

# Algebraic Conditions for Indirect Controllability in Quantum Coherent Feedback Schemes

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**Abstract**—In coherent feedback control schemes a target quantum system  $S$  is put in contact with an auxiliary system  $A$  and the coherent control can directly affect only  $A$ . The system  $S$  is controlled indirectly through the interaction with  $A$ . The system  $S$  is said to be indirectly controllable if every unitary transformation can be performed on the state of  $S$  with this scheme. In this paper we show how indirect controllability of  $S$  is equivalent to complete controllability of the combined system  $S + A$ , if the dimension of  $A$  is  $\geq 3$ . In the case where the dimension of  $A$  is equal to 2, it is possible to have indirect controllability without having complete controllability of  $S + A$  and we give sufficient conditions for this to happen. We conjecture that these conditions are also necessary. The results of the paper extend the result of [5] and expand the results of [6] to systems of arbitrary dimensions.

## I. INTRODUCTION

In the paper [11], S. Lloyd proposed a scheme for control of quantum systems where the controller itself was a quantum system which was affecting the target system via the interaction. This scheme, named *coherent feedback control* was later expanded in several ways (see [13] for a recent review) and it is currently object of intensive research. The consideration of this scheme motivates the fundamental question of to what extent one can control a system  $S$  indirectly through the interaction with an auxiliary system  $A$ . Controllability studies for systems where only one subsystem  $A$  can be directly accessed but another system  $S$  is the target of control have been carried out in several papers (see, e.g., [3], [7]), however always conditions have been given so that complete controllability of the whole system  $S + A$  (see Definition 2.1 below) is verified. In a recent paper [6], a study was started of indirect controllability (for a precise definition see Definition 2.2 below) and the case where both system  $S$  and  $A$  are two dimensional was treated in detail. It was shown that it is possible to have indirect controllability of system  $S$  without having complete controllability on the system  $S + A$

(while the converse implication is obvious). It was however showed later in [5] that if the system  $A$  is assumed to be in a perfectly mixed state at the beginning of the control procedure, then complete controllability is necessary to have indirect controllability. These results raised the general question of the conditions on the dynamics of the full system  $S + A$  and the initial state of the system  $A$  to have indirect controllability of  $S$ .

In this paper, we will extend the result of [5] by showing that the equivalence between indirect controllability and complete controllability is valid *independently of the initial state of  $A$* . However we will assume that the dimension of the auxiliary system  $A$  is greater than 2. This results will strengthen the results of papers such as [3], [7] that in dealing with indirect control schemes sought conditions of complete controllability.

The case  $\dim A = 2$  is more involved, and not yet completely understood. The two controllability conditions are not equivalent, when also  $\dim S = 2$  and the initial state of  $A$  is pure (see [6]). We will prove that this fact extends to any dimension of  $S$ . We shall give sufficient conditions on the *dynamical Lie algebra* underlying the system  $S + A$  and the initial state of  $A$ , to have indirect controllability. We conjecture in Section IV that these conditions are also necessary.

## II. DEFINITIONS AND MAIN RESULTS

We shall assume that both  $S$  and  $A$  are finite dimensional, and we denote by  $n_S$  and  $n_A$  the dimensions of  $S$  and  $A$ , respectively. According to the postulates of quantum mechanics, if we let  $\mathcal{H}_S$  the Hilbert space of the system  $S$ , and  $\mathcal{H}_A$  the Hilbert space of the system  $A$ , then the Hilbert space of total system  $S + A$  is given by  $\mathcal{H}_S \otimes \mathcal{H}_A$ , and so it has dimension  $n_S n_A$ . Recall that the state of a quantum mechanical system is described by a density matrix  $\rho$ , i.e., a Hermitian, trace 1, positive semi-definite operator (matrix) on the Hilbert space associated with the system. We shall denote by  $\rho_S$ ,  $\rho_A$  and  $\rho_{TOT}$ , the states of the systems  $S$ ,  $A$  and  $S + A$ , respectively. We also shall make the natural assumption that the system  $S + A$  has been prepared at the

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beginning of a control experiment in an uncorrelated state, i.e., at time 0, we have

$$\rho_{TOT}(0) = \rho_S(0) \otimes \rho_A(0). \quad (1)$$

The dynamics of the total system  $S + A$  is determined by a set of Hermitian operators on the Hilbert space associated with  $S + A$ . These are the *Hamiltonians* associated with the system. This set of Hamiltonians is parametrized by elements in a set of controls  $\mathcal{U}$ , so that it can be written as

$$\mathcal{F} := \{H_u | u \in \mathcal{U}\}. \quad (2)$$

A typical situation in experiments is when the  $H_u$  are linear in  $u$ , i.e., they have the form  $H_u := H_0 + \sum_j H_j u_j$  for some finite number of Hamiltonians  $H_0, H_j$ 's and control variables  $u_j$ 's. We shall assume in the following that all the Hamiltonians involved have zero trace. This is done without loss of generality because the introduction of the trace in the Hamiltonians only has the effect of a phase factor in the evolution of the state which has no physical meaning.

It is well known in quantum control theory (cf., e.g., [4], [9], [10]) that the controllability of a finite dimensional quantum system can be assessed by analyzing the (*dynamical*) *Lie algebra*  $\mathcal{L}$  generated by the Hamiltonians available in the evolution of the system. In particular, if  $e^{\mathcal{L}}$  denotes the Lie group associated with  $\mathcal{L}$ , then the set of possible evolutions for the quantum system is dense in  $e^{\mathcal{L}}$  and it is equal to  $e^{\mathcal{L}}$  if  $e^{\mathcal{L}}$  is compact. The dynamical Lie algebra  $\mathcal{L}$  associated with the system  $S + A$ , is the one generated by the elements in  $i\mathcal{F} := \{iH_u | u \in \mathcal{U}\}$ , i.e.,  $\mathcal{L}$  is the smallest subalgebra of  $su(n_S n_A)$  containing  $i\mathcal{F}$ .

*Definition 2.1:* A quantum system is said to be *completely controllable* if, for any special unitary transformation there exists a feasible evolution (i.e., a sequence of exponentials of the form  $e^{-iHt}$ , with  $t \geq 0$  and  $H$  in  $\mathcal{F}$ ) realizing that transformation.

From the above recalled controllability results of [9], [10], complete controllability can be characterized using the dynamical Lie algebra  $\mathcal{L}$ , i.e., it is verified if and only if  $\mathcal{L} := su(n_S n_A)$  (or  $\mathcal{L} := u(n_S n_A)$ ).

We make the following two assumptions on the dynamics of our model:

- (a) The set  $\mathcal{F}$  contains at least one element with nonzero component on the subspace

$$\text{span}\{S \otimes \sigma | S \in su(n_S), \sigma \in su(n_A)\}.$$

This is a natural assumption because it means that there exists an available Hamiltonian modeling the *interaction* between  $S$  and  $A$ .

- (b) The dynamical Lie algebra  $\mathcal{L}$  contains all matrices of the form  $\mathbf{1}_S \otimes \sigma$ , with  $\sigma \in su(n_A)$ .<sup>1</sup> This fact means that we have full unitary control on the auxiliary system  $A$ .

With initial condition  $\rho_{TOT} = \rho_S \otimes \rho_A$ , the set of available states for  $S + A$  is (dense in <sup>2</sup>)

$$\mathcal{R} := \{U \rho_S \otimes \rho_A U^\dagger | U \in e^{\mathcal{L}}\}. \quad (3)$$

The set of possible values for  $\rho_S$  is obtained by taking the partial trace with respect to the system  $A$  of the elements in  $\mathcal{R}$ , i.e., it is the set of elements

$$\text{Tr}_A(\mathcal{R}) := \{\text{Tr}_A(U \rho_S \otimes \rho_A U^\dagger) | U \in e^{\mathcal{L}}\}. \quad (4)$$

*Definition 2.2:* A quantum system  $S$  is said to be *indirectly controllable* given  $\rho_A$ , initial state of the probe  $A$ , if for every  $X \in SU(n_S)$  and  $\rho_S$ , initial state of  $S$ , there exists a reachable evolution  $U \in e^{\mathcal{L}}$  of the whole system  $S + A$  such that

$$\text{Tr}_A(U \rho_S \otimes \rho_A U^\dagger) = X \rho_S X^\dagger, \quad (5)$$

i.e.,  $X \rho_S X^\dagger \in \text{Tr}_A(\mathcal{R})$ .

Our goal is to give necessary and sufficient conditions for indirect controllability given  $\rho_A$ , in terms of the dynamical Lie algebra  $\mathcal{L}$  and  $\rho_A$ . The situation is quite different if  $n_A \geq 3$  and if  $n_A = 2$ . For  $n_A \geq 3$ , we are able to obtain a complete result as follows:

*Theorem 1:* Assume  $n_A \geq 3$ , and let  $\rho_A$  be any initial state of the probe  $A$ .  $S$  is indirectly controllable given  $\rho_A$  if and only if  $S + A$  is completely controllable, i.e.,  $\mathcal{L} = su(n_S n_A)$ .

Notice in particular as a consequence of this result that for  $n_A \geq 3$  indirect controllability does not depend on the initial state of  $A$ .

The case  $n_A = 2$ , where this equivalence is false, is not completely understood. We can however give a sufficient condition for indirect controllability which extends the one given in [6] for the case  $n_S = 2$ . In order to do that we need to give some more definitions (which we shall use for any  $n_A \geq 2$ ). Denote by  $\mathcal{K}$  the space of matrices  $K$  in  $su(n_S)$  such that  $K \otimes \mathbf{1}$  belongs to  $\mathcal{L}$  and denote by  $\mathcal{P}$  the space of matrices  $P$  of  $su(n_S)$  such that  $iP \otimes \sigma$  belongs to  $\mathcal{L}$  for every  $\sigma \in su(n_A)$ . Denote by  $\mathcal{L}_S$  the sum (not necessarily direct) of  $\mathcal{K}$  and  $\mathcal{P}$ .

<sup>1</sup> $\mathbf{1}_S$  and  $\mathbf{1}_A$  denote the  $n_S \times n_S$  and  $n_A \times n_A$  identity matrix. Sometimes we shall omit the subscript  $S$  or  $A$  since the position of the matrix in the tensor product suggests its dimension.

<sup>2</sup>We shall neglect in the following this distinction and refer to the set  $\mathcal{R}$  as the set of available states for  $S + A$ . In fact all the Lie groups we will encounter will be compact so that equality holds.

*Theorem 2:* Assume  $n_A = 2$ . If  $\rho_A$  is a pure state and  $\mathcal{L}_S = su(n_S)$  then  $S$  is indirectly controllable.

From the proofs of these two theorems it will follow that, for  $n_A \geq 3$ , indirect controllability is equivalent to  $\mathcal{L}_S = su(n_S)$ , which, in turns, is equivalent to  $\mathcal{L} = su(n_S n_A)$ . On the other hand, when  $n_A = 2$ , we know that  $\mathcal{L}_S = su(n_S)$  does not in general imply  $\mathcal{L} = su(n_S n_A)$ , (see Example 4.1 below), but still, if  $\rho_A$  is a pure state, this is sufficient for indirect controllability. We shall also conjecture that indirect controllability implies  $\mathcal{L}_S = su(n_S)$  and that if  $\mathcal{L}_S = su(n_S)$  but  $\mathcal{L} \neq su(n_S)$  then indirect controllability given  $\rho_A$  implies that  $\rho_A$  is a pure state.

### III. PROOFS OF THE MAIN RESULTS

Let us consider again the spaces  $\mathcal{K}$  and  $\mathcal{P}$  defined before the statement of Theorem 2 in the previous section. These spaces satisfy the commutation relations of a Riemannian symmetric space [8], i.e.,

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}. \quad (6)$$

The first two of these relations are obvious, while the third one is obtained by calculating, given  $P_1$  and  $P_2$  in  $\mathcal{P}$ ,

$$-\frac{1}{2} \sum_{j=1}^{n_A} [iP_1 \otimes \sigma_j, iP_2 \otimes \sigma_j] = [P_1, P_2] \otimes \mathbf{1} \in \mathcal{L}. \quad (7)$$

Here  $\sigma_j$ ,  $j = 1, \dots, n_A$ , denotes the matrix in  $su(n_A)$  with  $i$  and  $-i$  in position  $j$  and  $j+1 \pmod{n_A}$ , on the main diagonal, respectively and zeros everywhere else.

We also have the anti-commutation relation

$$i\{\mathcal{P}, \mathcal{P}\} \subseteq \mathcal{P} \oplus \text{span}\{i\mathbf{1}_S\}. \quad (8)$$

In order to see this consider  $\sigma_x$  and  $\sigma_y$  the standard Pauli matrices in  $su(2)$  which satisfy

$$[\sigma_x, \sigma_y] = \sigma_z, \quad \text{and} \quad \{\sigma_x, \sigma_y\} = 0. \quad (9)$$

In  $su(n_A)$ , with  $n_A \geq 3$ , we also denote, with some abuse in notation, by  $\sigma_{x,y,z}$  matrices which have the corresponding Pauli matrix in the block corresponding to the first two rows and columns and zeros everywhere else. They satisfy the same commutation and anti-commutation relation in (9). We have for any  $P_1$  and  $P_2$  in  $\mathcal{P}$

$$[iP_1 \otimes \sigma_x, iP_2 \otimes \sigma_y] = \frac{i}{2} i\{P_1, P_2\} \otimes \sigma_z \in \mathcal{L}. \quad (10)$$

From this, equation (8) follows, by assumption (b), since if an element  $iP \otimes \sigma_1$  is in  $\mathcal{L}$  for one non-zero  $\sigma_1 \in su(n_A)$  then  $iP \otimes \sigma$  in  $\mathcal{L}$  for every  $\sigma$ .<sup>3</sup>

<sup>3</sup>This was called a *simplicity Lemma* in [5] and it follow from the fact that  $\bigoplus_{k=0}^{\infty} ad_{su(n)}^k \text{span}\{\sigma\}$ , for any  $\sigma \in su(n)$  is a nontrivial ideal in  $su(n)$  and since  $su(n)$  is simple, then it must be the whole  $su(n)$ .

We denote by  $\tilde{\mathcal{P}} := \mathcal{P} \oplus \text{span}\{i\mathbf{1}_S\}$ .

*Lemma 3.1:* Let  $\mathcal{A}$  a maximal Abelian subalgebra in  $\tilde{\mathcal{P}}$ . After a possible change of coordinates in the Lie algebra  $\mathcal{L}_S$ , every element in  $\mathcal{A}$  can be written as

$$P = \begin{pmatrix} i\mu_1 \mathbf{1}_{n_1} & 0 & 0 & 0 \\ 0 & i\mu_2 \mathbf{1}_{n_2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & i\mu_l \mathbf{1}_{n_l} \end{pmatrix}, \quad (11)$$

where the real constants  $\mu_j$  are arbitrary.

An alternative statement of the Lemma is that positive integers  $n_1, n_2, \dots, n_l$  exist so that a basis of  $\mathcal{A}$  in appropriate coordinates is given by

$$D_1 := \text{diag}\{i\mathbf{1}_{n_1}, \mathbf{0}_{n_2}, \dots, \mathbf{0}_{n_l}\}, \dots, \quad (12)$$

$$D_l := \text{diag}\{\mathbf{0}_{n_1}, \mathbf{0}_{n_2}, \dots, i\mathbf{1}_{n_l}\}.$$

*Proof:* All matrices in  $\mathcal{A}$  can be simultaneously diagonalized via a change of coordinates. So we can assume that all matrices in  $\mathcal{A}$  are diagonal. Consider a basis of  $\mathcal{A}$ ,  $\mathcal{B}_S := \{A_1, \dots, A_l\}$ . With each element  $A_j$ ,  $j = 1, \dots, l$ , it is associated a partition  $\Pi_j$  of the indices  $\{1, 2, \dots, n_S\}$  where two integers are put in the same set if and only if there is the same value on the diagonal in  $A_j$  in the corresponding positions.<sup>4</sup> With any partition  $\Pi_j$ , it is associated a set of skew-Hermitian matrices  $\{\tilde{D}_1, \dots, \tilde{D}_{|\Pi_j|}\}$  where each matrix corresponds to a set in the partition  $\Pi_j$  and it has values equal to  $i$  on the diagonal element corresponding to that set and zero everywhere else. Consider the partition  $\Pi$  product of all the partitions  $\Pi_1, \dots, \Pi_l$ .<sup>5</sup> With  $\Pi$  it is associated a set of matrices  $\{D_1, \dots, D_w\}$  where  $D_j$ ,  $j = 1, \dots, w$ , has  $i$ 's on the entries on the diagonal corresponding to the  $j$ -th set of  $\Pi$ . We will show that  $\{D_1, \dots, D_w\}$  is a basis for  $\mathcal{A}$  (so in particular  $w = l$ ). From this, the lemma follows by a change of coordinates corresponding to a permutation of the indices.

It is clear that  $\{D_1, \dots, D_w\}$  are linearly independent. We only have to show that they span  $\mathcal{A}$ . We will prove that for all  $\tilde{\mathcal{B}}_S$  subset of  $\mathcal{B}_S$ , if we let  $\tilde{\Pi}$  the corresponding partition (product of all the partitions of the elements in  $\tilde{\mathcal{B}}_S$ ) and  $\{\tilde{D}_1, \dots, \tilde{D}_{|\tilde{\Pi}|}\}$  the corresponding set of matrices, then

$$\text{span } \tilde{\mathcal{B}}_S \subseteq \text{span}\{\tilde{D}_1, \dots, \tilde{D}_{|\tilde{\Pi}|}\} \subseteq \mathcal{A}. \quad (13)$$

<sup>4</sup>For example the partition corresponding to  $A := \text{diag}(2i, 2i, 1i, 0)$  is  $\Pi := \{1, 2\}\{3\}\{4\}$ .

<sup>5</sup>Recall that the product of two partitions  $\{K_1, \dots, K_f\}$ ,  $\{T_1, \dots, T_g\}$ , is defined as the partition given by the sets  $\{K_1 \cap T_1, K_1 \cap T_2, \dots, K_1 \cap T_g, K_2 \cap T_1, K_2 \cap T_2, \dots, K_2 \cap T_g, \dots, K_f \cap T_1, K_f \cap T_2, \dots, K_f \cap T_g\}$  and recursively one defines the product of three or more partitions.

By applying (13) for  $\tilde{\mathcal{B}}_S = \mathcal{B}_S$  and recalling that  $\mathcal{B}_S$  is a basis of  $\mathcal{A}$ , we get that the set  $\{D_1, \dots, D_w\}$  spans  $\mathcal{A}$ , as desired. We can prove (13) by induction on the cardinality  $\tilde{l}$  of  $\tilde{\mathcal{B}}_S$ . If  $\tilde{l} = 1$ , consider the matrix  $\tilde{A} := \text{diag}\{i\lambda_1 \mathbf{1}_{n_1}, \dots, i\lambda_s \mathbf{1}_{n_s}\}$  which is the only element in  $\tilde{\mathcal{B}}_S$ , where we have assumed, without loss of generality, that the  $\lambda_j$ 's are all different (i.e., we have grouped together entries that have the same value, assuming to be in appropriate coordinates). Applying (8) recursively it follows that  $\text{diag}\{i\lambda_1^k \mathbf{1}_{n_1}, \dots, i\lambda_s^k \mathbf{1}_{n_s}\}$  is in  $\mathcal{A}$  for every  $k \geq 1$ . A Vandermonde determinant argument, using the fact that,  $\lambda_1, \dots, \lambda_s$  are all different, shows that  $\tilde{D}_1 := \text{diag}\{i\mathbf{1}_{n_1}, 0, \dots, 0\}$ ,  $\tilde{D}_2 := \{0, \mathbf{1}_{n_2}, 0, \dots, 0\}, \dots, \tilde{D}_s := \{0, 0, \dots, 0, \mathbf{1}_{n_s}\}$  are also in  $\mathcal{A}$  and clearly they span  $\tilde{\mathcal{B}}_S$ .

Now assume that (13) holds for a certain set  $\tilde{\mathcal{B}}_S$  and a partition  $\tilde{\Pi}$  associated with it as well as the corresponding set of matrices  $\tilde{D}_1, \dots, \tilde{D}_s$ . Augment the set  $\tilde{\mathcal{B}}_S$  with another element of  $\mathcal{B}_S$ ,  $\tilde{A}$ . To  $\tilde{A}$ , it will correspond another partition  $\tilde{\Pi}_A$  and another set of matrices  $\tilde{D}_{A_1}, \dots, \tilde{D}_{A_s}$ . Consider now the product partition  $\tilde{\Pi}\tilde{\Pi}_A$  and let the set  $F_j \cap T_k$  be a given set of this partition. If  $\tilde{D}_j$  and  $\tilde{D}_{A_k}$  are the matrices corresponding to  $F_j$  and  $T_k$ , respectively, the matrix corresponding to  $F_j \cap T_k$  can be obtained from  $i\{\tilde{D}_j, \tilde{D}_{A_k}\}$  which according to (8) still belongs to  $\mathcal{A}$ . It is clear that the span of  $\tilde{\mathcal{B}}_S \cup \{\tilde{A}\}$  is included in the span of the matrices corresponding to  $\tilde{\Pi}\tilde{\Pi}_A$ . This concludes the inductive step and so the proof of the lemma. ■

Recall that the Lie algebra  $\mathcal{L}_S$  is defined as the sum of  $\mathcal{P}$  and  $\mathcal{K}$  and it is in general a subalgebra of  $su(n_S)$ . Next lemma is used in the proof of Theorem 1.

*Lemma 3.2:* Assume that  $n_A \geq 3$  and let  $\rho_A$  be any initial state of the auxiliary system  $A$ . If  $S$  is indirectly controllable given  $\rho_A$  then  $\mathcal{L}_S = su(n_S)$

*Proof:* Assume, by contradiction, that  $\mathcal{L}_S \neq su(n_S)$ .

If  $\mathcal{K} \cap \mathcal{P} \neq \{0\}$ , then given any matrix  $A \neq 0$  such that  $A \in \mathcal{K} \cap \mathcal{P}$  we choose as initial state  $\rho_S = \frac{1}{n_S} \mathbf{1} + \alpha iA$ , which for  $\alpha$  sufficiently small is a possible density matrix. Given any  $\rho_A = \frac{1}{n_A} \mathbf{1} + i\sigma$ , for  $\sigma \in su(n_A)$ , we have that  $i\rho_S \otimes \rho_A$  belongs to  $\tilde{\mathcal{L}} = (\text{span } i\mathbf{1} \otimes \mathbf{1}) \oplus \mathcal{L}$ , which is invariant under  $\mathcal{L}$ . Since we are assuming  $\mathcal{L}_S \neq su(n_S)$ , we have  $\text{Tr}_A(\tilde{\mathcal{L}}) \neq u(n_S)$  which contradicts the necessary condition for indirect controllability given in [6] (Theorem 1).

Now we assume that  $\mathcal{K} \cap \mathcal{P} = \{0\}$ .

If  $\tilde{\mathcal{P}}$  is Abelian we can assume that *all* the matrices in  $\tilde{\mathcal{P}}$  are of the form (11) of Lemma 3.1 and we can take the basis of  $\tilde{\mathcal{P}}$  as in (12). Partition any  $K \in \mathcal{K}$  according to the partition in the basis of  $\tilde{\mathcal{P}}$ . So, since  $[K, \tilde{\mathcal{P}}] \in \tilde{\mathcal{P}}$ , every matrix in  $[K, \tilde{\mathcal{P}}]$  must be of the type (11). From this fact,

it is easy to see that the matrices  $K \in \mathcal{K}$ , must have the following block diagonal structure:

$$K = \begin{pmatrix} K_{1,1} & 0 & 0 & 0 \\ 0 & K_{2,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & K_{r,r} \end{pmatrix}, \quad (14)$$

with  $K_{j,j} \in su(n_j)$ . Matrices in the Lie Algebra  $\mathcal{L}_S$  also have this block diagonal structure<sup>6</sup> and matrices in  $\mathcal{L}$  also have a block diagonal structure induced by this structure. Thus a matrix  $U \in e^{\mathcal{L}}$  is of the form:

$$U = \begin{pmatrix} U_1 & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & U_r \end{pmatrix}, \quad (15)$$

where the blocks  $U_j$ ,  $j = 1, \dots, r$  have dimension  $n_j n_A$ . Choose any initial state  $\rho_S \otimes \rho_A$  where  $\rho_S$  has the same block structure as in equation (14). Then for any  $U \in e^{\mathcal{L}}$ , also the matrix  $\text{Tr}_A(U\rho_S \otimes \rho_A U^\dagger)$  will have the same block diagonal structure as in equation (14). Since not all the matrices unitarily equivalent to  $\rho_S$  have this diagonal structure (recall that  $r \geq 2$  since  $\mathcal{P}$  is nonzero), the model is not indirectly controllable.

Now we prove that  $\tilde{\mathcal{P}}$  must be necessarily Abelian.<sup>7</sup> In fact, assume, by contradiction that  $\mathcal{P}$  is not Abelian and let  $P_1, P_2 \in \mathcal{P}$  such that  $[P_1, P_2] = K \neq 0$ , with  $K \in \mathcal{K}$ . Since  $n_A \geq 3$  there exist  $\sigma_1, \sigma_2 \in su(n_A)$  such that

$$[\sigma_1, \sigma_2] = 0, \quad \{\sigma_1, \sigma_2\} = \mathbf{1} + i\hat{\sigma}, \quad (16)$$

with  $\hat{\sigma} \in su(n_A)$ , different from zero. We have:

$$\begin{aligned} [iP_1 \otimes \sigma_1, iP_2 \otimes \sigma_2] &= -1/2[P_1, P_2] \otimes (\mathbf{1} + i\hat{\sigma}) \\ &= -1/2K \otimes (\mathbf{1} + i\hat{\sigma}) \in \mathcal{L}. \end{aligned} \quad (17)$$

Since  $K \otimes \mathbf{1} \in \mathcal{L}$ , it follows that  $iK \otimes \hat{\sigma} \in \mathcal{L}$ . Thus  $K \in \mathcal{P}$ , which contradicts the assumption  $\mathcal{K} \cap \mathcal{P} = \emptyset$ . So  $\mathcal{P}$  is Abelian, which concludes the proof of the lemma. ■

To prove the controllability result for  $n_A \geq 3$ , we need the following fact for the Lie algebra  $su(n_S)$ .

*Lemma 3.3:* Assume  $\mathcal{L}_S = su(n_S)$  Then the decomposition  $\mathcal{L}_S = \mathcal{K} \oplus \mathcal{P}$  is either trivial (one between  $\mathcal{K}$  or  $\mathcal{P}$  equal to  $\{0\}$ ) or it is a direct sum, with  $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$  and  $[\mathcal{K}, \mathcal{P}] = \mathcal{P}$  in (6).

<sup>6</sup>Notice that this structure is in a particular system of coordinates but the transformation to get in these coordinates is a local transformation acting on  $S$  only. So, it does not affect the indirect controllability properties of system  $S$ .

<sup>7</sup>Here the assumption  $n_A \geq 3$  is used.

For space reason we omit the prove of this lemma, which can be found in [2].

Now we prove the controllability result for  $n_S \geq 3$ .

*Proof of Theorem 1* Let  $\rho_A$  be any initial state of  $A$ , then the following are equivalent:

- 1)  $S$  is indirectly controllable given  $\rho_A$ ,
- 2)  $\mathcal{L}_S = su(n_S)$ ,
- 3)  $\mathcal{L} = su(n_S n_A)$ ,
- 4)  $S + A$  is completely controllable.

The equivalence between 3 and 4 follows from the standard controllability results recalled after Definition 2.1 (see [9], [10]). Lemma 3.2 gives that 1 implies 2. Since clearly 4 implies 1, to get equivalence it is sufficient to prove that 2 implies 3.

Assume that  $\mathcal{L}_S = su(n_S)$ . We will establish that all the matrices of the type  $iK \otimes \sigma$  and  $P \otimes \mathbf{1}$ , with  $K \in \mathcal{K}$ ,  $P \in \mathcal{P}$  and  $\sigma \in su(n_A)$ , are in  $\mathcal{L}$ . This implies that  $\mathcal{L} = su(n_S n_A)$ .

Since  $n_A \geq 3$ , we may take  $\sigma_1, \sigma_2 \in su(n_A)$  such that equation (16) is satisfied. Then, as computed in equation (17), given  $P_1, P_2 \in \mathcal{P}$ , we have:

$$[iP_1 \otimes \sigma_1, iP_2 \otimes \sigma_2] = -1/2K \otimes (\mathbf{1} + i\hat{\sigma}), \quad (18)$$

for  $K \in \mathcal{K}$ . We can assume  $K \neq 0$  since  $\mathcal{P}$  cannot be Abelian. Since  $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$  from Lemma 3.3, and  $K \otimes \mathbf{1} \in \mathcal{L}$ , we have

$$iK \otimes \hat{\sigma} \in \mathcal{L}, \text{ for all } K \in \mathcal{K}.$$

Since  $\mathbf{1} \otimes su(n_A) \in \mathcal{L}$ , from the previous equation we get that<sup>8</sup>

$$iK \otimes \sigma \in \mathcal{L}, \text{ for all } K \in \mathcal{K}, \text{ and } \sigma \in su(n_A). \quad (19)$$

Now let  $\sigma \in su(n_A)$ , such that  $\{\sigma, \sigma\} = \mathbf{1}$ , we have:

$$[iP \otimes \sigma, iK \otimes \sigma] = [P, K] \otimes \mathbf{1} \in \mathcal{L}.$$

Since  $[\mathcal{K}, \mathcal{P}] = \mathcal{P}$  from Lemma 3.3, we have :

$$P \otimes \mathbf{1} \in \mathcal{L}, \text{ for all } P \in \mathcal{P}. \quad (20)$$

From equations (19) and (20), the statement follows.  $\square$

This theorem generalizes (for the case  $n_A \geq 3$ ) the main result of [5]. That result stated that indirect controllability for  $\rho_A$  equal to a perfectly mixed state (i.e., a multiple of the identity) implies complete controllability. In particular, if we require *strong indirect controllability* i.e., indirect controllability for any state  $\rho_A$  of the auxiliary system (cf. [6]), this notion implies complete controllability. The

<sup>8</sup>From the simplicity Lemma in [5].

Theorem here is proved for an arbitrary state  $\rho_A$  of the auxiliary system.

Now we prove Theorem 2. The argument is a generalization to  $n_S \geq 2$  of the one given in [6].

*Proof of Theorem 2* Let the initial density operator be  $\rho_1 \otimes \rho_A$ , where  $\rho_A$  is a pure state. Assume that we want to steer  $\rho_1$  to  $\rho_2$ , i.e., we need to find a reachable evolution  $U \in e^{\mathcal{L}}$ , such that;

$$Tr_A(U\rho_1 \otimes \rho_A U^\dagger) = \rho_2. \quad (21)$$

Let  $Y \in SU(n_S)$  be such that  $\rho_2 = Y\rho_1 Y^\dagger$ .

Since  $\mathcal{L}_S = su(n_S)$ , then  $\mathcal{K}$  and  $\mathcal{P}$  provide a Cartan decomposition of  $su(n_S)$  (see equations (6)). Thus, we can write  $Y$  as  $Y = K_1 e^A K_2$ , where  $K_i \in e^{\mathcal{K}}$  and  $A \in \mathcal{A}$ , where  $\mathcal{A}$  is a maximal Abelian subalgebra (Cartan subalgebra) in  $\mathcal{P}$ .

Let  $\bar{\sigma} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ . The Lie group  $e^{\mathcal{L}}$  contains all elements of the form:

$$K \otimes \mathbf{1}, \mathbf{1} \otimes B, e^{itA \otimes \bar{\sigma}}, \quad (22)$$

with  $K \in e^{\mathcal{K}}$ ,  $B \in SU(2)$  and  $A \in \mathcal{A}$ .

Since  $\rho_A$  is a pure state, by applying a transformation  $\mathbf{1} \otimes B$ , we can steer the initial density operator to  $\rho_1 \otimes E_1$ , where  $E_1$  is the matrix with 1 in the (1,1) position and zero elsewhere. Since  $i\bar{\sigma}E_1 = iE_1\bar{\sigma} = E_1$ , one can verify that, for a general matrix  $\rho$  we have:

$$e^{itA \otimes \bar{\sigma}} (\rho \otimes E_1) e^{-itA \otimes \bar{\sigma}} = (e^{tA} \rho e^{-tA}) \otimes E_1 \quad (23)$$

Let  $Z_i = K_i \otimes \mathbf{1}$ , and  $X = e^{iA \otimes \bar{\sigma}}$ , then  $Z_i, X \in \mathcal{L}$ . We have:

$$\begin{aligned} Z_1 X Z_2 \mathbf{1} \otimes B (\rho_1 \otimes \rho_A) (Z_1 X Z_2 \mathbf{1} \otimes B)^\dagger = \\ Y \rho_1 Y^\dagger \otimes E_1 = \rho_2 \otimes E_1, \end{aligned}$$

taking the partial trace in the previous equality we get equation (21), as desired.  $\square$

#### IV. CONCLUDING REMARKS

In this work, we have given conditions for the indirect controllability of a target quantum system  $S$  via an auxiliary system  $A$ . In general such controllability conditions depend on the dynamical Lie algebra  $\mathcal{L}$  associated with the total system  $S + A$  and the initial state,  $\rho_A$  assumed for the auxiliary system  $A$ . When the auxiliary system  $A$  has dimension  $n_A \geq 3$ , indirect controllability of a target system  $S$  is equivalent to complete controllability of the system  $S + A$ , independently of the initial state  $\rho_A$  for  $A$ . This leaves open the case  $n_A = 2$  for which we know that it

is possible to have indirect controllability without having complete controllability [6].

In the proof of Theorem 1, we have proved the equivalence of the following for a fixed state  $\rho_A$  of  $A$ :

- 1)  $S$  is indirect controllable given  $\rho_A$ .
- 2)  $\mathcal{L}_S = su(n_S)$ .
- 3)  $\mathcal{L} = su(n_S n_A)$ .
- 4)  $S + A$  is completely controllable.

The equivalence between 3 and 4 holds in general for finite dimension quantum system, so it holds also if  $n_A = 2$ . The equivalence between 2 and 3 fails when  $n_A = 2$  (see [6] and the next example). Moreover, when  $n_A = 2$ , we know from [6] that  $\mathcal{L}_S = su(n_S)$  plus the condition that the initial state  $\rho_A$  of  $A$  is pure is sufficient to conclude indirect controllability. In [1] we proved that for the case  $n_S = 2$ , the condition of  $\rho_A$  being pure is also necessary for indirect controllability when  $\mathcal{L}_S = su(n_S)$  but  $\mathcal{L}$  is not equal to  $su(n)$ .

We believe that this holds in general. The characterization of indirect controllability for the case  $n_A = 2$  should therefore be as follows.

*Conjecture:* For the case  $n_A = 2$ , if  $S$  is indirect controllable given  $\rho_A$  then  $\mathcal{L}_S = su(n_S)$ . If  $\mathcal{L} = su(n_S n_A)$  then complete controllability is verified and indirect controllability is independent of  $\rho_A$ . If  $\mathcal{L}_S = su(n_S)$  but  $\mathcal{L} \neq su(n_S n_A)$ , then indirect controllability is verified if and only if  $\rho_A$  is a pure state.

One direction of the last statement is proved in Theorem 2.

*Example 4.1:* An example where  $\mathcal{L}_S = su(n_S)$ , but the dynamical Lie algebra  $\mathcal{L} \neq su(n_S n_A)$ , is given as follow (cf. [6]). Consider a two level quantum system  $S$  together with an auxiliary two level quantum system  $A$  (thus  $n_S = n_A = 2$ ), with the Hamiltonian  $H$  for the system  $S + A$  given by:

$$-iH = \omega_S \sigma_z \otimes \mathbf{1} + iJ \sigma_x \otimes \sigma_x + \mathbf{1} \otimes \sum_{k=x,y,z} \sigma_k u_k(t). \quad (24)$$

Here  $\omega_S$  represents the Larmor frequency of the spin  $S$ ,  $J$  is the coupling constant between the two spins which have an Ising interaction of the form  $\sigma_x \otimes \sigma_x$ . The three controls  $u_x, u_y, u_z$  give arbitrary control on the system  $A$ , while no direct control is present on system  $S$ . The dynamical Lie algebra  $\mathcal{L}$  for this model is given by:

$$\mathcal{L} := \text{span}\{\sigma_z \otimes \mathbf{1}, i\{\sigma_x, \sigma_y\} \otimes \sigma_1, \mathbf{1} \otimes \sigma_2 \mid \sigma_1, \sigma_2 \in su(2)\}, \quad (25)$$

This Lie algebra  $\mathcal{L}$  is a *proper* subalgebra of  $su(4)$ . It can be easily seen that it is isomorphic to the symplectic

Lie algebra  $sp(2)$ . Thus, for this model we have  $\mathcal{L} \neq su(n_S n_A) = su(4)$ , however  $\mathcal{L}_S = su(n_S) = su(2)$ . It has been proved in [1] that, even if the system  $S + A$  is not controllable, the system  $S$  is indirectly controllable if and only if the initial state  $\rho_A$  of  $A$  is pure.

## REFERENCES

- [1] F. Albertini and D. D'Alessandro Control of a two level quantum system in a coherent feedback scheme, *J. Phys. A, Math. Theor.* 460, No. 4, 045301 (8pp), 2013
- [2] F. Albertini, D. D'Alessandro, and R. Romano, Exact Algebraic Conditions for Indirect Controllability in Quantum Coherent Feedback Schemes, preprint.
- [3] D. Burgarth, S. Bose, C. Bruder and V. Giovannetti, Local controllability of quantum networks, *Physical Review A* 79, 060305(R) (2009)
- [4] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, CRC-Press, Boca Raton FL, 2007.
- [5] D. D'Alessandro, Equivalence Between Indirect Controllability and Complete Controllability for Quantum Systems, submitted to *Systems and Control Letters*.
- [6] D. D'Alessandro and R. Romano, Indirect controllability of quantum systems; A study of two interacting quantum bits, *IEEE Transactions on Automatic Control*, Vol. 57, No. 8, August 2012, pp. 2009-2020.
- [7] H.C. Fu, H. Dong, X.F. Liu and C.P. Sun, "Indirect control of quantum systems via an accessor: pure coherent control without system excitation", *Journal of Physics A* 42, 045303, 2009
- [8] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.
- [9] G. M. Huang, T. J. Tarn and J. W. Clark, "On the controllability of quantum mechanical systems", *Journal of Mathematical Physics* 24 no. 11, pp. 2608-2618, (1983).
- [10] V. Jurdjević and H. Sussmann, Control systems on Lie groups, *Journal of Differential Equations*, 12, 313-329, (1972).
- [11] S. Lloyd, Coherent quantum feedback, *Phys. Rev. A*, 2000, 62:022108.
- [12] T. Strubel, Basics on Hermitian Symmetric Spaces, Diploma-Thesis, ETH Zürich, Winter semester 2006/2007, available on line at <http://www.aimath.org/WWN/surfacegroups/strubel.pdf>.
- [13] G. Zhang and M.R. James, Quantum feedback networks and control: A brief survey, to appear in *Chinese Science Bulletin*, quant-ph.arXiv: 1201.6020.