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Computing the $L_\infty[0,h)$-induced norm of a compression operator

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***Abstract*— This paper is concerned with the computation of the induced norm of a compression operator defined on the Banach space $L_\infty[0,h)$, which is a difficult problem because it is an infinite-rank operator. This paper provides two methods for this problem, each of which can compute an upper bound and a lower bound of the induced norm by using an idea of staircase or piecewise linear approximation. Another key idea in both methods is to apply fast-lifting by which the interval $[0,h)$ is divided into M subintervals with equal width, and the computation errors in these methods are ensured to be reciprocally proportional to M or M^2. The effectiveness of the proposed methods is demonstrated through a numerical example.**

I. INTRODUCTION

Infinite-rank operators called compression operators [1] always appear in the lifting approach [2] to sampled-data systems [3],[4] and time-delay systems [5]. In view of the successes in the studies on the lifting approach to such systems and towards its further development, it is very important to study the properties of compression operators. In this paper, we consider the induced-norm as one of the most important properties of compression operators. While computing the induced-norm of a compression operator has been discussed in [1] when the operator is defined on the Hilbert space $L_2[0,h]$, we consider computing the induced-norm of the compression operator defined on the Banach space $L_\infty[0,h)$. Such a study is indeed expected to shed a new light on, e.g., the L_∞-induced norm problem of sampled-data systems [6],[7] and the monodromy operator approach to time-delay systems. A method for computing the $L_\infty[0,h)$-induced norm of the compression operator has been provided in [6] by applying the fast-sample/fast-hold (FSFH) approximation technique [8]; without essential loss of generality, only the case when the compression operator is associated with a single-input single-output (SISO) linear time-invariant (LTI) system is dealt with. This method takes M equally spaced points on the interval $[0,h)$, and the relation between the input and output sampled at these points is described by a matrix. Then, the ∞-norm of this matrix is ensured to converge to the L_∞-norm of the compression operator in the order of $1/M$. However, no explicit upper and lower bounds for the norm are given for each M.

In this paper, we provide two methods for computing the $L_\infty[0,h)$-induced norm of the compression operator associated with a multi-input multi-output (MIMO) system. Instead of taking the standpoint of FSFH approximation, we make use of the fast-lifting technique [9]. The latter technique also introduces an integer M as in the FSFH approximation technique, but it is used only to subdivide the interval $[0,h)$ into smaller pieces. In other words, no information is lost as to the function on $[0,h)$ because we do not sample the values of the function at M points in this interval but we only regard it as an M collection of functions defined over a smaller interval. Such a viewpoint allows us to approximate the compression operator directly with another tractable operator, while the FSFH technique inevitably introduces a matrix from the outset to 'approximate' the compression operator.

We first study a staircase (or piecewise constant) approximation type of arguments, and derive a result for the computation of the $L_\infty[0,h)$-induced norm of the compression operator in a very straightforward way. This result shows that the $L_\infty[0,h)$-induced norm can be approximated by the ∞-norm of a suitably constructed matrix, and an upper bound and a lower bound of the induced norm can also be obtained, which can be computed readily. Even though the arguments proceed in a different way, the result can also be regarded as a direct MIMO extension of the existing one with the FSFH technique, in which the computation error is ensured to converge to 0 in the order of $1/M$. We then study a piecewise linear approximation type of arguments, which can be easily applied under our fast-lifting treatment. This is because it does not introduce sampling of signals and thus we are viewing the signals themselves even after we have introduced the integer M. Through such an approximation technique, we derive another result for the computation of the $L_\infty[0,h)$-induced norm of the compression operator, in which the induced norm is approximated by the ∞-norm of another matrix. A computable upper bound as well as a lower bound are also given for the induced norm, and it is shown that the computation error tends to 0 in the order of $1/M^2$. Finally, we demonstrate the effectiveness of these computation methods through a numerical example.

In the following, we use the notations $\mathbb{N}$, $\mathbb{R}$ and $\mathcal{K}_\nu$ to denote the set of positive integers, the set of real numbers and $(L_\infty[0,h))^\nu$, respectively. The notation $\|\cdot\|$ is used to mean the either $L_\infty[0,T)$ norm of a matrix function (where T is either h or an integer fraction of it), i.e.,

$$\|F(\cdot)\| := \max_i \sup_{0\le t<T} \sum_j |F_{ij}(t)| \tag{1}$$

the $L_\infty[0,T)$-induced norm of an operator or the ∞-norm of a finite-dimensional matrix, whose distinction will be clear

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from the context.

II. Compression Operator and Its Fast-Lifting Treatment

Let us consider the linear-time invariant system described by

$$\frac{dx}{dt} = Ax + Bu,\ y = Cx \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$ and $C \in \mathbb{R}^{n_y\times n}$. Suppose we consider a sampled-data system with the sampling period h or a time-delay feedback system with the delay length h involving the above finite-dimensional system (in the latter case, we would assume $n_u = n_y$). When we take the lifting approach to such a system assuming that continuous-time signals belong to L_∞, we are led to dealing with the compression operator $\mathbf{D} : \mathcal{K}_{n_u} \to \mathcal{K}_{n_y}$ associated with the system (2) defined by

$$(\mathbf{D}u)(\theta) = \int_0^\theta C\exp(A(\theta-\tau))Bu(\tau)d\tau \tag{3}$$

where $0 \le \theta < h$. In addition, the lifting approach always leads us to dealing with two other finite-rank operators $\mathbf{B} : \mathcal{K}_{n_u} \to \mathbb{R}^n$ and $\mathbf{C} : \mathbb{R}^n \to \mathcal{K}_{n_y}$ associated with (2) defined as follows:

$$\mathbf{B}u = \int_0^h \exp(A(h-\theta))Bu(\theta)d\theta \tag{4}$$

$$(\mathbf{C}x)(\theta) = C\exp(A\theta)x \tag{5}$$

Even though they are not directly related to the compression operator $\mathbf{D}$ itself, very closely related operators $\mathbf{B}'$ and $\mathbf{C}'$ will play a significant role in the computation of the $L_\infty[0,h)$-induced norm $\|\mathbf{D}\|$. The above operators $\mathbf{B}'$ and $\mathbf{C}'$ are defined as $\mathbf{B}$ and $\mathbf{C}$, respectively, with the horizon $[0,h)$ replaced by $[0,h')$, where $h' := h/M$. Here, $M \in \mathbb{N}$ is called the fast-lifting parameter, which is introduced for the purpose of reducing the computation error of $\|\mathbf{D}\|$ (as M increases).

Now, to see how $\mathbf{B}'$ and $\mathbf{C}'$ are related to $\mathbf{D}$, let us review fast-lifting with the parameter M [9]. Let us denote $(L^\infty[0,h'))^\nu$ by $\mathcal{K}'_\nu$. For $f \in \mathcal{K}_\nu$, we define $f^{(i)} \in \mathcal{K}'_\nu$ $(i = 1,\cdots,M)$ by

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \quad (0 \le \theta' < h') \tag{6}$$

and further define $\check{f} := [(f^{(1)})^T\ \cdots\ (f^{(M)})^T]^T$. We call the mapping from $f \in \mathcal{K}_\nu$ to $\check{f} \in (\mathcal{K}'_\nu)^M$ fast-lifting, and denote it by $\check{f} = \mathbf{L}_M f$. It readily follows from the definition of $\mathbf{L}_M$ that

$$\|\mathbf{D}\| = \|\mathbf{L}_M\mathbf{D}\mathbf{L}_M{}^{-1}\| \tag{7}$$

where $\|\cdot\|$ on the right hand side denotes the induced norm on $(\mathcal{K}'_\nu)^M$. It is easy to see that the operator $\mathbf{L}_M\mathbf{D}\mathbf{L}_M{}^{-1}$ is described by

$$\mathbf{L}_M\mathbf{D}\mathbf{L}_M{}^{-1} = \overline{\mathbf{C}'}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_d)^{M-2} & \cdots & I & 0 \end{bmatrix}\overline{\mathbf{B}'} + \overline{\mathbf{D}'} \tag{8}$$

where $\mathbf{D}'$ is defined as $\mathbf{D}$ with $[0,h)$ replaced by $[0,h')$, the matrix A'_d is defined as $\exp(Ah')$, and $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot),\cdots,(\cdot)]$ consisting of M copies of $(\cdot)$.

It is difficult to compute $\|\mathbf{D}\|$ exactly since $\mathbf{D}$ is an infinite-rank operator between function spaces. However, suppose we are to compute it by computing $\|\mathbf{L}_M\mathbf{D}\mathbf{L}_M^{-1}\|$ instead. We then see that the first term on the right hand side of (8) is of finite-rank, while the second term consists of $\mathbf{D}'$, which is still of infinite-rank as the original $\mathbf{D}$ is, but is defined on a smaller interval than $\mathbf{D}$. It is hence expected that the difficulty of dealing with the infinite-rank operator $\mathbf{D}$ is somewhat alleviated by working instead on the fast-lifted counterpart. This paper aims at computing lower bounds and upper bounds of the $L_\infty[0,h)$-induced norm of $\mathbf{D}$ through staircase and piecewise linear approximations of $\mathbf{B}'$, $\mathbf{C}'$ and $\mathbf{D}'$.

III. Computation of $\|\mathbf{D}\|$ through Staircase and Piecewise Linear Approximations

This section gives the main results of this paper, i.e., two methods for computing $\|\mathbf{D}\|$ through staircase and piecewise linear approximations, and shows the associated convergence rates in M.

A. Staircase approximation

With the averaging operator $\mathbf{J}'_0$ defined by

$$(\mathbf{J}'_0 u)(\theta') = \frac{1}{h'}\int_0^{h'} u(\tau')d\tau' \tag{9}$$

for $0 \le \theta' < h'$, we introduce the operator $\mathbf{B}'_{p0} : \mathcal{K}'_{n_u} \to \mathbb{R}^n$ defined as $\mathbf{B}'_{p0} := \mathbf{B}'\mathbf{J}'_0$, or equivalently,

$$\mathbf{B}'_{p0}u = \int_0^{h'} \exp(A(h'-\theta'))B\cdot(\mathbf{J}'_0 u)(\theta')d\theta' \tag{10}$$

In other words, introducing the operator $\mathbf{B}'_{p0}$ corresponds to restricting the input of $\mathbf{B}'$ to constant functions and that $\mathbf{B}'_{p0}u = \mathbf{B}'u$ whenever u is a constant function. On the other hand, we introduce the operator $\mathbf{C}'_{p0}$ defined by

$$\left(\mathbf{C}'_{p0}x\right)(\theta') = Cx \quad (0 \le \theta' < h') \tag{11}$$

whose output is a constant function corresponding to the zero-order approximation of $(\mathbf{C}'x)(\theta') = C\sum_{i=0}^{\infty}\frac{1}{i!}(A\theta')^i x$.

We next consider the operator $\mathbf{D}_{M0}$ obtained by replacing $\mathbf{B}'$ and $\mathbf{C}'$ with $\mathbf{B}'_{p0}$ and $\mathbf{C}'_{p0}$, respectively, and ignoring $\mathbf{D}'$

in (8). In other words, $\mathbf{D}_{M0}$ is given by

$$\mathbf{D}_{M0} = \overline{\mathbf{C}'_{p0}} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_d)^{M-2} & \cdots & I & 0 \end{bmatrix} \overline{\mathbf{B}'_{p0}} \tag{12}$$

One of the main results of this paper is to show that $\|\mathbf{D}_{M0}\|$ can be computed exactly and tends to $\|\mathbf{D}\|$ as $M \to \infty$. The following two lemmas are important in establishing the latter fact.

Lemma 1: The inequality

$$\left\|\mathbf{C}'\left(A'_d\right)^k \mathbf{B}' - \mathbf{C}'_{p0}\left(A'_d\right)^k \mathbf{B}'_{p0}\right\| \le \frac{h^2}{M^2}\|A\|\cdot\|B\| e^{\|A\|h/M} \cdot \left(\left\|C(A'_d)^{k+1}\right\| + \left\|C(A'_d)^k\right\| e^{\|A\|h/M}\right) \tag{13}$$

holds for $k = 0, \cdots, M-2$.

Lemma 2: The following inequality holds:

$$\|\mathbf{D}'\| \le \frac{h}{M}\|C\|\cdot\|B\| e^{\|A\|h/M} \tag{14}$$

From Lemmas 1 and 2, we can obtain the following theorem.

Theorem 3: The inequality

$$\left\|\mathbf{L}_M \mathbf{D} \mathbf{L}_M{}^{-1} - \mathbf{D}_{M0}\right\| \le \frac{K_{M0}}{M} \tag{15}$$

holds with K_{M0} defined as

$$K_{M0} := h\|C\|\cdot\|B\| e^{\|A\|h/M} + \frac{h^2}{M}\|A\|\cdot\|B\| e^{\|A\|h/M} \cdot \sum_{k=0}^{M-2}\left\{\left\|C(A'_d)^{k+1}\right\| + \left\|C(A'_d)^k\right\| e^{\|A\|h/M}\right\} \tag{16}$$

Furthermore, K_{M0} has a uniform upper bound with respect to M given by

$$K_0^U := h\|C\|\cdot\|B\| e^{\|A\|h} + h^2\|C\|\cdot\|B\|\cdot\|A\| e^{2\|A\|h}\left(1 + e^{\|A\|h}\right) \tag{17}$$

Since $\|\mathbf{D}\| = \|\mathbf{L}_M \mathbf{D} \mathbf{L}_M^{-1}\|$, we readily have the following result.

Corollary 4: The following inequality holds:

$$\|\mathbf{D}_{M0}\| - \frac{K_{M0}}{M} \le \|\mathbf{D}\| \le \|\mathbf{D}_{M0}\| + \frac{K_{M0}}{M} \tag{18}$$

This implies that an upper bound and a lower bound of $\|\mathbf{D}\|$ can be obtained if $\|\mathbf{D}_{M0}\|$ can be computed, and as the fast-lifting parameter M becomes larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M$ (since K_{M0} has a uniform upper bound K_0^U).

Hence, the remaining task is to provide a method for computing $\|\mathbf{D}_{M0}\|$. Since $\|u\| \ge \|\mathbf{J}'_0 u\|$ whenever $u \in L_\infty[0, h')$ and since $\mathbf{J}'_0 u$ is a constant function, it follows immediately from (12) and $\mathbf{B}'_{p0} = \mathbf{B}'\mathbf{J}'_0$ that the input u of $\mathbf{D}_{M0}$ may always be assumed to be a constant function when we compute $\|\mathbf{D}_{M0}\|$. Again by (12) together with (11), it is also obvious that the output of $\mathbf{D}_{M0}$ is a constant function. Hence, we are immediately led to the conclusion that $\|\mathbf{D}_{M0}\|$ coincides with the ∞-norm of a finite-dimensional matrix. More precisely, it follows from (10) and (11) that $\|\mathbf{D}_{M0}\|$ coincides with the ∞-norm of the matrix obtained by replacing the operators $\mathbf{B}'_{p0}$ and $\mathbf{C}'_{p0}$ in (12) with the matrices B'_{0d} and C, respectively, where

$$B'_{0d} := \int_0^{h'} \exp\left(A\left(h' - \theta'\right)\right) B d\theta' \tag{19}$$

The resulting matrix, however, is a Toeplitz matrix, and thus its ∞-norm coincides with that of the last block row, i.e.,

$$P_{M0} := \begin{bmatrix} C\left(A'_d\right)^{M-2} B'_{0d} & \cdots & CB'_{0d} & 0 \end{bmatrix} \tag{20}$$

Summarizing the above arguments, we are led to the following result, which is the first main result in this paper.

Theorem 5: The inequality

$$\|P_{M0}\| - \frac{K_{M0}}{M} \le \|\mathbf{D}\| \le \|P_{M0}\| + \frac{K_{M0}}{M} \tag{21}$$

holds, and K_{M0} has a uniform upper bound K_0^U given by (17).

Remark 1: By a close inspection on the arguments, we can show that the lower bound of $\|\mathbf{D}\|$ in (21) can in fact be replaced simply by $\|P_{M0}\|$. Similarly for (18).

B. Piecewise linear approximation

We first introduce the 'linearizing' operator $\mathbf{J}'_1$ defined as

$$(\mathbf{J}'_1 u)(\theta') = \int_0^{h'} f_0(\tau')u(\tau')d\tau' + \theta' \int_0^{h'} f_1(\tau')u(\tau')d\tau' \tag{22}$$

with the scalar-valued functions $f_0(\tau')$ and $f_1(\tau')$ given by

$$f_0(\tau') = -\frac{6}{(h')^2}\tau' + \frac{4}{h'} \tag{23}$$

$$f_1(\tau') = \frac{12}{(h')^3}\tau' - \frac{6}{(h')^2} \tag{24}$$

By using the operator $\mathbf{J}'_1$, let us introduce the operator $\mathbf{B}'_{p1} := \mathbf{B}'\mathbf{J}'_1$, i.e.,

$$\mathbf{B}'_{p1} u = \int_0^{h'} \exp\left(A\left(h' - \theta'\right)\right) B \cdot (\mathbf{J}'_1 u)(\theta') d\theta' \tag{25}$$

Introducing the operator $\mathbf{B}'_{p1}$ is equivalent to restricting the input of $\mathbf{B}'$ to linear functions. In addition, we can see that $\mathbf{J}'_1 u = u$ and thus $\mathbf{B}'_{p1} u = \mathbf{B}' u$ whenever u is a linear function. Such functions f_0 and f_1 that lead to these properties alone have a large freedom; the particular functions in (23) and (24) are chosen specifically so that they further lead to the property in Lemma 6 given shortly, but the detailed arguments are omitted. We further introduce the operators $\mathbf{C}'_{p1}$ and $\mathbf{D}'_{p1}$ defined by

$$\left(\mathbf{C}'_{p1} x\right)(\theta') = C\left(I + A\theta'\right) x \quad (0 \le \theta' < h') \tag{26}$$

$$\left(\mathbf{D}'_{p1} u\right)(\theta') = CB \int_0^{\theta'} u(\tau')d\tau' \quad (0 \le \theta' < h') \tag{27}$$

Note that the output of $\mathbf{C}'_{p1}$ is a linear function corresponding to the first-order approximation of $(\mathbf{C}'x)(\theta') = C\sum_{i=0}^{\infty}\frac{1}{i!}(A\theta')^i x$.

We next consider the operator $\mathbf{D}_{M1}$ obtained by replacing $\mathbf{B}'$, $\mathbf{C}'$ and $\mathbf{D}'$ with $\mathbf{B}'_{p1}$, $\mathbf{C}'_{p1}$ and $\mathbf{D}'_{p1}$, respectively, in (8). In other words, $\mathbf{D}_{M1}$ is given by

$$\mathbf{D}_{M1} = \overline{\mathbf{C}'_{p1}} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_d)^{M-2} & \cdots & I & 0 \end{bmatrix} \overline{\mathbf{B}'_{p1}} + \overline{\mathbf{D}'_{p1}} \tag{28}$$

In this section, we show that $\|\mathbf{D}_{M1}\|$ can be computed exactly and converges to $\|\mathbf{D}\|$ as $M \to \infty$. The following two lemmas are significant in establishing the latter fact.

Lemma 6: The inequality

$$\begin{aligned} &\left\|\mathbf{C}'\left(A'_d\right)^k \mathbf{B}' - \mathbf{C}'_{p1}\left(A'_d\right)^k \mathbf{B}'_{p1}\right\| \\ &\qquad \leq \frac{1}{2}\|A\|^2 \cdot \|B\| e^{\|A\|h/M} \cdot \frac{h^3}{M^3} \\ &\cdot \left(\left\|C(A'_d)^k\right\| e^{\|A\|h/M} + \sup_{0\leq\theta'<h'} \left\|C(I+A\theta')\left(A'_d\right)^{k+1}\right\|\right) \end{aligned} \tag{29}$$

holds for $k = 0.\cdots, M-2$

Lemma 7: The following inequality holds:

$$\|\mathbf{D}' - \mathbf{D}'_{p1}\| \leq \frac{1}{2}\|C\|\cdot\|A\|\cdot\|B\|\frac{h^2}{M^2}e^{\|A\|h/M} \tag{30}$$

From Lemmas 6 and 7, we can obtain the following theorem.

Theorem 8: The inequality

$$\left\|\mathbf{L}_M \mathbf{D} \mathbf{L}_M{}^{-1} - \mathbf{D}_{M1}\right\| \leq \frac{K_{M1}}{M^2} \tag{31}$$

holds with K_{M1} defined as

$$\begin{aligned} K_{M1} &:= \frac{1}{2}\|A\|^2 \cdot \|B\| e^{\|A\|h/M} \frac{h^3}{M} \\ &\cdot \sum_{k=0}^{M-2} \left\{\left\|C(A'_d)^k\right\| e^{\|A\|h/M} + \sup_{0\leq\theta'<h'}\left\|C(I+A\theta')(A'_d)^{k+1}\right\|\right\} \\ &\qquad + \frac{1}{2}\|C\|\cdot\|A\|\cdot\|B\|h^2 e^{\|A\|h/M} \end{aligned} \tag{32}$$

Furthermore, K_{M1} has a uniform upper bound with respect to M given by

$$\begin{aligned} K_1^U &:= \frac{1}{2}\|C\|\cdot\|A\|^2\cdot\|B\|e^{2\|A\|h}h^3\left(e^{\|A\|h} + 1 + \|A\|h\right) \\ &\qquad + \frac{1}{2}\|C\|\cdot\|A\|\cdot\|B\|h^2 e^{\|A\|h} \end{aligned} \tag{33}$$

We readily have the following result in the same way as in staircase approximation.

Corollary 9: The following inequality holds:

$$\|\mathbf{D}_{M1}\| - \frac{K_{M1}}{M^2} \leq \|\mathbf{D}\| \leq \|\mathbf{D}_{M1}\| + \frac{K_{M1}}{M^2} \tag{34}$$

This shows that an upper bound and a lower bound of $\|\mathbf{D}\|$ can be obtained if $\|\mathbf{D}_{M1}\|$ can be computed, and by taking the fast-lifting parameter M larger, the gap between the upper and lower bounds tends to 0 at no slower convergence rate than $1/M^2$ (because K_{M1} has a uniform upper bound K_1^U).

We next provide a method for computing $\|\mathbf{D}_{M1}\|$. Since $\mathbf{D}_{M1}$ has a Toeplitz structure by (28), it follows readily that its $L_\infty[0,h)$-induced norm coincides with the induced norm of the its last block row, i.e.,

$$\mathbf{P}_{M1} = \begin{bmatrix}\mathbf{C}'_{p1}\left(A'_d\right)^{M-2}\mathbf{B}'_{p1} & \cdots & \mathbf{C}'_{p1}\mathbf{B}'_{p1} & \mathbf{D}'_{p1}\end{bmatrix} \tag{35}$$

The latter induced norm is given by

$$\|\mathbf{P}_{M1}\| = \sup_{\|u\|\leq 1}\sup_{0\leq\theta'<h'}\|(\mathbf{P}_{M1}u)(\theta')\| \tag{36}$$

and we note on the right hand side that

$$\begin{aligned} (\mathbf{P}_{M1}u)(\theta') = \sum_{i=1}^{M-1}&\left(\mathbf{C}'_{p1}\left(A'_d\right)^{M-1-i}\mathbf{B}'_{p1}u^{(i)}\right)(\theta') \\ &+ \left(\mathbf{D}'_{p1}u^{(M)}\right)(\theta') \end{aligned} \tag{37}$$

$$[(u^{(1)})^T, \cdots, (u^{(M)})^T]^T := \mathbf{L}_M u \tag{38}$$

By the definition of $\mathbf{B}'_{p1}$ and $\mathbf{C}'_{p1}$, the matrix function $\left(\mathbf{C}'_{p1}\left(A'_d\right)^{M-1-i}\mathbf{B}'_{p1}u^{(i)}\right)(\theta')$ in (37) is also described explicitly by the linear function

$$\begin{aligned} &\left(\mathbf{C}'_{p1}\left(A'_d\right)^{M-1-i}\mathbf{B}'_{p1}u^{(i)}\right)(\theta') \\ =&(H_{i0} + H_{i1}\theta')\int_0^{h'}\left(G_0 + G_1\tau'\right)u^{(i)}(\tau')d\tau' \end{aligned} \tag{39}$$

where the matrices H_{i0}, H_{i1}, G_0 and G_1 are defined as

$$H_{i0} := C\left(A'_d\right)^{M-1-i}, \quad i = 1,\cdots,M-1 \tag{40}$$

$$H_{i1} := CA\left(A'_d\right)^{M-1-i}, \quad i = 1,\cdots,M-1 \tag{41}$$

$$G_0 := -\frac{6}{(h')^2}B'_{1d} + \frac{4}{h'}B'_{0d} \tag{42}$$

$$G_1 := \frac{12}{(h')^3}B'_{1d} - \frac{6}{(h')^2}B'_{0d} \tag{43}$$

through the matrices B'_{0d} and

$$B'_{1d} := \int_0^{h'}\exp\left(A(h'-\theta')\right)\theta' B d\theta' \tag{44}$$

On the other hand, let us consider $\mathbf{D}'_{p1}u^{(M)}$ in (37) for a given $u^{(M)}$. It is easy to see from the definition of $\mathbf{D}'_{p1}$ that whenever $\theta' \in [0,h')$ is fixed, there exists a constant function $\widetilde{u}$ on $[0,h')$ such that $(\mathbf{D}'_{p1}u^{(M)})(\theta') = (\mathbf{D}'_{p1}\widetilde{u})(\theta')$ and $\|\widetilde{u}\| \leq \|u^{(M)}\|$; the constant value of $\widetilde{u}$ is given by the average of $u^{(M)}$ over $[0,\theta']$. This implies that $u^{(M)}$ may be assumed, without loss of generality, to be a constant function when we consider $\sup_{\|u\|\leq 1}\sup_{0\leq\theta'<h'}$ in (36). Then, the output $\mathbf{D}'_{p1}u^{(M)}$ becomes also a linear function, as is the case with the first term in (37). Hence, (36) simplifies to

$$\begin{aligned} \|\mathbf{P}_{M1}\| &= \sup_{\|u^{(i)}\|\leq 1}\max_{\theta'=0,h'}\|(\mathbf{P}_{M1}u)(\theta')\| \\ &= \sup_{\|u^{(i)}\|\leq 1}\left\|\begin{bmatrix}(\mathbf{P}_{M1}u)(0) \\ (\mathbf{P}_{M1}u)(h')\end{bmatrix}\right\| \end{aligned} \tag{45}$$

where $(\mathbf{P}_{M1}u)(h')$ is defined by continuity of a linear function (assuming that $u^{(M)}$ is a constant function).

It follows from (39) together with (27) that $(\mathbf{P}_{M1}u)(0)$ is determined by the mapping

$$u^{(i)} \mapsto \int_0^{h'} (S_{i0}^{(0)} + S_{i1}^{(0)}\tau')u(\tau')d\tau' \tag{46}$$

while $(\mathbf{P}_{M1}u)(h')$ is determined by the mapping

$$u^{(i)} \mapsto \int_0^{h'} (S_{i0}^{(h')} + S_{i1}^{(h')}\tau')u(\tau')d\tau' \tag{47}$$

and $CBh'u^{(M)}$ (provided that $u^{(M)}$ is constant), where

$$S_{i0}^{(0)} = H_{i0}G_0,\ S_{i1}^{(0)} = H_{i0}G_1 \tag{48}$$

$$S_{i0}^{(h')} = (H_{i0} + H_{i1}h')G_0,\ S_{i1}^{(h')} = (H_{i0} + H_{i1}h')G_1 \tag{49}$$

Let $T_i^{(0)}$ be the matrix consisting of the $L_1[0, h')$ norm of each entry of the matrix function $S_{i0}^{(0)} + S_{i1}^{(0)}\tau'$ involved in (46). Similarly, let $T_i^{(h')}$ be the matrix constructed in the same way from $S_{i0}^{(h')} + S_{i1}^{(h')}\tau'$ involved in (47). Note that each $L_1[0, h')$ norm can readily be computed exactly, because we only need to deal with linear functions. We can now readily see from (45) that $\|\mathbf{P}_{M1}\|$ coincides with the ∞-norm of the finite-dimensional matrix P_{M1} given by

$$P_{M1} := \begin{bmatrix} T_1^{(0)} & \cdots & T_{M-1}^{(0)} & 0 \\ T_1^{(h')} & \cdots & T_{M-1}^{(h')} & CBh' \end{bmatrix} \tag{50}$$

Summarizing the above arguments, we are led to the following results, which is the second main result in this paper.

Theorem 10: The inequality

$$\|P_{M1}\| - \frac{K_{M1}}{M^2} \le \|\mathbf{D}\| \le \|P_{M1}\| + \frac{K_{M1}}{M^2} \tag{51}$$

holds, and K_{M1} has a uniform upper bound K_1^U given by (33).

IV. Numerical Example

In this section, we examine through a numerical example the validity of the two approximation methods developed in the preceding section.

We consider the compression operator $\mathbf{D}$ associated with

$$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & -\sqrt{5} \end{bmatrix} \tag{52}$$

and $h = \sigma/2$, where $\tan(\sigma) = 2, 0 \le \sigma \le \pi$. This corresponds to an example in which the time-delay feedback system consisting of $C(sI - A)^{-1}B$ and e^{-sh} is in marginal stability. We compute estimates of $\|\mathbf{D}\|$ with the two methods by taking the fast-lifting parameter M ranging from 50 to 500. The results of the estimate $\|\mathbf{D}_{M0}\|$, the error bound K_{M0}/M and the computation time corresponding to the staircase approximation method are shown in Table I. In addition, the results of $\|\mathbf{D}_{M1}\|$, K_{M1}/M^2 and the computation time are shown in Table II.

We can see from Tables I and II that the error bounds for the computation of $\|\mathbf{D}\|$ through its estimates $\|\mathbf{D}_{M0}\|$ and $\|\mathbf{D}_{M1}\|$, which are given by K_{M0}/M and K_{M1}/M^2, respectively, are decreasing by taking M larger. Hence, we can confirm the validity of the staircase and piecewise linear approximation methods for computing the $L_\infty[0, h)$-induced norm $\|\mathbf{D}\|$. In particular, we can also observe that K_{M1}/M^2 is much smaller than K_{M0}/M under the same parameter M in Tables I and II. This demonstrates that the piecewise linear approximation method works much more effectively than the staircase approximation method [6] in the computation of $\|\mathbf{D}\|$. In this respect, it should be observed that the increase in the computation time for the piecewise linear approximation method is only modest, and for approximately the same computation time, the piecewise linear approximation method gives a much better result than the staircase approximation method. In particular, the estimate of $\|\mathbf{D}\|$ given by $\|\mathbf{D}_{M1}\|$ is rather close to the exact value of $\|\mathbf{D}\|$ even when M is relatively small. These observations suggest that the piecewise linear approximation method drastically outperforms the conventional staircase approximation.

TABLE I
Computation results with staircase approximation.

M	50	100	200	500
$\|\mathbf{D}_{M0}\|$	0.4518	0.4593	0.4630	0.4652
K_{M0}/M	0.2644	0.1290	0.0637	0.0253
time (sec)	2.5×10^{-3}	4.4×10^{-3}	8.7×10^{-3}	2.2×10^{-2}

TABLE II
Computation results with piecewise linear approximation.

M	50	100	200	500
$\|\mathbf{D}_{M1}\|$	0.4667	0.4667	0.4667	0.4667
K_{M1}/M^2	0.0059	0.0014	3.5×10^{-4}	5.6×10^{-5}
time (sec)	7.9×10^{-3}	1.6×10^{-2}	3.3×10^{-2}	8.7×10^{-2}

V. Conclusion

Stimulated by the success in the lifting approach to sampled-data systems and time-delay systems, we developed two methods for computing the $L_\infty[0, h)$-induced norm of a compression operator associated with a linear time-invariant system.

By using the fast-lifting technique, we approximated the compression operator directly with tractable operators that alleviated the difficulties for computing its $L_\infty[0, h)$-induced norm. We first studied staircase approximation, which corresponds to restricting the input of the compression operator to constant functions and approximating its output with constant functions. We then showed that the computation error of the induced norm is ensured to converge to 0 in the order of $1/M$, where M is the fast-lifting parameter with which the interval $[0, h)$ is divided into smaller subintervals. We next studied piecewise linear approximation, which corresponds to restricting the input of the compression operator to piecewise linear functions and approximating its output with piecewise linear functions. We then showed that the computation error is ensured to converge to 0 in the order of

$1/M^2$. Finally, we examined effectiveness of the developed methods through a numerical example, and it turned out that the piecewise linear approximation method works far more effectively than the staircase approximation method.

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