β) , let $\mathcal{S}_{(\alpha, \beta)} := \{(s_1(\omega; \beta), s_2(\omega; \alpha)) : \omega \in [0, \infty]\}$. The condition (18) is equivalent to existence of a linear function that strictly separates $\mathcal{S}_{(\alpha, \beta)}$ from the third quadrant. By this graphical interpretation, condition (18) can be easily verified by plotting $\mathcal{S}_{(\alpha, \beta)}$. For parameters $\nu_0 = \nu_1 = \xi_0 = 0.1$, $\xi_1 = 0.9$, using this graphical method, one can verify that the system is stable for various α and β . Figure 2 shows the set $\mathcal{S}_{(\alpha, \beta)}$ for four pairs of (α, β) and the corresponding linear separators. We note that, in particular, the system is stable for $(\alpha, \beta) = (1.22, 0)$ and $(\alpha, \beta) = (0.89, 0.99)$. In other words, for constant delay the system is stable if $d \leq 1.22$, and for fast time-varying delay the system is stable if $d(t) \leq 0.89 \forall t$.

On the other hand, if the delay is negligible and one would like to verify robust stability against varying channel gain, by setting $\alpha = \beta = 0$ one can simplify condition (17) to

$$x_3 s_3(\omega; \gamma_1, \gamma_2) + x_4 s_4(\omega; \gamma_3) > \epsilon \quad \forall \omega \in [0, \infty], \quad (19)$$

where

$$s_3(\omega; \gamma_1, \gamma_2) := \gamma_1 \gamma_2 \left| \frac{g_r(j\omega)}{0.1j\omega + 1} \right|^2 - (\gamma_1 + \gamma_2) \times \operatorname{Re} \left(\frac{j\omega g_r(j\omega)^*}{(-0.1j\omega + 1)(j\omega + 1)} \right) + \left| \frac{j\omega}{j\omega + 1} \right|^2,$$

$$s_4(\omega; \gamma_3) := -\frac{2\gamma_3^2}{\omega^2 + \gamma_3^2} \operatorname{Re} \left(\frac{j\omega g_r(j\omega)^*}{(-0.1j\omega + 1)(j\omega + 1)} \right).$$

Given $(\gamma_1, \gamma_2, \gamma_3)$, define the set $\mathcal{S}_{(\gamma_1, \gamma_2, \gamma_3)}$ to be $\{(s_3(\omega; \gamma_1, \gamma_2), s_4(\omega; \gamma_3)) : \omega \in [0, \infty]\}$. Again, the condition (19) is equivalent to strictly separation of $\mathcal{S}_{(\gamma_1, \gamma_2, \gamma_3)}$ from the third quadrant, and thus can be verified graphically. For parameters $\nu_0 = \nu_1 = \xi_0 = 0.1$, $\xi_1 = 0.9$, robust stability against different varying channel gains can be verified using this graphical criterion. For this example, the system is stable for arbitrarily fast varying channel gains, as long as the channel gain is strictly positive and does not exceed 4.1. Figure VII shows the set $\mathcal{S}_{(\gamma_1, \gamma_2, \gamma_3)}$ for two sets of $(\gamma_1, \gamma_2, \gamma_3)$ and the corresponding linear separators.

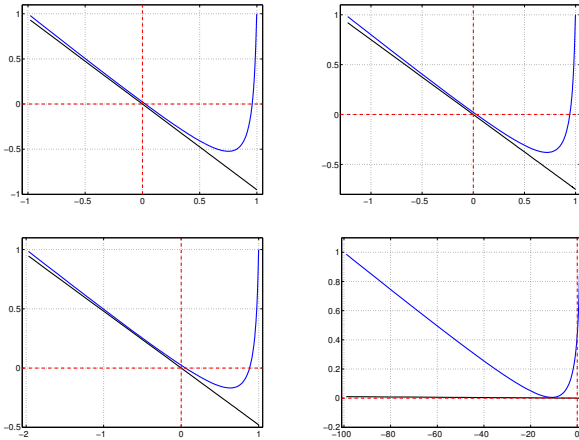


Fig. 2. Four $\mathcal{S}_{(\alpha, \beta)}$ sets (blue curves) and the corresponding linear separators (black lines). Upper left: $S_{(1.22, 0)}$. Upper right: $S_{(1.14, 0.2)}$. Lower left: $S_{(1.01, 0.5)}$. Lower right: $S_{(0.89, 0.99)}$.

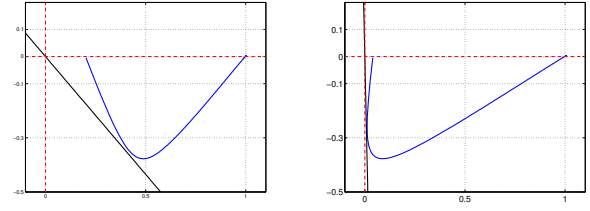


Fig. 3. Two $\mathcal{S}_{(\gamma_1, \gamma_2, \gamma_3)}$ sets (blue curves) and the corresponding linear separators (black lines). Left: $S_{(0.1, 2.0, 1000)}$. Right: $S_{(0.01, 4.1, 100)}$.

VIII. CONCLUSION

This paper considers the problem of verifying the stability of a structured interconnection of distributed-parameter transfer functions via feedback channels with time-varying delays and time-varying channel gains. A sufficient condition is established via separating the graph of the distributed-parameter system from the graphs of the time-varying uncertainty. The result is applied to a numerical example.

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