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H_∞-control with state feedback of an inclined cable

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***Abstract*— In this study we consider the robust control of an inclined cable modelled using partial differential equations and subjected to external disturbances. This paper focuses on the construction of a standard linear infinite dimensional state space system and an H_∞ feedback control with full observation of the state.**

***Index Terms*— partial differential equations, robust control, inclined cable, state space model.**

I. INTRODUCTION

Inclined cables are a critical component in cable stayed bridges, and one way their vibrations can be induced is from deck movements. The increasing span of the bridges makes cable and deck vibrations more sensitive to wind and traffic induced vibrations and this situation has become a major design issue. Since cables are very flexible and lightly damped, cable structure systems usually have a range of dynamic problems. Their modelling is therefore very important in predicting and controlling the response to excitation. A good review about vibration suppression in civil structures and many references on this topic can be found in [11].

In this article, we work on a linearized model of an inclined cable using partial differential equations (pde) and we aim at designing robust control laws for this kind of vibrating system. In order to do so, we will first build the corresponding standard state space model of infinite dimension and describe the H_∞-control of the system with state feedback. This has to be understood as a first step toward the measurement feedback study of an inclined cable using an active tendon, which is ongoing work.

In civil engineering, the usual study of the vibrations of inclined cables is made through the consideration of a few structural modes. A novel aspect of this current work is to consider, as far as we can, the complete pde model of the system in order to study its robust control. The objective is to construct the standard infinite-dimensional model so that we can apply a robust control strategy based on modern control tools for distributed parameter systems [2]. We will then prove the H_∞-control of the cable, robust with respect to deck movements, by the means of the resolution of a Riccati equation.

It should be noted that in this current derivation we do not consider the non-linearities that arise in modelling the vibrations of an inclined cable. For example [17] (see also [16]), use a detailed non-linear pde model which is decomposed into the first few vibration modes from which the precise non-linear coupling between in-plane and out-of-plane vibrations can be seen. Neither do we consider a finite element modelling approach as in [9], where one can find an introduction to active tendon control of cables. The development of active devices for future large bridges is justified by the difficulty of damping the stay cable vibration. Here our approach is to use the case of a feedback controller with full observation of the state, as a preliminary study on the way to partial measurement feedback control through an active tendon where actuator and sensor are collocated at one end of the cable. The novelty of our approach resides in the active control study at the "pde level" of the system.

Among the numerous possibilities for modelling the motions of inclined cables with small sag, we adopt the pde modelling presented in [16] using the derivation from [17]. The derivation includes the effects of support motion at both ends of the cable. The cable is supported at end points a and b and the direction of the chord line from a to b, of length ℓ is defined as x, see Figure 1. The cable equilibrium sag position and the chord line both lie in the $x-z$ plane, therefore w represents in-plane motion and v represents out-of-plane motion. The angle of inclination of the chord line relative to the horizontal is defined as θ. We set ρ to be the density of the cable, E the Young's modulus and g is gravity. We then define $\varrho = \rho g \cos\theta$ as the distributed weight perpendicular to the cable cord.

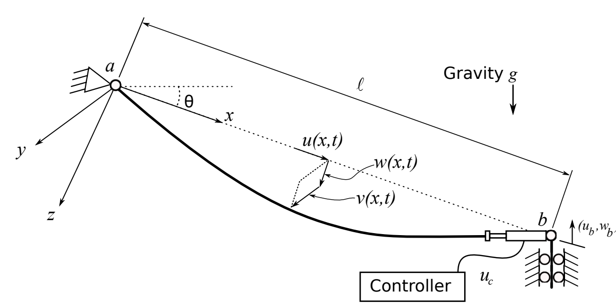

Fig. 1. Inclined Cable

The notations used here broadly follows the approach of [16] and one can refer to this book for further details about the modelling of the inclined cable movement:

- $u = u_q(x,t) + u_m(x,t)$ is the dynamic axial displacement (in x-direction) of the cable ;

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- $v = v_q(x,t) + v_m(x,t)$ is the dynamic out-of-plane transverse displacement (in y-direction) ;
- $w = w_q(x,t) + w_m(x,t)$ is the dynamic in-plane transverse displacement (in z-direction) ;
- T_s is the static tension of the cable and is assumed to be constant (w.r.t. x and t) ;
- $T_d = T_q + T_m$ is the dynamic tension of the cable;
- $w_s(x) = \frac{\varrho\mathcal{A}}{2T_s}\left(\ell x - x^2\right)$ is the static in-plane displaced shape of the cable and gives the sag displacement relationship ($\mathcal{A}$ being the cross sectional area of the cable).

Each displacement is the sum of a quasi-static component (subscript q) and a modal component (subscript m). We denote by ∂_z the partial derivative with respect to z. Since we neglect the axial inertial force ($\partial_{tt}u = 0$) we have the following equation for the dynamic tension:

$$T_d = \mathcal{A}E\left[\partial_x u + \frac{1}{2}(\partial_x v)^2 + \frac{1}{2}(\partial_x w)^2 + \frac{dw_s}{dx}\partial_x w\right].$$

The equation of motion for the dynamic analysis of this inclined cable, are then given by: $\forall(x,t) \in (0,\ell)\times(0,\infty)$,

$$\left\{\begin{array}{lcl} \rho\mathcal{A}\partial_{tt}v & = & (T_s + T_d)\partial_{xx}v, \\ \rho\mathcal{A}\partial_{tt}w & = & \partial_x\left((T_s+T_d)\partial_x w + T_d\frac{dw_s}{dx}\right). \end{array}\right. \quad (1)$$

Outline: In Section II, one will read how we manage to obtain the preliminary equations for the construction of the linear standard model of the inclined cable. Section III describes what we define as H_∞-control with state feedback in an infinite dimensional setting. Finally, in Section IV, we gather all these parts to build our standard model and apply the results of [2] to prove the robust control of the system. In Section V we discuss how this model used to produce some numerical results.

II. Modelling of a controlled inclined cable

Considering an inclined cable as a structural element of a cable-stayed bridge, we are interested in the modelling of the robust control of the vibrations of the cable subjected to the oscillation of the bridge deck. From equation (1), our aim is to first define a linearized system that could fit in the standard state space model.

A. Partial differential equation model of the inclined cable

In addition to the pde we are going to study, we impose the following boundary conditions corresponding to this support motion (see Figure 1)

$$\left\{\begin{array}{ll} u(0,t) = 0, & u(\ell,t) = u_b(t), \\ v(0,t) = 0, & v(\ell,t) = 0, \\ w(0,t) = 0, & w(\ell,t) = w_b(t), \end{array}\right. \quad \forall t \in (0,\infty). \quad (2)$$

with $u_b, w_b \in L^\infty(0,\infty)$, and some compatible initial conditions in $(0,\ell)$

$$\begin{array}{l} (u,v,w)(0,x) = (u^0,v^0,w^0), \\ \partial_t(u,v,w)(0,x) = (u^1,v^1,w^1). \end{array} \quad (3)$$

One can read in [16] the details about the splitting of the equations into quasi-static and modal motions equations and how this influences the analysis of the inclined cable vibrations (taking into account the non-linear dynamics or not). Here, we only give a summary of the results.

The first step is to solve the quasi-static equations, that correspond to the motion of the cable subjected to the boundary conditions but without taking into account any dynamic response (initial conditions are not considered and we assume that $T_d << T_s$) : $\forall(x,t) \in (0,\ell)\times(0,\infty)$,

$$\left\{\begin{array}{lcl} T_s\partial_{xx}v_q & = & 0 \\ T_s\partial_{xx}w_q + T_q\frac{d^2w_s}{dx^2} & = & 0 \end{array}\right. \quad (4)$$

with $T_q = \mathcal{A}E\left(\partial_x u_q + \frac{dw_s}{dx}\partial_x w_q\right)$.

As one can read in [16], using the boundary conditions (2), these equations can be solved and one obtains the values of the quasi-static components, for $x \in (0,\ell)$ and $t \in (0,\infty)$:

$$\begin{array}{lcl} u_q(x,t) & = & \frac{E_q}{E}u_b(t)\frac{x}{\ell} - \frac{\varrho\mathcal{A}\ell}{2T_s}w_b(t)\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] \\ & & +\frac{\lambda^2E_q}{4E}u_b(t)\left[\frac{x}{\ell} - 2\left(\frac{x}{\ell}\right)^2 + \frac{4}{3}\left(\frac{x}{\ell}\right)^3\right] \\ v_q(x,t) & = & 0 \\ w_q(x,t) & = & w_b(t)\frac{x}{\ell} - \frac{\varrho E_q\ell\mathcal{A}^2}{2T_s^2}u_b(t)\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right] \\ T_q(t) & = & \frac{\mathcal{A}E_q}{\ell}u_b(t) \end{array} \quad (5)$$

where $E_q = E/(1+\lambda^2/12)$ is the equivalent modulus of the cable and $\lambda^2 = E\varrho^2\ell^2\mathcal{A}^3/T_s^3$ is Irvine's parameter.

In the second step, we consider the linearized dynamic modal partial differential equations that consist of the following pair of 1-dimensional non-linear wave equations on $(0,\ell)\times(0,\infty)$:

$$\left\{\begin{array}{lcl} \rho\mathcal{A}\partial_{tt}v_m & = & T_s\partial_{xx}v_m \\ \rho\mathcal{A}\partial_{tt}w_m & = & \partial_x\left(T_s\partial_x(w_m + w_q) + T_d\frac{dw_s}{dx}\right) \\ & & +\rho\mathcal{A}\partial_{tt}w_q \end{array}\right. \quad (6)$$

subjected to homogeneous Dirichlet boundary conditions

$$\left\{\begin{array}{ll} v_m(0,t) = 0, & v_m(\ell,t) = 0, \\ w_m(0,t) = 0, & w_m(\ell,t) = 0, \end{array}\right. \quad \forall t \in (0,\infty) \quad (7)$$

initial conditions (3) and where the linearized dynamic tension satisfies (since it is assumed that the modal axial displacement u_m can be neglected)

$$T_d = T_q + \frac{\varrho\mathcal{A}^2E}{2T_s}(\ell - 2x)\,\partial_x w_m.$$

In order to be able to derive a state space model from this setting, we have to linearize equations (6) and use equations (5). The linearized system we will work with is derived for

$(x,t) \in (0,\ell) \times (0,\infty)$ and reads:

$$\begin{cases} \partial_{tt} v_m = \dfrac{T_s}{\rho \mathcal{A}} \partial_{xx} v_m, \\ \partial_{tt} w_m = \partial_x \left(\left[\dfrac{T_s}{\rho \mathcal{A}} + \dfrac{\varrho^2 \mathcal{A}^2 E}{4 \rho T_s^2} (\ell - 2x)^2 \right] \partial_x w_m \right) \\ \qquad\qquad - \dfrac{x}{\ell} w_b'' + \dfrac{\varrho E_q \ell \mathcal{A}^2}{2 T_s^2} \left[\dfrac{x}{\ell} - \left(\dfrac{x}{\ell} \right)^2 \right] u_b''. \end{cases} \tag{8}$$

We want to define a standard infinite dimensional state space model, of the shape (10), of the robust control of this inclined cable. The goal is to prove that we can construct a feedback law that is robust to several perturbations to be defined later, such as model uncertainties (the non-linearities we are neglecting for instance) or traffic induced vibrations (deck movement).

We will come back later to the legitimate questions of existence and regularity of the solutions to the pde system, but one can observe that after the linearization of (6), we obtain a system (8) of two decoupled wave equations, whose solutions are well studied in applied analysis math literature. The existence of a unique solution relies for instance on semi-group theory (see [8]) and one can read in [13] some details on the w_m specific Sturm-Liouville operator equation.

B. Control and perturbations of the cable

We are considering an inclined cable as in Figure 1 connected at its bottom end b with an active tendon aligned with the cable and perturbed by in-plane oscilations (u_b, w_b). An active tendon consists of a displacement actuator (e.g. piezoelectric) collocated with a force sensor (see [3], [9], [11]). In this paper, we initiate the study of this device with the standard modelling of a state feedback controller. Nevertheless a tendon (our controller) is principally meant to have an axial movement, corresponding to a control u_c which is then an additive displacement term to the perturbation u_b in equation (5). But as mentioned in [9], if we only consider this *inertial* control, we only interact with the symmetric modes of vibration. And because of the linear framework, we lose the complementary *parametric* control. To overcome this issue, we will then consider that the control also acts through the in-plane bottom displacement as a term w_c which will be added to the perturbation w_b and we change (8) accordingly.

It can be noticed in the non-linear model (1) of the inclined cable, that there is a non-linear coupling between v and w through the dynamic tension T_d and when we linearized these equations, we completely lost this affect and obtain decoupled equations. As it is explained in [7] or [17] for instance, out-of-plane dynamic stability of inclined cables subjected to in-plane vertical support excitation is an ongoing subject of investigation that relies on the analysis of the non-linear model.

Here, because we choose to focus on studying the pde model and its robust control, we need to continue developing a linear standard model. Therefore, we can only consider the non-linearities as a perturbation of the model, that will be denoted by W_{mod} in Section IV. Moreover, it is our choice to let the pde in v_m evolve conservatively by itself, without any control and perturbation, and therefore to take it out of the modelling. The study will focus now on the robust control of the in-plane modal motion w_m and the final model reads

$$\partial_{tt} w_m = \partial_x \left(\left[\frac{T_s}{\rho \mathcal{A}} + \frac{\varrho^2 \mathcal{A}^2 E}{4 \rho T_s^2} (\ell - 2x)^2 \right] \partial_x w_m \right) \\ - \frac{x}{\ell}(w_b'' + w_c'') + \frac{\varrho E_q \ell \mathcal{A}^2}{2 T_s^2} \left[\frac{x}{\ell} - \left(\frac{x}{\ell} \right)^2 \right] (u_b'' + u_c''). \tag{9}$$

Nevertheless, we wanted to let the out-of-plane movement v_m be detailed in the model, because this study is also a first step toward more intricate situations where the decoupling between w_m and v_m may not be true any more.

III. H_∞ CONTROL WITH STATE-FEEDBACK

This section is devoted to recall the H_∞-robust control result proved in [2] (and also in [15]) that we want to apply to the pde model we have already presented. These results give an equivalence between the H_∞-robust control with state-feedback of an infinite-dimensional system and the resolution of a Riccati equation. We will explain later how they can be applied to the model we consider, following the presentation and assumptions detailed in [2].

The articles [1], [2] and [15] give a survey of the H_∞-control theory with state-feedback in the infinite-dimensional case. The dynamic measurement-feedback (or partial information case) for the same class of linear infinite-dimensional systems is also addressed in [2] and [14] and will be considered for the particular setting of the inclined cable controlled by an active tendon in a future study. The main results in all these articles is a generalization of the finite-dimensional regular H_∞-control problem (see for instance [6] and [12]) presented in a standard state-space approach. In particular, the solution is given in terms of the solvability of two Riccati equations.

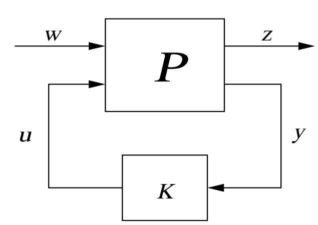

Fig. 2. Closed-loop system. State feedback: Y= X.

Let A be the infinitesimal generator of a C_0-semigroup $T(t) = e^{At}$ on a real separable Hilbert space $\mathcal{X}$ and let be the following linear bounded operators $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $D \in \mathcal{L}(\mathcal{W}, \mathcal{X})$ and $H \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ where $\mathcal{U}$ (space of controls), $\mathcal{W}$ (space of perturbations), and $\mathcal{Z}$ are also real separable Hilbert spaces. We consider the dynamic system governed by the equations, $\forall t \geq 0$

$$\begin{cases} X'(t) &= AX(t) + BU(t) + DW(t), \\ X(0) &= 0, \\ Z(t) &= HX(t) + U(t). \end{cases} \tag{10}$$

Here, $X(t) \in \mathcal{X}$ is the state of the system, $U(t) \in \mathcal{U}$ is the control input, $W(t) \in \mathcal{W}$ stands for the disturbance input and $Z(t) \in \mathcal{Z}$ is the "to be controlled output".

Let us begin by recalling the following usual definition (relying on a result of R. Datko [5]):

Definition 1: The pair A, B is stabilizable (ie $\exists F \in L(\mathcal{X},\mathcal{U})$ such that $A+BF$ generates an exponentially stable semigroup) if for any $h \in \mathcal{X}$, there exists $V \in L^2(0,\infty;\mathcal{U})$ such that the solution X of $X' = AX + BV$ with $X(0) = h$ satisfies $X \in L^2(0,\infty;\mathcal{X})$.
Moreover, we say that the pair A, H is detectable (*resp.* observable) if the pair A^*, H^* is stabilizable (*resp.* controllable).

Let us now precisely explain what is meant by H_∞-optimal control (or robust control) with state feedback. The standard description (10) of the system we consider implies that, as in Figure 2, the plant $\mathcal{P}$ is to be controlled under the cost function (related to the output Z)

$$K_0(U,W) = \int_0^\infty \left(|HX(t)|^2 + |U(t)|^2\right) dt$$

and the full observation of its state X. The goal is to construct a state-feedback controller $\mathcal{K}$ insuring that the influence of the unknown disturbances W on the "to be controlled output" Z is kept small, *i.e.* the ratio

$$\rho(U,W) = \frac{K_0(U,W)}{\int_0^\infty |W(t)|^2 dt}$$

shall be held below a fixed value by the controller $\mathcal{K}$, regardless of W.

We define $\mathcal{M}$ as the class of all applications μ from $\mathcal{X}$ to $\mathcal{U}$ that are such that

$$X' = AX + B\mu(X) + DW, \qquad X(0) = h$$

has a solution over $(0,\infty)$ for all $h \in \mathcal{X}$ and for all $W \in L^2(0,\infty)$, and such that the control $U = \mu(X)$ generated belongs also to $L^2(0,\infty)$.

The main result that we will apply to the dynamic control of an inclined cable is the following.

Theorem 1: [Proof to be read in [2] or [15]]
Let $\gamma > 0$ and assume that the pair A, B is stabilizable and that A, H is detectable. The following assertions are equivalent:

(i) The γ^2-robustness property with full observation holds for the system (10) under the cost function K_0, *i.e.*:

$$\inf_{\mu\in\mathcal{M}} \sup_{W\in L^2(0,\infty)} \rho(\mu(X),W) < \gamma^2,$$

(ii) There exists a nonnegative definite symmetric operator $P \in \mathcal{L}(X)$ solution of the Riccati equation

$$PA + A^*P + P(BB^* - \gamma^{-2}DD^*)P + H^*H = 0 \quad (11)$$

and $A - (BB^* - \gamma^{-2}DD^*)P$ generates an exponentially stable semigroup.

Moreover, in this case, the state feedback controller given by

$$\mu(X) \quad = \quad -B^*PX \quad (12)$$

gives an exponentially stable operator $A - BB^*P$ and guarantees that $\sup_W \rho(\mu(X),W) < \gamma^2$. Finally, if the solutions to the Riccati equation exist, then it is unique.

IV. State space model of an inclined cable's vibrations

A linear infinite-dimensional model derived from the partial differential equation model presented previously will be used in the sequel. In order to fit in the formalism presented in the previous subsection, the following notations are introduced:

- The state vector $X = (w_m, \partial_t w_m)$ where the data were defined in the introduction;
- The exogenous disturbance input $W = (W_{\text{mod}}, u_b'', w_b'')$ which represents the different types of perturbations on the model (uncertainty affecting dynamics of the model, non-linearities) and the perturbation from the deck movement (u_b, w_b);
- The control input $U = (u_c'', w_c'')$ is the acceleration vector of the active tendon actuator, set at the lower end of the cable;
- The controlled output Z is related to the cost functional K_0 and contains the modal vertical movement of the cable and the control amplitude ;

We consider the H_∞-control with state feedback of the inclined cable using a displacement actuator. In other words, we want the control u_c of the system of state $(w_m, \partial_t w_m)$ to be robust with respect to the perturbations corresponding to the deck movement (u_b, w_b) and model uncertainties (W_{mod}).

Therefore, the operators defining the standard form (10) are built from the linear partial differential equations we derived in Section II-A such that

$$A = \begin{pmatrix} 0 & I \\ \partial_x\left(a(x)\partial_x(\cdot)\right) & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ \alpha(x) & \beta(x) \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ d & \alpha(x) & \beta(x) \end{pmatrix}, H = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} a(x) &= \frac{T_s}{\rho\mathcal{A}} + \frac{\varrho^2\mathcal{A}^2E}{4\rho T_s^2}(\ell - 2x)^2 \geq \frac{T_s}{\rho\mathcal{A}} > 0, \\ \alpha(x) &= \frac{\varrho E_q \mathcal{A}^2\ell}{2T_s^2}\left[\frac{x}{\ell} - \left(\frac{x}{\ell}\right)^2\right], \quad \beta(x) = -\frac{x}{\ell}, \\ d &\in L^2(0,\ell). \end{aligned}$$

The appropriate functional Hilbert spaces associated with the infinite-dimensional model are now precisely defined. We consider the state space

$$\mathcal{X} = H_0^1(0,\ell) \times L^2(0,\ell)$$

and the output and input Hilbert spaces $\mathcal{W} = \mathbb{R}^3$, $\mathcal{U} = \mathbb{R}^2$, $\mathcal{Z} = (L^2(0,\ell))^2$.

In order to prove that A is the infinitesimal generator of a C_0-semigroup on the real separable Hilbert space $\mathcal{X}$, one can use the classical theory of semi-groups as in [8] or refer to the book [13] or the article [12]. Since the partial differential equation we have to deal with is a damped wave equation, we do not give detail of the proof here. Indeed, equation (9) can either be seen as a non-homogeneous wave equation with space dependent potential or as a Sturm-Liouville operator based equation (see [13], using that $a(x) \geq T_s/\rho\mathcal{A} > 0$). Either way, one can prove existence, uniqueness and regularity of the solution when $w_0 \in H_0^1(0,\ell)$, $w_1 \in L^2(0,\ell)$ and $u_b, w_b \in W^{2,\infty}(0,\infty)$: then $w \in C([0,\infty), H_0^1(0,\ell)) \cap C^1([0,\infty), L^2(0,\ell))$.

In addition, B, D and H are bounded operators well defined in the appropriate spaces allowing us to apply Theorem 1 if we can confirm that (A,B) is stabilizable and (A,H) is detectable.

The two main difficulties to prove that (A,B) is controllable in a time T large enough (implying stabilizability) come from the Sturm-Liouville operator and the fact that the control input has a prescribed shape in space (see B, where $\alpha(x)$ is such that we can't control the symmetric modes using only u_c'' as a control, e.g. [9]). One can refer to [13] where the Sturm-Liouville operator is considered in several chapters and one manages to obtain the desired proof that we do not wish to detail here. The fact that we have two controls u_c'' and w_c'' is in itself strong information. Therefore, (A,B) is stabilizable after a time T large enough. Finally, since HX contains w_m on the whole domain, one knows it implies observability and we obtain that (A,H) is detectable. This closes the verification of the assumptions of Theorem 1.

Therefore, considering that we now have a well-posed robust control problem in infinite dimension, we would like to perform some numerical experiments to illustrate the results we can obtain. Of course, we first need to define an appropriate finite-dimensional model.

V. A TRUNCATED MODEL FOR NUMERICAL DESIGN

A. Truncation

The goal of this section is to give the first step toward numerical experiments that will be presented in the final part of the article. The truncation of the pde system can be seen as a way of coming back to the structural vibrations of the system. The corresponding finite dimensional model of (10) can be presented as :

$$\begin{cases} X_N'(t) &= A_N X_N(t) + B_N U(t) + D_N W(t), \\ X_N(0) &= 0 \\ Z_N(t) &= H_N X_N(t) + U(t), \end{cases} \tag{13}$$

where the operators of system (10) are replaced by real-valued matrices computed on a truncated basis of the N first eigenfunctions precisely defined below. $X_N \in \mathbb{R}^{2N}$ is the state vector, $W \in \mathbb{R}^3$ is the exogenous perturbation vector, $U \in \mathbb{R}^2$ is the control vector and $Z_N \in \mathbb{R}^2$ is the controlled output vector. The matrices $A_N \in \mathcal{M}_{2N\times 2N}$, $B_N \in \mathcal{M}_{2N\times 2}$, $D_N \in \mathcal{M}_{2N\times 3}$, $H_N \in \mathcal{M}_{2\times 2N}$ are of appropriate dimensions.

In order to compute these objects, we choose to use everywhere the Hermitian basis of $L^2(0,\ell)$ given by the eigenfunctions of the (compact self-adjoint) operator $\frac{T_s}{\rho\mathcal{A}}\partial_{xx}$. The basis $(\phi_n)_{n\in\mathbb{N}^*}$ is defined by:

$$\phi_n(x) = \sqrt{\frac{2}{\ell}} \sin\left(n\pi\frac{x}{\ell}\right), \quad \omega_n = \frac{n\pi}{\ell}\sqrt{\frac{T_s}{\rho\mathcal{A}}}$$

and satisfies for all $x \in (0,\ell)$ and $n \in \mathbb{N}^*$,

$$\frac{T_s}{\rho\mathcal{A}}\partial_{xx}\phi_n(x) = -\omega_n^2\phi_n(x).$$

This approach meets the Galerkin method used in [16] (chapter 7) and the point is that

$$\forall y \in L^2(0,\ell),\ y(x) = \sum_{n\geq 1} y_n\phi_n(x),$$

$(y_n)_{n\in\mathbb{N}^*}$ being a sequence of real numbers satisfying $y_n = \langle y, \phi_n\rangle_{L^2} := \int_0^\ell y(x)\phi_n(x)\,dx$ and $\sum_{n\geq 1} y_n^2 < \infty$.

Given $N \in \mathbb{N}$, we compute A_N, B_N, D_N and H_N, using the truncated basis $\{\phi_1, \dots, \phi_N\}$. We make the assumption that the tuning parameter $d = (d_n)$ is a vector of real numbers and recall that it is a weighting function of the disturbance signal W_{mod} that corresponds for instance to the "forgotten" non-linearities. We obtain:

$$A_N = \text{block}_{n,m}\left(\begin{bmatrix} 0 & \delta_{nm} \\ a_{nm} & 0 \end{bmatrix}\right),$$

where δ_{nm} is the Kronecker delta symbol
and $a_{nm} = \langle \partial_x\left(a(x)\partial_x\phi_m\right), \phi_n\rangle_{L^2}$;

$$B_N = \text{vect}_n\left(\begin{bmatrix} 0 & 0 \\ \alpha_n & \beta_n \end{bmatrix}\right) \text{ and}$$

$$D_N = \text{vect}_n\left(\begin{bmatrix} 0 & 0 & 0 \\ d_n & \alpha_n & \beta_n \end{bmatrix}\right)$$

where $\alpha_n = \langle\alpha(x), \phi_n\rangle_{L^2}$ and $\beta_n = \langle\beta(x), \phi_n\rangle_{L^2}$ and finally

$$H_N = \left(\text{vect}_n\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right)^\top.$$

When computing all the terms a_{nm}, α_n and β_n, one will observe that we get terms different from zero in the non diagonal blocks of the matrices A_N, B_N and D_N. This is due to the choice of the eigenfunctions of the operator $\frac{T_s}{\rho\mathcal{A}}\partial_{xx}$ which is different from the Sturm-Liouville operator $\phi \mapsto \partial_x\left(a(x)\partial_x\phi\right)$. Reference [13] explains how to construct an Hermitian base of eigenfunctions of this operator, but we choose here an easier way to make the calculations, even if the price to pay are non diagonal terms.

The numerical simulations will be presented using the parameter values given in Table I that were chosen in [7] to approximately match a typical full-scale bridge cable

Cable length	ℓ	1.98 m
Density	ρ	1.34×10^6 kg.m^{-3}
Cross sectional area	$\mathcal{A}$	0.5×10^{-6} m^3
Static tension	T_s	205 N
Steel Young's modulus	E	200×10^9 N.m^{-2}

TABLE I
CABLE CHARACTERISTICS

inclined at $\theta = 20°$ to the horizontal, of length 400m, mass per unit length 130kg.m^{-1} and tension 8000kN:
These values correspond to an inclined steel cable experiment that could be used in a next step to implement our controllers.

We can illustrate the results of the closed loop control based on this truncated state model of the PDE modelling in several ways. Through the singular values of the frequency response, Figure 3 presents the attenuation of the first modes of vibrations for the case $N = 5$, with respect to the uncontrolled open loop. The H_∞ optimal controller is computed using the *hinfsyn* Matlab function.

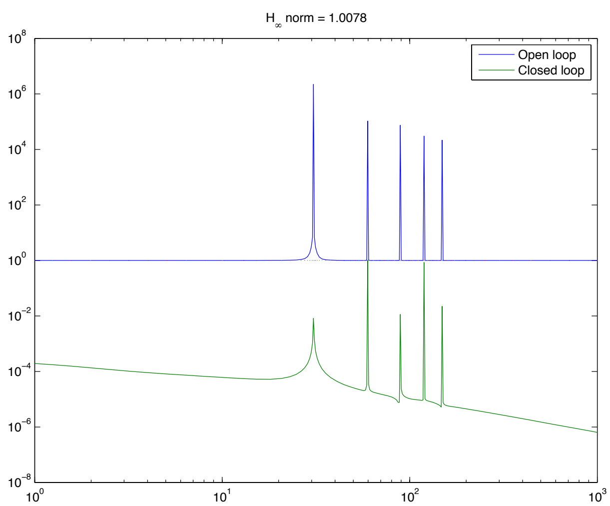

Fig. 3. Singular values of the frequency response of the truncated model in open and closed loop.

VI. DISCUSSION AND CONCLUSION

The goal of this article was to present an infinite-dimensional state space model of an inclined cable in order to study the robust control of the corresponding system. After dealing with several modelling issues, we were able to ensure the H_∞-control of the infinite dimensional system under the condition of solvability of a Riccati equation, thanks to the result of [2] and [15]. This allowed us to perform some numerical computations on the truncated model in order to illustrate the action of the robust controller. We examined the possibility of connecting the inclined cable to an active tendon in order to bring active damping into cable structure and as far as we know, there exists no such study of the robust control of an inclined cable when the partial differential model is used.

We can mention several new directions to follow that we intend to pursue as future extensions of this study. To begin with, when linearizing the model, we lost the coupling between v_m and w_m. A first objective is to find a way, maybe using the dynamic tension, to recover a link between them in the system we construct. Then, one could be interested in studying the feedback control with partial observation, using now the active tendon as a sensor (in addition to it's actuator function). We hope to obtain good results when considering measurement feedback control with an active tendon since the collocation of actuator and sensor has proved great effectiveness in active damping of cables [3], [11]. Moreover, a next step of this preliminary study would be to compare the control result we manage to obtain here applied to the non-linear model of 2 or more modes that one can read in [16]. This will be covered in future studies of this system.

REFERENCES

[1] V. Barbu. H_∞ boundary control with state feedback: the hyperbolic case. *SIAM J. Control Optim.*, 33(3):684–701, 1995.

[2] A. Bensoussan and P. Bernhard. On the standard problem of H_∞-optimal control for infinite-dimensional systems. In *Identification and control in systems governed by partial differential equations (South Hadley, MA, 1992)*, pages 117–140. SIAM, Philadelphia, PA, 1993.

[3] F Bossens and A Preumont. Active tendon control of cable-stayed bridges: a large-scale demonstration. *Earthquake Engineering & Structural Dynamics*, 30(7):961–979, JUL 2001.

[4] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[5] Richard Datko. Extending a theorem of A. M. Liapunov to Hilbert space. *J. Math. Anal. Appl.*, 32:610–616, 1970.

[6] J. C. Doyle, K. Zhou, and K. Glover. *Robust and Optimal Control Robust and Optimal Control Robust and Optimal Control*. Prentice Hall, 1996.

[7] A. Gonzalez-Buelga, S.A. Neild, D.J. Wagg, and J.H.G. Macdonald. Modal stability of inclined cables subjected to vertical support excitation. *Journal of Sound and Vibration*, 318(3):565 – 579, 2008.

[8] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[9] André Preumont. *Vibration control of active structures*, volume 50 of *Solid Mechanics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997. An introduction.

[10] Andre J. Preumont and Frederic Bossens. Active tendon control of cable-stayed bridges. *SPIE proceedings series*, Smart Structures and Materials:188–198, 2000.

[11] G. Song, V. Sethi, and H.-N. Li. Vibration control of civil structures using piezoceramic smart materials: A review. *Engineering Structures*, 28(11):1513 – 1524, 2006.

[12] G. Tadmor. Worst-case design in the time domain: the maximum principle and the standard H_∞ problem. *Math. Control Signals Systems*, 3(4):301–324, 1990.

[13] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*, volume XI of *Birkäuser Advanced Texts*. Springer, 2009.

[14] B. van Keulen. H_∞-control with measurement-feedback for linear infinite-dimensional systems. *J. Math. Systems Estim. Control*, 3(4):373–411, 1993.

[15] B. van Keulen, M. Peters, and R. Curtain. H_∞-control with state-feedback: the infinite-dimensional case. *J. Math. Systems Estim. Control*, 3(1):1–39, 1993.

[16] D. Wagg and S. Neild. *Nonlinear Vibration with Control: For Flexible and Adaptive Structures*. Solid mechanics and its applications. Springer, 2010.

[17] P. Warnitchai, Y. Fujino, and T. Susumpow. A non-linear dynamic model for cables and its application to a cable structure system. *Journal of Sound and Vibration*, 187(4):695 – 712, 1995.