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An Instability Condition for Uncertain Systems toward Robust Bifurcation Analysis

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Abstract— **In this paper, we consider instability analysis of uncertain feedback systems. First, we present an instability counterpart of the small gain stability theorem. Based on the instability theorem, we solve the problem of instability analysis for systems with dynamic uncertainties. Then, we propose a novel concept of robust bifurcation analysis. An illustrative example is presented for bifurcation analysis of uncertain genetic network models.**

I. INTRODUCTION

Robustness analysis of complex behavior in biosystems attracts many interests recently, e.g., [1], [2], [3]. Functions or characteristic of bio-molecular systems such as regulations, genetic oscillations [4], genetic switches [5] have been studied. They consider how robust the complex behavior is against parameter variations (static perturbations). Bifurcation theory [6], [7] is widely used for analysis of complex behavior. A bifurcation diagram is drawn and the robustness of oscillations is identified with the volume of a parameter region in which the system has unstable equilibria, e.g., [2]. However, bifurcation theory is not applicable to systems with dynamic perturbations (dynamic uncertainties). There inevitably exist in actual systems not only static but dynamic uncertainties [8], [9]. Robustness analysis based on bifurcation theory does not efficiently work in such uncertain systems.

Analysis methods for systems with dynamic uncertainties have been developed in the feedback control and robust control theories [8]. Almost all of them consider stability analysis and stabilization problems. There were only a few studies on instability of feedback systems in 1960's –1970's [10], [11], [12], [13], [14]. Some papers [4], [5], [15], [16], [17], [18] pointed out recently that the existence of complex behavior, oscillating or switching behavior for example, requires some equilibria to be unstable. However, they do not consider dynamic uncertainties in instability analysis of complex behavior. To analyze such behavior in more actual situations, developments of instability analysis methods for systems with dynamic uncertainties are necessary.

In this paper, we consider instability analysis of uncertain feedback systems. First, we present an instability counterpart of the small gain stability theorem. Based on the instability theorem, we solve the problem of instability analysis for systems with dynamic uncertainties. Finally, we consider local bifurcation of an equilibrium and propose a novel concept of robust bifurcation analysis. An illustrative example for analysis of uncertain genetic network models is presented.

Notations: $\mathbb{R}_+ = [0, \infty)$. For a function f defined on $\mathbb{R}_+$, f_T is the function obtained by truncating f after T. L_2^m and L_{2e}^m are the L_2 space and the extended L_2 space of $f : \mathbb{R}_+ \mapsto \mathbb{R}^m$, respectively. $\|\cdot\|$ denotes the L_2 norm of a signal. $\mathcal{RH}_\infty$ is the space consists of all proper and real rational stable transfer matrices.

II. INSTABILITY OF UNCERTAIN SYSTEMS

This section provides a tool for instability analysis of uncertain systems.

A. Preliminaries: System and Definition

We consider the feedback system of Fig. 1.

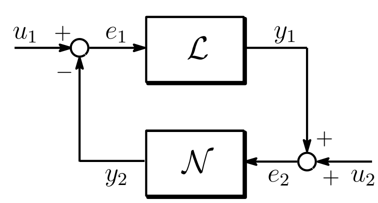

Fig. 1. Feedback system

In the figure, signals $e_1, e_2, u_1, u_2, y_1, y_2 \in \mathbb{R}^m$ satisfy

$$e_1 = u_1 - \mathcal{N}e_2, \quad e_2 = u_2 + \mathcal{L}e_1, \tag{1}$$

where $\mathcal{N} : L_{2e}^m \to L_{2e}^m$ is causal and $\mathcal{L} : L_{2e}^m \to L_{2e}^m$ is linear and causal. They can be infinite dimensional systems. Mainly in this paper, $\mathcal{L}$ and $\mathcal{N}$ are regarded as a linear time-invariant certain system and a nonlinear time-varying uncertain system, respectively. We assume zero initial states for all systems in this paper.

First, we give the definitions of the input-output stability and instability of a system $\mathcal{S} : L_{2e}^m \to L_{2e}^m$.

Definition 1. (L_2 stability) $\mathcal{S}$ is said to be L_2 stable if $\mathcal{S}w \in L_2^m$ holds for any $w \in L_2^m$.

Definition 2. (L_2 instability) $\mathcal{S}$ is said to be L_2 unstable if there exists $w \in L_2^m$ such that $\mathcal{S}w \notin L_2^m$ holds.

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The L_2 stability and L_2 instability of feedback systems are defined as well. The feedback system (1) is said to be L_2 stable if $(y_1, y_2) \in L_2^{2m}$ holds for any $(u_1, u_2) \in L_2^{2m}$. The system (1) is said to be L_2 unstable if there exists a pair $(u_1, u_2) \in L_2^{2m}$ such that $(y_1, y_2) \notin L_2^{2m}$.

Remark 1: L_2 stability of a linear time invariant system is equivalent in a sense to the asymptotic stability of the unique equilibrium. Suppose that $\mathcal{S}$ is represented by the finite dimensional linear state space equation

$$\dot{x}_s = A_s x_s + B_s w_s, \quad z_s = C_s x_s + D_s w_s, \tag{2}$$

which is stabilizable and detectable. Then, the unique equilibrium (the origin) is asymptotically stable if and only if $\mathcal{S}$ is L_2 stable. The equivalence between L_2 instability of a linear system and instability of the origin holds as well.

Next, we define a linear manifold and gains of systems [13], [14].

Definition 3. (linear manifold $M(\mathcal{S})$) Consider $\mathcal{S} : L_{2e}^m \to L_{2e}^m$. A linear manifold $M(\mathcal{S})$ in L_2^m is defined by

$$M(\mathcal{S}) \triangleq \{w \in L_2^m : \ \mathcal{S}w \in L_2^m\}. \tag{3}$$

Definition 4. (L_2 gain) Consider $\mathcal{S} : L_2^m \to L_2^m$. The L_2 gain of $\mathcal{S}$ is defined by

$$\mu(\mathcal{S}) := \inf\{\, \mu > 0 : \ \|\mathcal{S}w\| \le \mu\|w\|, \ \forall w \in L_2^m \,\}. \tag{4}$$

(incremental L_2 gain) Consider $\mathcal{S} : L_{2e}^m \to L_{2e}^m$. The incremental L_2 gain of $\mathcal{S}$ is defined by

$$\begin{aligned} \mu_I(\mathcal{S}) := \inf\{\, \mu > 0 : \ & \|(\mathcal{S}w_1)_T - (\mathcal{S}w_2)_T\| \\ & \le \mu\|w_{1T} - w_{2T}\|, \\ & \forall T \in \mathbb{R}_+, \forall w_1, w_2 \in L_2^m \,\}. \end{aligned} \tag{5}$$

(L_2 gain on M) Consider $\mathcal{S} : L_{2e}^m \to L_{2e}^m$. The L_2 gain on M of $\mathcal{S}$ is defined by

$$\begin{aligned} \mu_M(\mathcal{S}) := \inf\{\, \mu > 0 : \ & \|\mathcal{S}w\| \le \mu\|w\|, \\ & \forall w \in M(\mathcal{S}) \,\}. \end{aligned} \tag{6}$$

For any L_2 unstable system $\mathcal{L}$, there exists a signal $w_1 \in L_2^m$ such that $\mathcal{S}w_1 \notin L_2^m$ holds. However, there also exists a signal $w_2 \in L_2^m$ satisfying $\mathcal{S}w_2 \in L_2^m$. Then, we can define the linear manifold $M(\mathcal{S})$ in the L_2^m space and the L_2 gain on M exists. When the system $\mathcal{S}$ is L_2 stable, $M(\mathcal{S})$ is equivalent to the L_2 space itself and $\mu_M(\mathcal{S}) = \mu(\mathcal{S})$ holds if exists. The incremental L_2 gain can be defined even when a system is not L_2 stable. For example, let us consider affine systems $\mathcal{S}$ satisfying $\mathcal{S}w = \mathcal{L}w + b$, where $\mathcal{L}$ is an L_2 stable linear operator and b is a non-zero constant. We cannot define the L_2 gain of $\mathcal{S}$ but can define the incremental L_2 gain.

Some elements of $M(\mathcal{S})$ are characterized as follows. Consider a finite-dimensional system $\mathcal{S}$ represented by a rational transfer function $G(s)$ with unstable poles. Then, all elements of $M(\mathcal{S})$ have unstable zeros to cancel the unstable poles of $G(s)$. Such signals are represented by $G^{-1}(s)r(s)$, where $r(s)$ is a proper stable rational function.

We set some assumptions on the feedback system (1).

A1) For every external signals u_1, $u_2 \in L_{2e}^m$, the internal signals y_1, y_2, e_1, $e_2 \in L_{2e}^m$ are uniquely determined, that is, the feedback system (1) is well-posed [8].

A2) $\mathcal{N}$ is L_2 stable and has an L_2 gain.

A2′) $\mathcal{N}$ has an incremental L_2 gain.

A3) $\mathcal{L}$ is L_2 unstable and has an L_2 gain on M.

B. Instability of Uncertain Systems

We derive an instability condition for systems with dynamic uncertainties. To simplify the notation, we define

$$\Delta(\gamma) := \{\, \mathcal{N} : \ \mu(\mathcal{N}) \le \gamma \,\}, \tag{7}$$

$$\Delta_I(\gamma) := \{\, \mathcal{N} : \ \mu_I(\mathcal{N}) \le \gamma \,\}. \tag{8}$$

$\Delta(\gamma)$ and $\Delta_I(\gamma)$ are the sets of nonlinear time-varying dynamic uncertainties. An upper bound of $\mathcal{N} \in \Delta(\gamma)$ and that of $\mathcal{N} \in \Delta_I(\gamma)$ are known, respectively. Then, the following problem is considered.

Problem: Instability Analysis of Uncertain Systems. Suppose that $\mathcal{L}$ is a certain system and $\mathcal{N}$ is an uncertainty. Then, determine whether the system (1) remains unstable for all $\mathcal{N}$ in $\Delta(\gamma)$ or $\Delta_I(\gamma)$.

First, we derive the small gain instability theorem based on the idea of [13], i.e., the orthogonal decomposition of the L_2 space. The following theorem plays a central role for instability analysis of uncertain systems as well as the small gain stability theorem [8], [14], [19] for robust stability analysis.

Theorem 1: Suppose that at least one of (a) or (b) holds:

(a) A1, A2, and A3 are satisfied and $\mu_M(\mathcal{L})\mu(\mathcal{N}) \le 1$.

(b) A1, A2′, and A3 are satisfied and $\mu_M(\mathcal{L})\mu_I(\mathcal{N}) \le 1$.

Then, the system (1) is L_2 unstable.

When A3 is replaced by the L_2 stability of $\mathcal{L}$, Theorem 1 becomes the small gain stability theorem [14].

Theorem 1 provides a solution to the problem above. By utilizing the small gain instability theorem, we can solve the problem of instability analysis for systems with uncertainties. Therefore, the condition of Theorem 1 is an instability counterpart of the robust stability condition [8] which is given in Proposition 1 of Section III. In this sense, Theorem 1 may as well be called *robust instability condition*.

Next we present a proof of Theorem 1. To this end, define

$$M^{\perp}(\mathcal{L}) = \{z \in L_2^m : \ \langle z, w\rangle = 0, \forall w \in \bar{M}(\mathcal{L})\}, \tag{9}$$

where $\bar{M}(\mathcal{L})$ is the closure of $M(\mathcal{L})$. Then, $L_2 = \bar{M} \oplus M^{\perp}$ holds [20].

Lemma 1: [13] Suppose A3 holds and the L_2 gain on M of $\mathcal{L}$ is bounded. Then, $M^{\perp}(\mathcal{L})$ is non-empty.

Proof of Theorem 1. We prove the case (b) based on the idea of the direct sum decomposition [13].

From Lemma 1, $M^{\perp}(\mathcal{L})$ is non-empty. We choose u_1, $u_1'(\neq u_1) \in M^{\perp}(\mathcal{L})$ and $u_2 = u_2' = 0$. From A1, there exist y_1, y_2, e_1, e_2, y_1', y_2', e_1', $e_2' \in L_{2e}^m$.

Suppose y_1, y_1', y_2, $y_2' \in L_2^m$ to prove the theorem by contradiction. Then, $e_1 = u_1 - y_2$, $e_1' = u_1' - y_2' \in M(\mathcal{L})$

hold, which are orthogonal to u_1, $u_1' \in M^{\perp}(\mathcal{L})$. We have $e_1 - e_1' \in M(\mathcal{L})$ and $u_1 - u_1' \in M^{\perp}(\mathcal{L})$. Using $u_1 \neq u_1'$, we obtain

$$\begin{aligned} \|y_2 - y_2'\|^2 &= \|u_1 - u_1' - (e_1 - e_1')\|^2 \\ &= \|u_1 - u_1'\|^2 + \|e_1 - e_1'\|^2 \\ &> \|e_1 - e_1'\|^2. \end{aligned} \tag{10}$$

However, since $\mu_M(\mathcal{L})\mu_I(\mathcal{N}) \leq 1$ and

$$\|y_1 - y_1'\| = \|\mathcal{L}(e_1 - e_1')\| \leq \mu_M(\mathcal{L})\|e_1 - e_1'\| \tag{11}$$

hold, we have

$$\begin{aligned} \|y_2 - y_2'\|^2 &= \|\mathcal{N}e_2 - \mathcal{N}e_2'\|^2 \leq \mu_I^2(\mathcal{N})\|y_1 - y_1'\|^2 \\ &\leq \mu_M^2(\mathcal{L})\mu_I^2(\mathcal{N})\|e_1 - e_1'\|^2 \\ &\leq \|e_1 - e_1'\|^2. \end{aligned} \tag{12}$$

This contradicts (10). Then, $y_1 \notin L_2^m$ or $y_2 \notin L_2^m$ holds.

In the same way, we can prove the case (a). We omit the details. □

Further remarks on the instability condition of Theorem 1 are given in the followings.

We consider computation methods of the L_2 gain on M.

Remark 2: (Computation of L_2 gain on M) To evaluate the L_2 instability of uncertain systems by Theorem 1, we need to compute the L_2 gain on $M(\mathcal{L})$. By simple calculations, we can show that the gain $\mu_M(\mathcal{L})$ is equivalent to the L_∞ norm of the system $\mathcal{L}$; see e.g., [8]. When $\mathcal{L}$ is represented by the transfer function matrix $\mathcal{L}(s)$ with no poles on the imaginary axis, the L_∞ norm is defined by

$$\|\mathcal{L}\|_{L_\infty} \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{\mathcal{L}(j\omega)\}, \tag{13}$$

where $\sigma_{\max}\{\cdot\}$ is the maximum singular value of a matrix. Some efficient L_∞ norm computational algorithms are presented. For example, the Hamiltonian matrix approach is proposed by [21] and the Riccati equation and linear matrix inequality approaches are by [22], [23].

In the case that both $\mathcal{L}$ and $\mathcal{N}$ are finite dimensional linear dynamical systems, a robust stability condition [8] does not require the well-posedness of (1). Similarly, we can remove A1 from Theorem 1 by a small modification. The well-posedness of (1) is guaranteed from the strict inequality version. We note that when $\mathcal{N}$ is linear, $\mathcal{N} \in \Delta(\gamma)$ if and only if $\mathcal{N} \in \Delta_I(\gamma)$. We summarize the instability condition for linear feedback systems as follows.

A4) $\mathcal{L}$ and $\mathcal{N}$ are finite dimensional linear dynamical systems and whose transfer function matrices denoted by $\mathcal{L}(s)$ and $\mathcal{N}(s)$ are rational, proper, and with no poles on the imaginary axis.

Corollary 1: Suppose that $\mathcal{L}$ and $\mathcal{N}$ satisfy A2, A3, and A4. Then, for any $\mathcal{N} \in \Delta(\gamma)$, the feedback system (1) is well-posed and L_2 unstable if

$$\mu_M(\mathcal{L}) < 1/\gamma \tag{14}$$

holds.

Proof of Corollary 1 (Outline). The inequality (14) and $\mathcal{N} \in \Delta(\gamma)$ guarantee that for any $\omega \in \mathbb{R} \cup \{\infty\}$

$$\rho(\mathcal{L}(j\omega)\mathcal{N}(j\omega)) \leq \sigma_{\max}\{\mathcal{L}(j\omega)\}\sigma_{\max}\{\mathcal{N}(j\omega)\} < 1 \tag{15}$$

holds, where $\rho(\cdot)$ is the spectral radius. All eigenvalues of the rational transfer function matrix $\mathcal{L}(s)\mathcal{N}(s)$ are less than 1 when $|s| \to \infty$. Then, the well-posedness condition [8], [14] of

$$\det\{I_n + \mathcal{L}(\infty)\mathcal{N}(\infty)\} \neq 0 \tag{16}$$

is satisfied. From Theorem 1, (1) is L_2 unstable. □

Remark 3: (Necessity for instability) Theorem 1 and Corollary 1 provide sufficient conditions for the L_2 instability of (1). We may expect that they are necessary conditions as well. Unfortunately, this is not true at least in the situation of Corollary 1. That is, on the assumptions A2, A3, and A4, there exists $\mathcal{L}$ such that for any $\mathcal{N} \in \Delta(\gamma)$, (1) is well-posed and L_2 unstable but (14) does not hold. We show this fact. Let us consider the system

$$\mathcal{L}(s) = \frac{1}{\gamma}\frac{4(s-3)(s-5)}{5(s-2)(s-4)}, \tag{17}$$

which satisfies A3. Obviously $\mathcal{L}(s)$ has unstable zeros at $s =$ 3, 5 and unstable poles at $s = 2$, 4 on the real axis (Fig. 2). There is only one real pole between two different real zeros. This does not satisfy the parity interlacing property (PIP) condition [24], which is necessary (and sufficient) for the strong stabilizability of $\mathcal{L}(s)$. Then, for any $\mathcal{N}(s) \in \mathcal{RH}_\infty$ on A2 and A4, (1) is L_2 unstable. Since $\mathcal{L}(\infty) = 4/(5\gamma)$, any arbitrary $\mathcal{N}(s) \in \Delta(\gamma)$ satisfies (16). The closed-loop system (1) is well-posed. However, since $\mu_M(\mathcal{L}) = 3/(2\gamma) > 1/\gamma$ holds, $\mathcal{L}(s)$ does not satisfy (14).

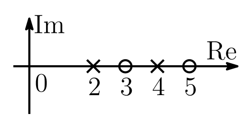

Fig. 2. PIP condition (×: poles, ∘: zeros)

III. CONCEPT OF ROBUST BIFURCATION

A. Motivating Example

First, let us carry out conventional bifurcation analysis [6], [7] of the genetic network model [4] of Fig. 3. Each node in the figure has the ordinary differential equations

$$\dot{m}_i = -m_i + \frac{\alpha}{1 + q_j^2}, \quad q_j = p_j, \tag{18}$$

$$\dot{p}_i = -\beta(p_i - m_i) \tag{19}$$

$$(i, j) = (1, 3), (2, 1), (3, 2),$$

where m_i, $i = 1, 2, 3$ are the messenger RNA (mRNA) concentrations, p_i, $i = 1, 2, 3$ are the repressor-protein concentrations, and α and β with $\alpha, \beta > 0$ and denote the protein copies per cell when repressor is absent, the ratio of the protein decay rate to the mRNA decay rate, respectively. In the model, a negative feedback loop is constructed. As

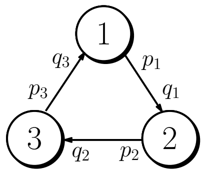

Fig. 3. Genetic network model (three nodes)

illustrated in Fig. 3, p_1 inhibits m_2 and p_2, and p_2 inhibits m_3 and p_3, and p_3 inhibits m_1 and p_1.

The behavior of the system is dependent on the parameter values α and β. Two types of behavior are possible and we can find them by numerical simulations of time responses $p_i(t)$ and $m_i(t)$. We draw a bifurcation diagram for the parameters; see Fig. 4. The parameter space $\mathcal{P} = \{\,\alpha, \beta \in [\,1,\,10^4\,]\,\}$ is divided into two regions, the stability region and the instability region[1]. We can know rough behavior of $m_i(t)$ and $p_i(t)$ from the diagram. The behavior converge to fixed values for any parameter pair on the stability region. Conversely, sustainable oscillations appear on the instability region.

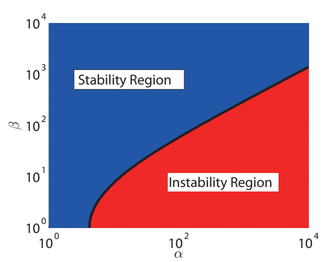

Fig. 4. Bifurcation diagram

As shown above, the bifurcation analysis is well-employed to analyze the properties of dynamical systems depending on parameters in question or to synthesize roughly the desired behavior, e.g., converging or oscillating one. When the mathematical model is completely known, it is possible to perform *robustness* analysis against parametric perturbations (static perturbations) for qualitative properties such as stability, instability, and so on. In addition, control in the sense of parameter tunings based on the bifurcation analysis realizes the desired behavior.

In conventional bifurcation theory, the existence of unanticipated or dynamic perturbations (or *dynamic uncertainties*) cannot be considered. However, there inevitably exist in actual systems dynamic uncertainties such as signal distortions, unforeseen signal delays, approximation errors from PDEs to ODEs, and truncated errors of high index ODEs [8], [9]. Bifurcation diagrams do not efficiently work when a small dynamic perturbation is inserted into nominal models.

Consider the situation that signals are distorted in a loop of Fig. 3 due to the existence of a filter not known in advance as Fig. 5. Then, the link $q_2 = p_2$ is replaced by

$$\dot{\xi} = -0.02\xi + 0.02p_2, \quad q_2 = 0.3\xi + 0.7p_2, \tag{20}$$

which can be represented by the transfer function

$$q_2(s) = \frac{1+35s}{1+50s}p_2(s). \tag{21}$$

We choose $\alpha = 10$ and $\beta = 2$ on the instability region of Fig. 4 and compute the behavior of $m_i(t)$. In the nominal model, sustainable oscillations appear in Fig. 6. However, in the model with a dynamic perturbation, oscillations disappear and the behavior converges to fixed values in Fig. 7.

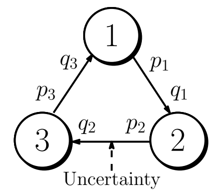

Fig. 5. Uncertain genetic network model (three nodes)

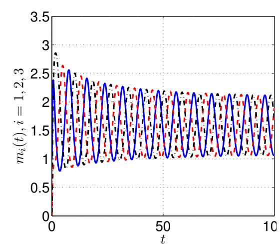

Fig. 6. Nominal system case

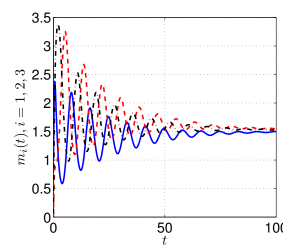

Fig. 7. Perturbed system case

B. Robust Bifurcation Analysis

As illustrated above, bifurcation analysis based on nominal models does not work for systems with dynamic uncertainties. It is impossible for such uncertain systems to accurately compute bifurcation boundaries. We derive a bifurcation theory that is applicable to uncertain systems. The concept of the bifurcation of an equilibrium (local bifurcations, [7]) is extended to the robust bifurcation of an equilibrium. We consider the following problem.

Problem: Robust Bifurcation Analysis. Consider a dynamical system with static parameters in $\mathcal{P}$ and dynamic uncertainties in Δ and an equilibrium depends only on static parameters. Then, divide the parameter space $\mathcal{P}$ into the following three regions. *Stability region*: The region in which the equilibrium is stable independently of the elements of Δ. *Instability region*: The region in which the equilibrium is unstable independently of the elements of Δ. *Uncertain region*: The region in which the stability of the equilibrium may change dependently on the elements of Δ.

There is a similar concept called imperfect bifurcation analysis [25] that is used for analysis of systems with uncertain *static parameters*. On the other hand, the robust bifurcation analysis defined in this paper is applicable to

[1]The boundary separating the parameter space $\mathcal{P}$ is called a bifurcation boundary [7]. In the example, at any point on the boundary, a Hopf-type bifurcation of the unique equilibrium [6], [7] occurs.

systems with uncertain *dynamics* represented by input-output operators in Δ. This section presents a solution to the robust bifurcation analysis problem. To this end, we also use the following well-known result under the assumption:

A5) $\mathcal{L}$ is L_2 stable and has an L_2 gain.

Proposition 1: [8] Suppose that $\mathcal{L}$ and $\mathcal{N}$ satisfy A2, A4, and A5. Then, for any $\mathcal{N} \in \Delta(\gamma)$, (1) is well-posed and L_2 stable if and only if

$$\mu(\mathcal{L}) < 1/\gamma \tag{22}$$

holds.

C. Analysis of Genetic Network Model

We illustrate a procedure of robust bifurcation analysis by Corollary 1 and Proposition 1. For simplicity of notation, transfer function representations are used in the following.

Consider the gene network model (18), (19) with the dynamic uncertainty

$$\begin{aligned} q_2(s) &= \{1 + w(s)\delta(s)\}p_2(s), \quad w(s) = \frac{s}{s+1}, \\ \delta(s) &\in \{d(s) \in \mathcal{RH}_\infty : \ \mu(d) \le \gamma\}, \end{aligned} \tag{23}$$

which represents the unmodelled dynamics at high frequencies. The equilibrium of the nominal model is independent of the uncertainty $\delta(s)$ since $w(0)\delta(0) = 0$ holds and then the output of $w(s)\delta(s)p_2(s)$ is zero for any steady state $p_2(s) = p^e$. Therefore, the uncertain system has the equilibrium $m_i = m^e$ and $p_i = p^e$, $i = 1, 2, 3$, satisfying

$$m^e = p^e = \frac{\alpha}{1 + (p^e)^2}, \tag{24}$$

which is independent of the dynamic uncertainty $\delta(s)$. In this section, we draw a robust bifurcation diagram for the unique equilibrium.

Now, let us transform the network model of Fig. 5 to the feedback form of Fig. 1. First, we employ new variables $\tilde{p}_i = p_i - p^e$, $\tilde{m}_i = m_i - m^e$, $i = 1, 2, 3$, and w_1, w_2 to obtain the error dynamics

$$\begin{aligned} \dot{\tilde{m}}_i &= -\tilde{m}_i + f(\tilde{q}_j), & (25)\\ \dot{\tilde{p}}_i &= -\beta(\tilde{p}_i - \tilde{m}_i), & (26)\\ f(\tilde{q}_j) &= \frac{\alpha}{1 + (\tilde{q}_j + p^e)^2} - \frac{\alpha}{1 + (p^e)^2}, & (27)\\ \tilde{q}_1 &= \tilde{p}_1, \quad \tilde{q}_3 = \tilde{p}_3, & (28)\\ &(i,j) = (1,3), (2,1), (3,2) & \end{aligned}$$

and the dynamic uncertainty

$$\begin{aligned} \dot{w}_1 &= -w_1 + w_2, \\ \tilde{q}_2 &= \tilde{p}_2 - w_1 + w_2, \\ w_2(s) &= \delta(s)\tilde{p}_2(s). \end{aligned} \tag{29}$$

We next define $x = [\, \tilde{m}_1 \ \tilde{p}_1 \ \tilde{m}_2 \ \tilde{p}_2 \ \tilde{m}_3 \ \tilde{p}_3 \ w_1 \,]^{\mathrm{T}}$, $y_1 = \tilde{p}_2$, $e_1 = -y_2 = -w_2$, and $e_2 = y_1$. Then, we can describe the linearized certain part $\mathcal{L}$ and the uncertain part $\mathcal{N}$ by

$$\mathcal{L}: \begin{cases} \dot{x} = \begin{bmatrix} \tilde{A} & 0 & r\tilde{b}\tilde{c} & 0 \\ r\tilde{b}\tilde{c} & \tilde{A} & 0 & 0 \\ 0 & r\tilde{b}\tilde{c} & \tilde{A} & -r\tilde{b} \\ 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -r\tilde{b} \\ -1 \end{bmatrix} e_1 \\ y_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} x \end{cases} \tag{30}$$

$$\tilde{A} = \begin{bmatrix} -1 & 0 \\ \beta & -\beta \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\mathcal{N}: \ y_2(s) = \delta(s)e_2(s), \tag{31}$$

where r is the derivative of f at the equilibrium, that is, $r = df/dp|_{p^e}$. $\mathcal{L}$ is stabilizable and detectable, and therefore may be stabilized by $y_2(s) = \delta(s)e_2(s)$ if there is no assumptions on $\delta(s)$. $\mathcal{N}$ is L_2 stable and satisfies A2. The feedback system composed of $\mathcal{L}$ and $\mathcal{N}$ satisfies A1. In addition, $\mathcal{L}$ satisfies either A3 or A5, depending on the parameters α, β, and the derivative of f.

We draw a robust bifurcation diagram on the parameter space $\mathcal{P} = \{\, \alpha, \beta \in [\, 1,\ 10^4 \,] \,\}$ by the following procedure. Divide $\mathcal{P}$ into three regions, i.e., the stability region, the instability region, and the uncertain region, using stability and instability tests by Corollary 1 and Proposition 1.

i) Choose a parameter pair $(\, \alpha,\ \beta \,)$ in $\mathcal{P}$.
ii) Solve the algebraic equation (24) to find the unique equilibrium of the system.
iii) Define the linearized system $\mathcal{L}$ of (30).
iv) Divide the parameter space $\mathcal{P}$.
  - If $\mathcal{L}$ is L_2 stable and the stability test by Corollary 1 is positive, the stability region includes $(\, \alpha,\ \beta \,)$.
  - If $\mathcal{L}$ is L_2 unstable and the instability test by Proposition 1 is positive, the instability region includes $(\, \alpha,\ \beta \,)$.
  - Else if both tests are negative, the uncertain region includes $(\, \alpha,\ \beta \,)$.
v) Choose another pair $(\, \alpha,\ \beta \,)$ in $\mathcal{P}$. If no different new pair $(\, \alpha,\ \beta \,)$ remains in $\mathcal{P}$, finish. Otherwise go to ii).

Illustrative figures are given in Figs. 8 $\sim$ 10. In the figures, the L_2 gain in the uncertain set (23) are given by $\gamma = 0.01$, 0.1, 2. The blue region indicates that the system of Fig. 1 is L_2 stable independently of the uncertainty $\delta(s)$ in (23). On the red region, the system is L_2 unstable independently of $\delta(s)$. On the white region, it is not determined by the proposed procedure if the system is stable or unstable without further information on $\delta(s)$. The solid lines are the bifurcation boundaries for the nominal model, which are included in the white region.

By using the diagrams, we can know rough behavior of the systems and design desired functions such as robust regulations or robust oscillations. The diagrams show us guidelines for analysis or synthesis of gene network models. For example, the parameter β should be as small as possible for a robust oscillator design. Because when the uncertainties are large enough, only on small values of β the instability regions exist generating robust oscillators.

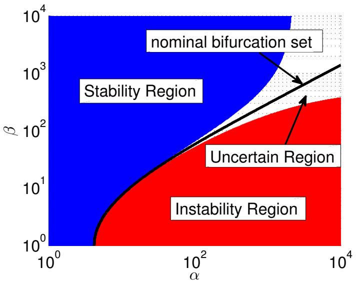

Fig. 8. Robust bifurcation diagram ($\gamma = 0.01$)

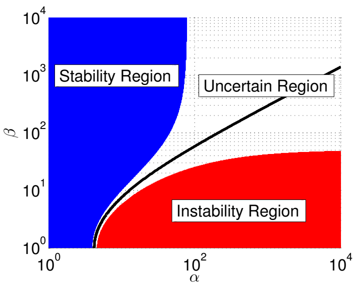

Fig. 9. Robust bifurcation diagram ($\gamma = 0.1$)

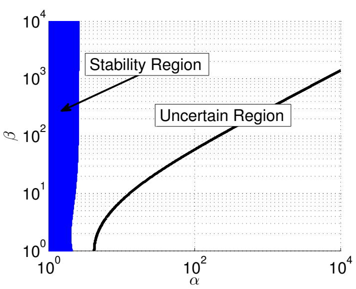

Fig. 10. Robust bifurcation diagram ($\gamma = 2$)

As shown in Fig. 10, the instability region disappears when the uncertainty is large enough, although the stability region does not. This is because that each sub-node in Fig. 5 is stable when the network is disconnected. That is, $m_i(t)$ and $p_i(t)$, $i = 1, 2, 3$ converge to fixed values when one of $q_i = 0$, $i = 1, 2, 3$ holds. Then, for the system of Fig. 5, the stability property is more robust than the instability against dynamic uncertainties in the network edges.

IV. CONCLUDING REMARKS

This paper derived an instability condition for linear systems with dynamic uncertainties. The condition is an instability counterpart of the well-known robust stability condition. Two types of gain-bounded uncertainties are considered in the condition. One is written by the incremental L_2 gain form and the other is the L_2 gain form. Then, a concept of robust bifurcation analysis and the analysis method were newly proposed. Although the proposed method was applied to only a Hopf bifurcation model in this paper, it is applicable to other types of bifurcation models.

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