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Stabilizing switched T-S fuzzy systems using a fuzzy Lyapunov function approach*

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***Abstract*— In this work, sufficient conditions for the existence of switching laws for stabilizing switched T-S fuzzy systems via a fuzzy Lyapunov function are proposed. The conditions are found by exploring properties of the membership functions and are formulated in terms of linear matrix inequalities (LMIs). A state feedback design is also used to extend the applicability of the results. Numerical examples illustrate the effectiveness of the proposed design methods.**

I. INTRODUCTION

Recently, Takagi-Sugeno (T-S) fuzzy models have been used to investigate the dynamic behaviour of switched nonlinear systems [1], [2], [3], [4]. The main attractiveness of T-S fuzzy modeling is that the stability analysis and the controller design can be formulated in the framework of LMIs, which can be efficiently solved by convex programming techniques [5].

Stability and stabilization of switched T-S fuzzy models are usually investigated via a common quadratic Lyapunov function [1], [2], [3], [4]. This approach consists of finding a symmetric positive definite matrix to satisfy all Lyapunov inequalities associated to each fuzzy rule of the system. However, to find a common matrix satisfying all Lyapunov inequalities, for a T-S fuzzy system with a large number of rules, may be too conservative. To overcome this problem, in this paper, sufficient conditions for the existence of switching laws that stabilize the switched T-S fuzzy systems using a fuzzy Lyapunov function are presented. Fuzzy Lyapunov function approaches allow to explore membership function properties to reduce the conservativeness in the LMI-based stability conditions. Differing from [4], the presented results are obtained without any requirement on the matrices of the local models. In addiction, a state feedback design is also presented.

Throughout this paper $\mathbf{M} > \mathbf{0}$ ($\mathbf{M} \geq \mathbf{0}$) means that the matrix $\mathbf{M}$ is a real symmetric and positive definite (semidefinite) matrix. Similarly, $\mathbf{M} < \mathbf{0}$ ($\mathbf{M} \leq \mathbf{0}$) means that the matrix $\mathbf{M}$ is symmetric and negative definite (semidefinite) matrix. The symbol "$\star$" within a matrix represents the symmetric terms of the matrix.

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II. PRELIMINARIES

Let us consider the following switched nonlinear system

$$\dot{\mathbf{x}}(t) = f_{\sigma(\mathbf{x}(t))}\Big(\mathbf{x}(t), \mathbf{u}(t)\Big) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ and $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ are the state and input vectors, respectively. Define a set $\mathscr{P} = \{1, 2, \cdots, N\}$, where N is the number of subsystems, $\sigma(\mathbf{x}(t)) : \mathbb{R}^{n_x} \to \mathscr{P}$ is a piecewise constant function of the state called switching signal and f_p is a smooth nonlinear function for all $p \in \mathscr{P}$. It is assumed that the state of (1) does not jump at the switching instants, that is, the solution $\mathbf{x}(t)$ is everywhere continuous. The smooth nonlinear function f_p can be exactly represented by a T-S fuzzy model in the following subset of the state space [6]:

$$S := \{\mathbf{x}(t) \in \mathbb{R}^{n_x} : |x_j(t)| \leq \bar{x}_j\},\ j \in \mathscr{I} \tag{2}$$

where $\mathscr{I} \subset \{1, 2, \cdots, n_x\}$ and $\bar{x}_j$ is a known positive real number for all $j \in \mathscr{I}$. Then, the switched nonlinear system (1) can be described by fuzzy IF-THEN rules, as follows [2]:

Model rule k for subsystem $\sigma(\mathbf{x}(t)) = p$:

IF $x_1(t)$ is M_{pk1} and $x_2(t)$ is M_{pk2} and $\cdots$ and $x_q(t)$ is M_{pkq}

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_{pk}\mathbf{x}(t) + \mathbf{B}_{pk}\mathbf{u}(t), \quad k = 1, 2, \cdots, r_p$

where $M_{pkj},\ j = 1, 2, \cdots, q,\ q \leq n_x$, are the fuzzy sets. The overall fuzzy subsystem $\sigma(\mathbf{x}(t)) = p$ is obtained by fuzzy blending k-rules as follows:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{r_p} h_{pk}(\mathbf{x}(t))\Big(\mathbf{A}_{pk}\mathbf{x}(t) + \mathbf{B}_{pk}\mathbf{u}(t)\Big) \tag{3}$$

where $\mathbf{A}_{pk} \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{B}_{pk} \in \mathbb{R}^{n_x \times n_u}$ are the matrices of the local models, r_p is the number of model rules of the subsystem $\sigma(\mathbf{x}(t)) = p$ and

$$h_{pk}(\mathbf{x}(t)) = \frac{w_{pk}(\mathbf{x}(t))}{\sum_{k=1}^{r_p} w_{pk}(\mathbf{x}(t))}$$

with $w_{pk}(\mathbf{x}(t)) = \prod_{j=1}^{q} M_{pkj}(x_j(t))$ the normalized weight function for each local model. In $h_{pk}(\mathbf{x}(t))$ the term $M_{pkj}(x_j(t))$ represents the grade of membership of $x_j(t)$ in the fuzzy set M_{pkj}.

Remark 1: If no constraint on the state $x_j(t),\ j \in \mathscr{I}$ is needed, then $S = \mathbb{R}^{n_x}$.

It should be noted from the properties of membership functions that the following relations hold:

$$\forall p \in \mathscr{P},\ k \in \mathscr{R}_p,\ h_{pk}(\mathbf{x}(t)) \geq 0 \text{ and } \sum_{k=1}^{r_p} h_{pk}(\mathbf{x}(t)) = 1 \tag{4}$$

with $\mathscr{R}_p = \{1, 2, \cdots, r_p\}$. From (4), it follows

$$\sum_{k=1}^{r_p} \dot{h}_{pk}(\mathbf{x}(t)) = 0, \quad \forall p \in \mathscr{P}, \tag{5}$$

$$h^2_{p\beta}(\mathbf{x}(t)) = 1 - \sum_{\substack{k=1\\k\neq\beta}}^{r_p} h^2_{pk}(\mathbf{x}(t)) - 2\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} h_{pk}(\mathbf{x}(t))h_{pi}(\mathbf{x}(t)) \tag{6}$$

for any $p \in \mathscr{P}$ and $\beta \in \bigcap_{p=1}^{N} \mathscr{R}_p$, and

$$\sum_{p=1}^{N}\left[\left(\sum_{k=1}^{r_p} h_{pk} - \frac{1}{N-1}\sum_{\substack{\ell=1\\\ell\neq p}}^{N}\sum_{i=1}^{r_\ell} h_{\ell i}\right)\sum_{k=1}^{r_p} h_{pk}\right] = \sum_{p=1}^{N}\sum_{k=1}^{r_p} h^2_{pk} + 2\sum_{p=1}^{N}\sum_{k=1}^{r_p-1}\sum_{i=k+1}^{r_p} h_{pk}h_{pi} - \frac{2}{N-1}\sum_{p=1}^{N-1}\sum_{\ell=p+1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_\ell} h_{pk}h_{\ell i} = 0. \tag{7}$$

For simplicity, in what follows, $h_{pk}(\mathbf{x}(t))$ will be denoted by h_{pk}.

To establish the main results of this paper, consider real numbers $\alpha_p,\ p \in \mathscr{P}$ satisfying

$$\alpha_p \geq 0, \quad \forall p \in \mathscr{P} \quad \text{and} \quad \sum_{p=1}^{N} \alpha_p = 1 \tag{8}$$

and the following proposition.

Proposition 1: If there exist real numbers $\alpha_p,\ p \in \mathscr{P}$ satisfying (8) such that

$$\dot{\mathbf{x}}(t) = \sum_{p=1}^{N} \alpha_p \sum_{k=1}^{r_p} h_{pk}\Big(\mathbf{A}_{pk}\mathbf{x}(t) + \mathbf{B}_{pk}\mathbf{u}(t)\Big) \tag{9}$$

is asymptotically stable, then there exists a switching law that ensures the asymptotic stability of the switched T-S fuzzy system (3).

Proof: See [4]. ■

III. Stabilization of switched T-S fuzzy systems

This section provides sufficient conditions for the existence of a state-dependent switching law $\sigma(\mathbf{x})$ that ensures the asymptotic stability of the switched fuzzy system (3) for the case $\mathbf{u}(t) = \mathbf{0}$. The results are developed using the following Lyapunov function candidate:

$$V(\mathbf{x}(t)) = \mathbf{x}(t)'\mathbf{P}(h)\mathbf{x}(t) \tag{10}$$

where $\mathbf{P}(h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} h_{pk}\mathbf{P}_{pk}$.

The first-order time-derivative of the normalized membership function h_{pk} appears in the expression for the derivative of (10) given as

$$\dot{V}(\mathbf{x}(t)) = \dot{\mathbf{x}}(t)'\mathbf{P}(h)\mathbf{x}(t) + \mathbf{x}(t)'\left(\dot{\mathbf{P}}(h)\mathbf{x}(t) + \mathbf{P}(h)\dot{\mathbf{x}}(t)\right) \tag{11}$$

where $\dot{\mathbf{P}}(h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} \dot{h}_{pk}\mathbf{P}_{pk}$. Therefore, in order to obtain LMI conditions some assumptions are needed. Let h_{pk} be C^1 functions and assume that $\dot{h}_{pk}$ is bounded. For known positive real numbers ϕ_{pk} define

$$D_{pk} = \{\mathbf{x}(t) \in \mathbb{R}^{n_x} : |\dot{h}_{pk}| \leq \phi_{pk}\},\ p \in \mathscr{P} \ \text{ and } \ k \in \mathscr{R}_p. \tag{12}$$

Using Proposition 1, sufficient LMI-based conditions for local stabilization of system (3) with $\mathbf{u}(t) = \mathbf{0}$ is proposed next.

Theorem 1: Let α_p and ϕ_{pk} be known real numbers satisfying (8) and (12), respectively. If for some fixed $\beta \in \bigcap_{p=1}^{N} \mathscr{R}_p$ there exist matrices $\mathbf{L}_{pk} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{R}_{pk} \in \mathbb{R}^{n_x \times n_x}$, and symmetric matrices $\mathbf{M}_p \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{P}_{pk} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{T} \in \mathbb{R}^{2n_x \times 2n_x}$, satisfying (13)-(18) in the next page top. Then, there exists a switching law such that system (3) with $\mathbf{u}(t) = \mathbf{0}$ is asymptotically stable.

Proof: Multiplying (15) by $h_{pk}h_{\ell i}$, (16) by $h_{pk}h_{pi}$, (17) by h^2_{pk} and adding to (18) multiplied by $h^2_{p\beta}$, it follows that

$$\begin{aligned}&\sum_{p=1}^{N-1}\sum_{\ell=p+1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_\ell} h_{pk}h_{\ell i}\left(\Upsilon_{pk\_\ell i} + \Upsilon_{\ell i\_pk} - \frac{2}{N-1}\mathbf{T}\right) \\ &+ \sum_{p=1}^{N}\sum_{k=1}^{r_p-1}\sum_{i=k+1}^{r_p} h_{pk}h_{pi}\Big(\Upsilon_{pk\_pi} + \Upsilon_{pi\_pk} + 2\mathbf{T}\Big) \\ &+ \sum_{p=1}^{N}\sum_{\substack{k=1\\k\neq\beta}}^{r_p} h^2_{pk}\Upsilon_{pk\_pk} + \sum_{p=1}^{N}\sum_{k=1}^{r_p} h^2_{pk}\mathbf{T} + \hat{\mathbf{P}}_\phi \\ &+ \sum_{p=1}^{N}\left(1 - \sum_{\substack{k=1\\k\neq\beta}}^{r_p} h^2_{pk} - 2\sum_{k=1}^{r_p-1}\sum_{i=k+1}^{r_p} h_{pk}h_{pi}\right)\Upsilon_{p\beta\_p\beta} < \mathbf{0}\end{aligned} \tag{19}$$

with $\hat{\mathbf{P}}_\phi = \sum_{p=1}^{N}\sum_{k=1}^{r_p} \phi_{pk}\begin{bmatrix}\mathbf{P}_{pk} + \mathbf{M}_p & \star \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$. Replacing (6) into (19) and considering (7), it yields

$$\begin{aligned}&\sum_{p=1}^{N-1}\sum_{\ell=p+1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_\ell} h_{pk}h_{\ell i}\Big(\Upsilon_{pk\_\ell i} + \Upsilon_{\ell i\_pk}\Big) \\ &+ \sum_{p=1}^{N}\sum_{k=1}^{r_p-1}\sum_{i=k+1}^{r_p} h_{pk}h_{pi}\Big(\Upsilon_{pk\_pi} + \Upsilon_{pi\_pk}\Big) \\ &+ \sum_{p=1}^{N}\sum_{\substack{k=1\\k\neq\beta}}^{r_p} h^2_{pk}\Upsilon_{pk\_pk} + \sum_{p=1}^{N} h^2_{p\beta}\Upsilon_{p\beta\_p\beta} + \hat{\mathbf{P}}_\phi \\ &= \sum_{p=1}^{N}\sum_{\ell=1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_\ell} h_{pk}h_{\ell i}\Upsilon_{pk\_\ell i} + \hat{\mathbf{P}}_\phi \\ &= \begin{bmatrix}\upsilon_{11} & \upsilon'_{21} \\ \upsilon_{21} & -\mathbf{R}(h) - \mathbf{R}(h)'\end{bmatrix} < \mathbf{0}\end{aligned} \tag{20}$$

where $\upsilon_{11} = \mathbf{L}(h)\mathbf{A}(\alpha,h) + \mathbf{A}(\alpha,h)'\mathbf{L}(h)' + \mathbf{P}_\phi$, $\upsilon_{21} = \mathbf{P}(h) - \mathbf{L}(h)' + \mathbf{R}(h)\mathbf{A}(\alpha,h)$, $\mathbf{L}(h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} h_{pk}\mathbf{L}_{pk}$, $\mathbf{R}(h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} h_{pk}\mathbf{R}_{pk}$, $\mathbf{A}(\alpha,h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} \alpha_p h_{pk}\mathbf{A}_{pk}$ and

$$\mathbf{P}_{pk} > \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p, \tag{13}$$

$$\mathbf{P}_{pk} + \mathbf{M}_p \geq \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p, \tag{14}$$

$$\Upsilon_{pk\_\ell i} + \Upsilon_{\ell i\_pk} - \frac{2}{N-1}\mathbf{T} < \mathbf{0}, \quad p < \ell, \quad p, \ell \in \mathscr{P}, \quad i \in \mathscr{R}_\ell \quad \text{and} \quad k \in \mathscr{R}_p, \tag{15}$$

$$\Upsilon_{pk\_pi} + \Upsilon_{pi\_pk} - 2\Upsilon_{p\beta\_p\beta} + \frac{2}{N}\tilde{\mathbf{\Upsilon}}_\phi + 2\mathbf{T} < \mathbf{0}, \quad k < i, \quad p \in \mathscr{P} \quad \text{and} \quad i, k \in \mathscr{R}_p, \tag{16}$$

$$\Upsilon_{pk\_pk} - \Upsilon_{p\beta\_p\beta} + \frac{1}{N}\tilde{\mathbf{\Upsilon}}_\phi + \mathbf{T} < \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p - \{\beta\}, \tag{17}$$

$$\frac{1}{N}\tilde{\mathbf{\Upsilon}}_\phi + \mathbf{T} < \mathbf{0} \tag{18}$$

with $\Upsilon_{pk\_\ell i} = \begin{bmatrix} \alpha_\ell(\mathbf{L}_{pk}\mathbf{A}_{\ell i} + \mathbf{A}'_{\ell i}\mathbf{L}'_{pk}) & \star \\ \mathbf{P}_{pk} - \mathbf{L}'_{pk} + \alpha_\ell \mathbf{R}_{pk}\mathbf{A}_{\ell i} & -\mathbf{R}_{pk} - \mathbf{R}'_{pk} \end{bmatrix}$ and $\tilde{\mathbf{\Upsilon}}_\phi = \sum_{g=1}^{N} \Upsilon_{g\beta\_g\beta} + \sum_{p=1}^{N}\sum_{k=1}^{r_p} \phi_{pk} \begin{bmatrix} \mathbf{P}_{pk} + \mathbf{M}_p & \star \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

$\mathbf{P}_\phi = \sum_{p=1}^{N}\sum_{k=1}^{r_p} \phi_{pk}\left(\mathbf{P}_{pk} + \mathbf{M}_p\right)$. Pre-multiplying and post-multiplying LMI (20) by matrix $[\mathbf{I} \quad \mathbf{A}(\alpha,h)']$ and its transpose, respectively, it yields

$$\mathbf{P}_\phi + \mathbf{A}(\alpha,h)'\mathbf{P}(h) + \mathbf{P}(h)\mathbf{A}(\alpha,h) < \mathbf{0}. \tag{21}$$

From (5), it follows that

$$\sum_{k=1}^{r_p} \dot{h}_{pk}\mathbf{P}_{pk} = \sum_{k=1}^{r_p} \dot{h}_{pk}\left(\mathbf{P}_{pk} + \mathbf{M}_p\right) \tag{22}$$

for any matrix $\mathbf{M}_p$, $p \in \mathscr{P}$. Thus, under assumptions $|\dot{h}_{pk}| \leq \phi_{pk}$, $\forall p \in \mathscr{P}$, $k \in \mathscr{R}_p$ and (22), one has that

$$\sum_{p=1}^{N}\sum_{k=1}^{r_p} \dot{h}_{pk}\mathbf{P}_{pk} + \mathbf{A}(\alpha,h)'\mathbf{P}(h) + \mathbf{P}(h)\mathbf{A}(\alpha,h) < \mathbf{0}. \tag{23}$$

Pre-multiplying and post-multiplying (23) by $\mathbf{x}(t)'$ and its transpose, respectively, one concludes that (11) along the trajectories of the T-S fuzzy system (9) is negative for all $\mathbf{x}(t) \neq 0$. Therefore, from Proposition 1, there exists a switching law that ensures the asymptotic stability of the switched T-S fuzzy system (3) with $\mathbf{u}(t) = \mathbf{0}$. ■

From Theorem 1, a stabilizing switching law can be established via the following condition.

Switching condition 1: For $\mathbf{u}(t) = \mathbf{0}$, switched nonlinear (1) can be switched to or can stay at mode p if at time t

$$\mathbf{x}'(t)\left\{\sum_{k=1}^{r_p} h_{pk}\left(\mathbf{A}'_{pk}\mathbf{P}(h) + \mathbf{P}(h)\mathbf{A}_{pk}\right) + \mathbf{P}_\phi\right\}\mathbf{x}(t) < 0. \tag{24}$$

Remark 2: The LMI solution of Theorem 1 can also be used to determine an estimate of the basin of attraction of system (1). By definition, the basin of attraction of system (1) is the following set

$$B_A := \left\{\mathbf{x}_0 \in S : \lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}\right\}$$

where $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{0}$ is the origin of the $\mathbb{R}^{n_x}$. Now, let $Z = \bigcap_{p=1}^{N}\left[\left(\bigcap_{k=1}^{r_p} D_{pk}\right)\bigcap S\right]$ be the set where the switched nonlinear system (1) can be exactly represented by T-S fuzzy model (3), $\Omega(c) = \{\mathbf{x}_0 \in \mathbb{R}^{n_x} : V(\mathbf{x}(t)) < c\}$ the sublevel set of Lyapunov function candidate (10) and $c^* = \max\{c \in \mathbb{R} : \Omega(c) \subseteq Z\}$. By Theorem 1, for all $\mathbf{x}(t) \in Z$ such that $\mathbf{x}_0 \in \Omega(c^*)$, $\dot{V}(\mathbf{x}(t)) < 0$ and $V(\mathbf{x}(t)) < c^*$, thus $\lim_{t\to+\infty} V(\mathbf{x}(t)) = 0$. As a consequence $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}$ and therefore $\Omega(c^*)$ is an estimate of B_A.

Remark 3: The stability conditions of Theorem 1 are based on the Finsler Lemma. This lemma was first used in control in the analysis of stability [7]. Since then, it has been used in several control problems [8], [9], [10]. In this lemma, the Lyapunov matrices are decoupled from the system matrices. In [11], it was shown that matrices decoupling increases the estimates of the basin of attraction.

The efficiency of Theorem 1 is illustrated by a numerical example.

EXAMPLE 1

Consider a T-S fuzzy system (3) with local models

$$\mathbf{A}_{11} = \begin{bmatrix} 10 & -10 \\ 6.2 & 1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 10 & -10 \\ 6.2 & 1.9 \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} -10 & 1 \\ 3 & 0.73 \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} -10 & 1 \\ 3 & 0.27 \end{bmatrix} \tag{25}$$

and membership functions

$$h_{11} = \frac{1 + \sin\big(x_1(t)\big)}{2}, \quad h_{12} = 1 - h_{11}, \quad h_{21} = \frac{1}{1 + e^{x_1(t)}}, \quad h_{22} = 1 - h_{21} \tag{26}$$

for the following subset of the state space: $S = \{\mathbf{x}(t) \in \mathbb{R}^{n_x} : |x_1(t)| \leq 1\}$.

Using MATLAB toolboxes YALMIP [12] and SeDuMi [5] to solve (13)-(18) for $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\phi_{pk} = 5.5$ for all $p \in \mathscr{P}, k \in \mathscr{R}_p$ and $\beta = 1$, the following solution is obtained:

$$\mathbf{T} = \begin{bmatrix} 30.68 & -10.28 & -2.71 & 3.68 \\ -10.28 & 3.49 & 0.97 & -1.34 \\ -2.71 & 0.97 & -0.01 & -0.08 \\ 3.68 & -1.34 & -0.08 & 0.06 \end{bmatrix}, \tag{27}$$

$$\mathbf{P}_{11} = \begin{bmatrix} 2.72 & 0.99 \\ 0.99 & 1.73 \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} 2.73 & 0.99 \\ 0.99 & 1.73 \end{bmatrix},$$
$$\mathbf{P}_{21} = \begin{bmatrix} 2.69 & 0.92 \\ 0.92 & 1.65 \end{bmatrix}, \quad \mathbf{P}_{22} = \begin{bmatrix} 2.69 & 0.92 \\ 0.92 & 1.65 \end{bmatrix}, \quad (28)$$
$$\mathbf{M}_{1} = \begin{bmatrix} -2.72 & -0.99 \\ -0.99 & -1.72 \end{bmatrix}, \quad \mathbf{M}_{2} = \begin{bmatrix} -2.69 & -0.92 \\ -0.92 & -1.65 \end{bmatrix}.$$

The eigenvalues of matrices $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$ are $5.5 \pm 6.46i$ and $5.95 \pm 5.2i$, respectively. As the eigenvalues have positive real parts, then, by the T-S fuzzy model, the subsystem 1 is unstable. On the other hand, the eigenvalues of matrices $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are -10.27, 1 and -10.28, 0.55, respectively. Again, the origin is not a stable equilibrium point of subsystem 2 for any solution with initial condition $\mathbf{x}_0 \in \Omega(c^*)$. Although, subsystems 1 and 2 are not asymptotically stable, Theorem 1 shows that there exist a Lyapunov function (10) with membership functions (26) and matrices (28) for system (9). Therefore, for all $\mathbf{x}_0 \in \Omega(c^*)$, switching law (24) with matrices (25), (28) and MFs (26) ensures that switched T-S fuzzy system (3) with local models (25) and MFs (26) is asymptotically stable. Figures 1(a) and 1(b) show the system state and the respective stabilizing switching law. Figure 2 shows the estimates of the basin of attraction for $\Omega(c^*)$, with $c^* = 2.5$. The sets Z and $\Omega(c^*)$ were obtained numerically for a 0.01 grid.

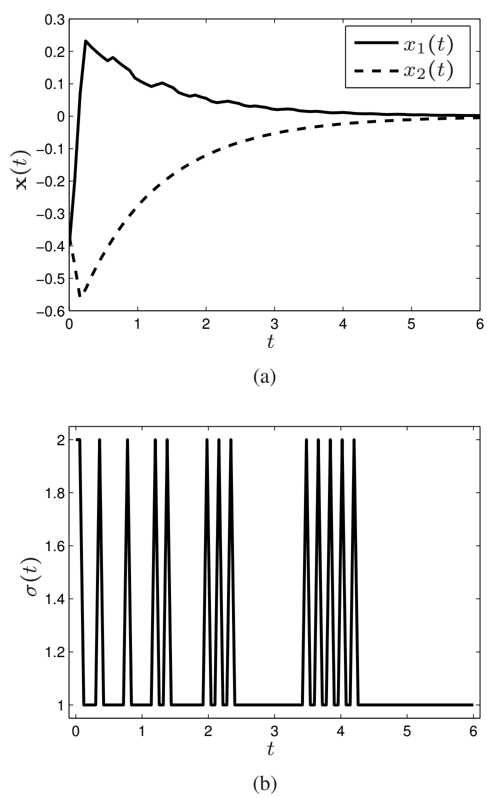


Fig. 1. a) Switching solution with initial condition $x_0 = [-0.4 \ -0.37]'$ for system (3) with local models (25) and MFs (26), (b) switching law satisfying (24).

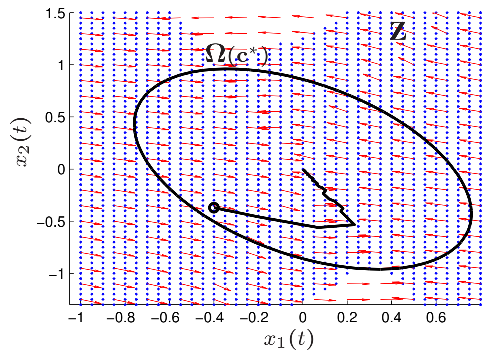


Fig. 2. Phase plane and basin of attraction for switched system (3) with local models (25) and MFs (26).

Notice that switching law (24) allows some flexibility in choosing the system to activate. For instance, Figure 1(a) was obtained by checking (24) on the subsystem 1 first, for any instant t. Thus, subsystem 2 is activated only when (24) is not satisfied for the case $p = 1$. This represents a preference on the activation of subsystem 1. Therefore, switching law (24) allows the designer to use his experience to select the order that subsystems will be checked. This characteristics can decrease the number of switching needed to stabilize nonlinear system (1).

Remark 4: Property (7) allows the addition of a slack matrix variable $\mathbf{T}$ in conditions (15)-(18), improving the feasible region of the LMIs. For $\mathbf{T} = \mathbf{0}$, LMIs (13)-(18) with matrices (25) are infeasible. Thus, Theorem 1 can not ensure the asymptotic stability of Example 1.

IV. State feedback design

In this section, the parallel distributed compensation (PDC) procedure [6] is used to design fuzzy controllers for switched T-S fuzzy systems (3). With this procedure, the designed fuzzy controller for fuzzy models (3), is of the form:

$$\mathbf{u}(t) = \sum_{k=1}^{r_p} h_{pk}\mathbf{K}_{pk}\mathbf{x}(t), \quad \forall p \in \mathscr{P}. \tag{29}$$

Replacing (29) into (3), the overall closed-loop system is given by:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^{r_p}\sum_{i=1}^{r_p} h_{pk}h_{pi}(\mathbf{A}_{pk} + \mathbf{B}_{pk}\mathbf{K}_{pi})\mathbf{x}(t), \quad \forall p \in \mathscr{P}. \tag{30}$$

Now, sufficient conditions for the existence of gains $\mathbf{K}_{pi}$, $p \in \mathscr{P}$, $i \in \mathscr{R}_p$ and a stabilizing switching law $\sigma(\mathbf{x})$ for system (30) is proposed using the same strategy of Theorem 1, that is, verifying the existence of a Lyapunov function (10) for system (9).

Theorem 2: Let α_p and ϕ_{pk} be known real numbers satisfying (8) and (12), respectively. If for some fixed $\beta \in \bigcap_{p=1}^{N} \mathscr{R}_p$ and a positive constant μ there exist matrices $\mathbf{W} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{Y}_{pk} \in \mathbb{R}^{n_u \times n_x}$, and symmetric matrices $\mathbf{X}_p \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{Q}_{pk} \in \mathbb{R}^{n_x \times n_x}$, satisfying (31)-(35) in the next page top.

$$\mathbf{Q}_{pk} \succ \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p, \tag{31}$$

$$\mathbf{Q}_{pk} + \mathbf{X}_p \geq \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p, \tag{32}$$

$$\Lambda_{pk\_pi} + \Lambda_{pi\_pk} - 2\Lambda_{p\beta\_p\beta} + \frac{2}{N}\tilde{\mathbf{\Lambda}}_\phi \prec \mathbf{0}, \quad i < k, \quad p \in \mathscr{P} \quad \text{and} \quad i, k \in \mathscr{R}_p, \tag{33}$$

$$\Lambda_{pk\_pk} - \Lambda_{p\beta\_p\beta} + \frac{1}{N}\tilde{\mathbf{\Lambda}}_\phi \prec \mathbf{0}, \quad p \in \mathscr{P} \quad \text{and} \quad k \in \mathscr{R}_p - \{\beta\}, \tag{34}$$

$$\frac{1}{N}\tilde{\mathbf{\Lambda}}_\phi \prec \mathbf{0} \tag{35}$$

with $\Lambda_{pk\_pi} = \begin{bmatrix} \alpha_p(\mathbf{A}_{pk}\mathbf{W} + \mathbf{W}'\mathbf{A}'_{pk} + \mathbf{B}_{pk}\mathbf{Y}_{pi} + \mathbf{Y}'_{pi}\mathbf{B}'_{pk}) & \star \\ \mathbf{Q}_{pk} - \mathbf{W}' + \mu\alpha_p(\mathbf{A}_{pk}\mathbf{W} + \mathbf{B}_{pk}\mathbf{Y}_{pi}) & -\mu(\mathbf{W} + \mathbf{W}') \end{bmatrix}$ and $\tilde{\mathbf{\Lambda}}_\phi = \sum_{p=1}^{N} \Lambda_{p\beta\_p\beta} + \sum_{p=1}^{N}\sum_{k=1}^{r_p} \phi_{pk} \begin{bmatrix} \mathbf{Q}_{pk} + \mathbf{X}_p & \star \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Then, there exists a switching law such that the switched T-S fuzzy system (30) with local gains

$$\mathbf{K}_{pk} = \mathbf{Y}_{pk}\mathbf{W}^{-1} \tag{36}$$

is asymptotically stable for all $\mathbf{x}(t) \in Z$ such that $\mathbf{x}_0 \in \Omega(c^*)$.

Proof: Multiplying (33) by $h_{pk}h_{pi}$, (34) by h^2_{pk}, (35) by $h^2_{p\beta}$ and following the same steps from (19)-(20), it yields

$$\begin{bmatrix} \lambda_{11}(\alpha, h) & \lambda_{21}(\alpha, h)' \\ \lambda_{21}(\alpha, h) & -\mu(\mathbf{W} + \mathbf{W}') \end{bmatrix} \prec \mathbf{0} \tag{37}$$

where $\lambda_{11}(\alpha, h) = \mathbf{A}(\alpha, h)\mathbf{W} + \mathbf{W}'\mathbf{A}(\alpha, h)' + \mathbf{B}_Y(\alpha, h) + \mathbf{B}_Y(\alpha, h)' + \mathbf{Q}_\phi$, $\lambda_{21}(\alpha, h) = \mathbf{Q}(h) - \mathbf{W}' + \mu\big(\mathbf{A}(\alpha, h)\mathbf{W} + \mathbf{B}_Y(\alpha, h)\big)$, $\mathbf{B}_Y(\alpha, h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_p} \alpha_p h_{pk} h_{pi} \mathbf{B}_{pk}\mathbf{Y}_{pi}$, $\mathbf{Q}(h) = \sum_{p=1}^{N}\sum_{k=1}^{r_p} h_{pk}\mathbf{Q}_{pk}$, $\mathbf{Q}_\phi = \sum_{p=1}^{N}\sum_{k=1}^{r_p} \phi_{pk}(\mathbf{Q}_{pk} + \mathbf{X}_p)$ and $\mathbf{A}(\alpha, h)$ as already defined. Performing the matrix transformation

$$\mathbf{B}_Y(\alpha, h) = \left(\sum_{p=1}^{N}\sum_{k=1}^{r_p}\sum_{i=1}^{r_p} \alpha_p h_{pk} h_{pi} \mathbf{B}_{pk}\mathbf{K}_{pi}\right)\mathbf{W} = \mathbf{B}_K(\alpha, h)\mathbf{W} \tag{38}$$

and pre-multiplying and post-multiplying (37) by $\begin{bmatrix} \mathbf{W}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{-1} \end{bmatrix}'$ and its transpose, respectively, it yields

$$\begin{bmatrix} \bar{\lambda}_{11}(\alpha, h) & \bar{\lambda}_{21}(\alpha, h)' \\ \bar{\lambda}_{21}(\alpha, h) & -\mu\Big(\mathbf{W}^{-1} + (\mathbf{W}')^{-1}\Big) \end{bmatrix} \prec \mathbf{0} \tag{39}$$

with $\bar{\lambda}_{11}(\alpha, h) = (\mathbf{W}')^{-1}\Big(\mathbf{A}(\alpha, h) + \mathbf{B}_K(\alpha, h)\Big) + \Big(\mathbf{A}(\alpha, h) + \mathbf{B}_K(\alpha, h)\Big)'\mathbf{W}^{-1} + \mathbf{P}_\phi$, $\bar{\lambda}_{21}(\alpha, h) = \mathbf{P}(h) - \mathbf{W}^{-1} + \mu(\mathbf{W}')^{-1}\Big(\mathbf{A}(\alpha, h) + \mathbf{B}_K(\alpha, h)\Big)$, $\mathbf{P}(h) = (\mathbf{W}')^{-1}\mathbf{Q}(h)\mathbf{W}^{-1}$ and $\mathbf{P}_\phi = (\mathbf{W}')^{-1}\mathbf{Q}_\phi\mathbf{W}^{-1}$. Which is equivalent to (20) for the particular case $\mathbf{L}(h) = (\mathbf{W}')^{-1}$ and $\mathbf{R}(h) = \mu(\mathbf{W}')^{-1}$. Thus, when (31)-(35) hold, by Theorem 1 there exists a switching law ensuring the asymptotic stability of the switched T-S fuzzy system (30). ■

Switching condition 2: Let $\mathbf{P}_{pk}$ and $\mathbf{K}_{pi}$ be solutions of Theorem 2. Switched system (30) with arbitrary N individual subsystems can be switched to or can stay at mode p if at time t

$$\mathbf{x}'(t)\left\{\sum_{k=1}^{r_p}\sum_{i=1}^{r_p} h_{pk}h_{pi}\Big[(\mathbf{A}_{pk} + \mathbf{B}_{pk}\mathbf{K}_{pi})'\mathbf{P}(h) + \mathbf{P}(h)(\mathbf{A}_{pk} + \mathbf{B}_{pk}\mathbf{K}_{pi})\Big] + \mathbf{P}_\phi\right\}\mathbf{x}(t) < 0. \tag{40}$$

The efficiency of Theorem 2 is illustrated in the next example.

EXAMPLE 2

Consider the switched T-S fuzzy system (30) with matrices:

$$\mathbf{A}_{11} = \begin{bmatrix} 10 & -10 \\ 6.2 & 1 \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} 10 & -10 \\ 6.2 & -1 \end{bmatrix}, \quad \mathbf{A}_{21} = \begin{bmatrix} 10 & 1 \\ 3 & 0.73 \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} 10 & 1 \\ 3 & 0.27 \end{bmatrix}, \tag{41}$$

$$\mathbf{B}_{11} = \mathbf{B}_{12} = \mathbf{B}_{21} = \mathbf{B}_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{42}$$

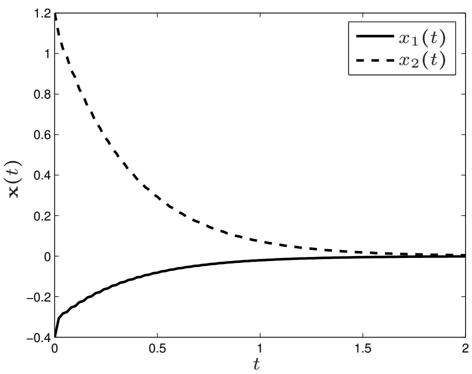
and membership functions (26). For the case $\mathbf{u}(t) = \mathbf{0}$, LMIs (13)-(18) are infeasible. Thus, Theorem 1 can not be used to stabilize this example. Now, using MATLAB to solve (31)-(35) from Theorem 2 with parameters $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\phi_{pk} = 5.5$ for all $p \in \mathscr{P}, k \in \mathscr{R}_p$, $\beta = 1$ and $\mu = 0.1$, the following matrices are obtained:

$$\mathbf{P}_{11} = \begin{bmatrix} 45.01 & 13.87 \\ 13.87 & 4.55 \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} 32.08 & 9.80 \\ 9.80 & 3.27 \end{bmatrix}, \quad \mathbf{P}_{21} = \begin{bmatrix} 45.01 & 13.87 \\ 13.87 & 4.55 \end{bmatrix}, \quad \mathbf{P}_{22} = \begin{bmatrix} 32.08 & 9.80 \\ 9.80 & 3.27 \end{bmatrix}, \tag{43}$$

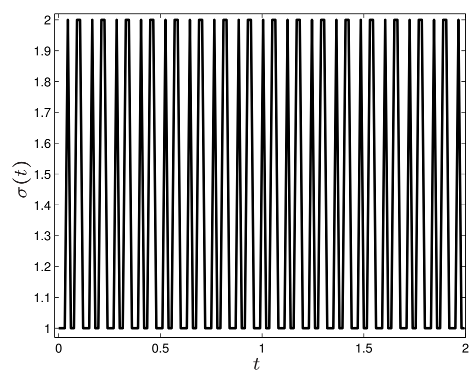
$$\mathbf{P}_\phi = \begin{bmatrix} 853.60 & 299.39 \\ 299.39 & 106.04 \end{bmatrix}, \tag{44}$$

$$\mathbf{K}_{11} = \begin{bmatrix} -280.14 & -92.90 \end{bmatrix}, \ \ \mathbf{K}_{21} \approx \mathbf{0},$$
$$\mathbf{K}_{12} = \begin{bmatrix} -276.06 & -89.68 \end{bmatrix}, \ \ \mathbf{K}_{22} = \begin{bmatrix} 2.72 & 1.27 \end{bmatrix}. \quad (45)$$

Thus, by Theorem 2, switching law (40) makes system (30) with local models (41), (42), MFs (26) and local gains (45) asymptotically stable. For initial condition $x_0 = [-0.4 \ \ 1.2]'$, Figure 3 shows the state and switching law of the closed-loop switched T-S fuzzy (30). Notice that fuzzy controller (29) associated with conditions of Proposition 1 was able to stabilize this example. Figure 4 shows the estimates of the basin of attraction for $\Omega(c^*)$, with $c^* = 1.8$.



(a)



(b)

Fig. 3. a) Switching solution with initial condition $x_0 = [-0.4 \ \ 1.2]'$ for system (30) with local models (41), (42), MFs (26) and gains (45), (b) switching law satisfying (40).

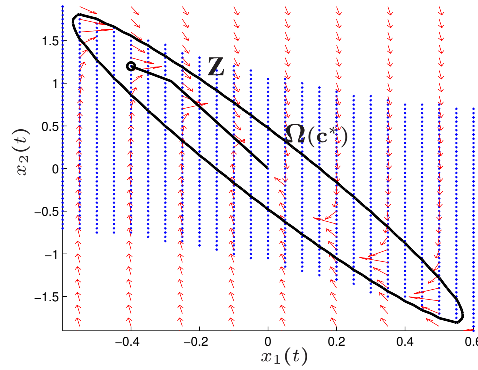


Fig. 4. Phase plane and basin of attraction for switched system (30) with local models (41), (42), MFs (26) and gains (45).

V. CONCLUSIONS

Using T-S fuzzy modeling, LMI-based conditions for the existence of a stabilizing switching law were proposed. It was showed that stabilization of a switched T-S fuzzy system can be obtained by finding a Lyapunov function for a convex combination of all its T-S fuzzy subsystems. In addition, the LMI solution was extended to the state feedback design. One advantage of the approach is that the proposed stabilizing switching laws allow the designer to impose a preference over which subsystem activate.

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