2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Consensus Control Protocols for Nonlinear Dynamical Systems via Hybrid Stabilization of Sets

Wassim M. Haddad, Sergey G. Nersesov, and Masood Ghasemi

Abstract— **In this paper, we develop a hybrid control framework for addressing multiagent consensus control protocols for general nonlinear dynamical systems using stabilization of sets. The proposed framework develops a novel class of fixed-order, energy-based hybrid controllers that combine a logical switching architecture with the continuous system dynamics to guarantee that a system generalized energy function whose zero level set characterizes a specified system formation is strictly decreasing across switchings. The proposed approach addresses general nonlinear dynamical systems and is not limited to systems involving single integrator dynamics for consensus control.**

I. Introduction

Using system-theoretic thermodynamic concepts, an energy- and entropy-based hybrid controller architecture was proposed in [1], [2] as a means for achieving enhanced energy dissipation in lossless and dissipative dynamical systems. These dynamic controllers combined a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework developed in [1] leads to closed-loop systems described by impulsive differential equations [2]. In particular, the authors in [1], [2] construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. Specifically, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy-based and entropy-based hybrid controllers involving state-dependent switching were also developed to show the efficacy of the approach.

Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks [3]. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, and congestion control in communication networks. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents, wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus* [4], [5], [6]. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [7], [8], and not asymptotic stability, is the relevant notion of stability. In addition, system-theoretic thermodynamic concepts [9], [4], [5], [6] have proved invaluable in addressing Lyapunov stability and convergence for nonlinear dynamical networks.

Convergence and state equipartitioning also arise in numerous complex large-scale dynamical networks that demonstrate a degree of synchronization. System synchronization typically involves coordination of events that allows a dynamical system to operate in unison resulting in system self-organization. The onset of synchronization in populations of coupled dynamical networks have been studied for various complex networks including network models for mathematical biology, statistical physics, kinetic theory, bifurcation theory, as well as plasma physics [10]. Synchronization of firing neural oscillator populations also appears in the neuroscience literature [11].

Since a specified formation of multiagent systems, which can include flocking, cyclic pursuit, rendezvous, or consensus, can be characterized by a hyperplane or manifold in the state space, in this paper we extend the results of [1], [2] to develop a state-dependent hybrid control framework for addressing multiagent formation control protocols for general nonlinear dynamical systems using hybrid stabilization of sets. The proposed framework involves a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations. These dynamic controllers combine a logical switching architecture with continuous dynamics to guarantee that a system generalized energy function, whose zero level set characterizes a specified system formation, is strictly decreasing across switchings. The general framework leads to hybrid closed-loop systems described by impulsive differential equations and addresses general nonlinear dynamical systems without limiting consensus protocols to single integrator models.

II. Hybrid Control and Impulsive Dynamical Systems

In this section, we establish definitions, notation, and review some basic results on impulsive dynamical systems [2]. Let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{Z}_+$ denote the set of positive integers, $(\cdot)^{\mathrm{T}}$ denote transpose, I_n denote the $n \times n$ identity matrix, and $0_{n\times n}$ denote the $n \times n$ zero matrix. Furthermore, let $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, and $\overline{\mathcal{S}}$ denote the boundary, the interior, and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the *open ball centered* at α with *radius* ε, and $V'(x)$ for the Fréchet derivative of V at x. Finally, we write $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists

This research was supported in part by AFOSR under Grant FA9550-12-1-0192 and ONR under Grant N00014-09-1-1195.

W. M. Haddad is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA; S. G. Nersesov and M. Ghasemi are with the Department of Mechanical Engineering, Villanova University, Villanova, PA 19085-1681, USA (wm.haddad@aerospace.gatech.edu; sergey.nersesov@villanova.edu; mghase01@villanova.edu).

$T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$.

In this paper, we consider continuous-time nonlinear dynamical systems of the form

$$\dot{x}_{\mathrm{p}}(t) = f_{\mathrm{p}}(x_{\mathrm{p}}(t), u(t)), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}0}, \quad t \geq 0, \tag{1}$$

$$y(t) = h_{\mathrm{p}}(x_{\mathrm{p}}(t)), \tag{2}$$

where $t \geq 0$, $x_{\mathrm{p}}(t) \in \mathcal{D}_{\mathrm{p}} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{p}}$ is an open set, $u(t) \in \mathbb{R}^m$, $f_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^m \to \mathbb{R}^{n_{\mathrm{p}}}$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}_{\mathrm{p}} \times \mathbb{R}^m$, and $h_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \to \mathbb{R}^l$ is smooth. Furthermore, we consider hybrid (i.e., resetting) dynamic controllers of the form

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), y(t)) \notin \mathcal{Z}_{\mathrm{c}}, \tag{3}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \quad (x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{4}$$

$$u(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \tag{5}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathcal{D}_{\mathrm{c}} \subseteq \mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set, $\Delta x_{\mathrm{c}}(t) \triangleq x_{\mathrm{c}}(t^+) - x_{\mathrm{c}}(t)$, where $x_{\mathrm{c}}(t^+) \triangleq x_{\mathrm{c}}(t) + f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)) = \lim_{\varepsilon \to 0^+} x_{\mathrm{c}}(t+\varepsilon)$, $(x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}$, $f_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is smooth on $\mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l$, $h_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^m$ is smooth, $f_{\mathrm{dc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is continuous, and $\mathcal{Z}_{\mathrm{c}} \subset \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l$ is the *resetting set*. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_{c}, which may be less than the plant order n_{p}.

The equations of motion for the closed-loop dynamical system (1)–(5) have the form

$$\dot{x}(t) = f_{\mathrm{c}}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \tag{6}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{7}$$

where $x \triangleq [x_{\mathrm{p}}^{\mathrm{T}}, x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$,

$$f_{\mathrm{c}}(x) \triangleq \begin{bmatrix} f_{\mathrm{p}}(x_{\mathrm{p}}, h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))) \\ f_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \end{bmatrix}, \tag{8}$$

$$f_{\mathrm{d}}(x) \triangleq \begin{bmatrix} 0 \\ f_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \end{bmatrix}, \tag{9}$$

and $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \in \mathcal{Z}_{\mathrm{c}}\}$, with $n \triangleq n_{\mathrm{p}} + n_{\mathrm{c}}$ and $\mathcal{D} \triangleq \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$. We refer to the differential equation (6) as the *continuous-time dynamics*, and we refer to the difference equation (7) as the *resetting law*. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from (9) that only those states associated with the controller are reset. To ensure well-posedness of the solutions to (6) and (7), we make the following additional assumptions [2].

Assumption 1: If $x \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $\psi(\delta, x) \notin \mathcal{Z}$, where $\psi(\cdot, \cdot)$ denotes the solution to the continuous-time dynamics (6).

Assumption 2: If $x \in \mathcal{Z}$, then $x + f_{\mathrm{d}}(x) \notin \mathcal{Z}$.

Assumption 1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$; that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, Assumption 2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_{\mathrm{d}}(x_0) \notin \mathcal{Z}$, which serves as the initial condition for the continuous-time dynamics (6).

A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is a *solution* to the impulsive dynamical system (6) and (7) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$, where $\mathcal{I}_{x_0}$ denotes the maximal interval of existence of a solution to (6) and (7), if $x(\cdot)$ is left-continuous and $x(t)$ satisfies (6) and (7) for all $t \in \mathcal{I}_{x_0}$. For further discussion on solutions to impulsive differential equations, see [12], [13]. For convenience, we use the notation $s(t, x_0)$ to denote the solution $x(t)$ of (6) and (7) at time $t \geq 0$ with initial condition $x(0) = x_0$.

For a particular closed-loop trajectory $x(t)$, we let $t_k \triangleq \tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (6) and (7) from the initial condition $x(0) = x_0$ is given by $\psi(t, x_0)$ for $0 < t \leq t_1$. If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_{\mathrm{d}}(x_1)$ according to the resetting law (7). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t - t_1, x_1^+)$, and so on. Our convention here is that the solution $x(t)$ of (6) and (7) is left continuous, that is, it is continuous everywhere except at the resetting times t_k, and $x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon)$ and $x_k^+ \triangleq x(t_k) + f_{\mathrm{d}}(x(t_k)) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$ for $k = 1, 2, \ldots$.

It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times $t_k = \tau_k(x_0)$ are distinct and well defined [2]. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, impulsive dynamical systems of the form (6) and (7) are time-invariant systems. These systems are called *state-dependent impulsive dynamical systems* [2]. Since the resetting times are well defined and distinct, and since the solution to (6) exists and is unique, it follows that the solution of the impulsive dynamical system (6) and (7) also exists and is unique over a forward time interval. For details on the existence and uniqueness of solutions of impulsive dynamical systems in forward time see [12], [13].

Assumption 3: Consider the impulsive dynamical system (6) and (7), and let $s(t, x_0)$, $t \geq 0$, denote the solution to (6) and (7) with initial condition x_0. Then, for every $x_0 \notin \mathcal{Z}$ and every $\varepsilon > 0$ and $t \neq t_k$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $\|x_0 - z\| < \delta(\varepsilon, x_0, t)$, $z \in \mathcal{D}$, then $\|s(t, x_0) - s(t, z)\| < \varepsilon$.

Proposition 2.1 ([1]): Consider the impulsive dynamical system $\mathcal{G}$ given by (6) and (7). Assume that Assumptions 1 and 2 hold, $\tau_1(\cdot)$ is continuous at every $x \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x) < \infty$, and if $x \in \mathcal{Z}$, then $x + f_{\mathrm{d}}(x) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$. Furthermore, for every $x \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ such that $0 < \tau_1(x) < \infty$, assume that the following statements hold:

i) If a sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ is such that $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} \tau_1(x_i)$ exists, then either both $f_{\mathrm{d}}(x) = 0$ and $\lim_{i \to \infty} \tau_1(x_i) = 0$, or $\lim_{i \to \infty} \tau_1(x_i) = \tau_1(x)$.

ii) If a sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ is such that $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} \tau_1(x_i)$ exists, then $\lim_{i \to \infty} \tau_1(x_i) = \tau_1(x)$.

Then $\mathcal{G}$ satisfies Assumption 3.

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, that is, $\lim_{i \to \infty} \tau_1(x_i) = \tau_1(x_0)$ for $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ and $\lim_{i \to \infty} x_i = x_0$. For this result, the following definition is needed. First, however, recall that the *Lie derivative* of a smooth function $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ along the vector field of the continuous-time dynamics $f_{\mathrm{c}}(x)$ is given by $L_{f_{\mathrm{c}}} \mathcal{X}(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{X}(\psi(t, x))|_{t=0} = \frac{\partial \mathcal{X}(x)}{\partial x} f_{\mathrm{c}}(x)$, and the *zeroth* and *higher-order Lie derivatives* are, respectively, defined by $L_{f_{\mathrm{c}}}^0 \mathcal{X}(x) \triangleq \mathcal{X}(x)$ and $L_{f_{\mathrm{c}}}^k \mathcal{X}(x) \triangleq L_{f_{\mathrm{c}}}(L_{f_{\mathrm{c}}}^{k-1} \mathcal{X}(x))$, where $k \geq 1$.

Definition 2.1 ([1]): Let $\mathcal{Q} \triangleq \{x \in \mathcal{D} : \ \mathcal{X}(x) = 0\}$, where $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ is an infinitely differentiable function. A point $x \in \mathcal{Q}$ such that $f_{\rm c}(x) \neq 0$ is *k-transversal* to (6) if there exists $k \in \{1, 2, \ldots\}$ such that

$$L_{f_{\rm c}}^{r}\mathcal{X}(x) = 0, \quad r = 0, \ldots, 2k-2, \quad L_{f_{\rm c}}^{2k-1}\mathcal{X}(x) \neq 0. \tag{10}$$

Proposition 2.2 ([1]): Consider the impulsive dynamical system (6) and (7). Let $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ be an infinitely differentiable function such that $\overline{\mathcal{Z}} = \{x \in \mathcal{D} : \mathcal{X}(x) = 0\}$, and assume that every $x \in \overline{\mathcal{Z}}$ is k-transversal to (6). Then at every $x_0 \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x_0) < \infty$, $\tau_1(\cdot)$ is continuous. Furthermore, if $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\tau_1(x_0) \in (0, \infty)$ and *i)* $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ or *ii)* $\lim_{i\to\infty} \tau_1(x_i) > 0$, where $\{x_i\}_{i=1}^{\infty} \notin \overline{\mathcal{Z}}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

Remark 2.1: The notion of k-transversality introduced in Definition 2.1 differs from the well-known notion of transversality [14], [15] involving an orthogonality condition between a vector field and a differentiable submanifold. In the case where $k = 1$, Definition 2.1 coincides with the standard notion of transversality and guarantees that the solution of the closed-loop system (6) and (7) is not tangent to the closure of the resetting set $\mathcal{Z}$ at the intersection with $\overline{\mathcal{Z}}$ [2]. In general, however, k-transversality guarantees that the sign of $\mathcal{X}(x(t))$ changes as the closed-loop system trajectory $x(t)$ transverses the closure of the resetting set $\mathcal{Z}$ at the intersection with $\overline{\mathcal{Z}}$.

Remark 2.2: Proposition 2.2 is a nontrivial generalization of Lemma 3 of [16]. Specifically, Proposition 2.2 establishes the continuity of $\tau_1(\cdot)$ in the case where the resetting set $\mathcal{Z}$ is not a closed set. In addition, the k-transversality condition given in Definition 2.1 is also a generalization of the transversality conditions given in [16] by considering higher-order derivatives of the function $\mathcal{X}(\cdot)$ rather than simply considering the first-order derivative as in [16].

The next result characterizes impulsive dynamical system limit sets in terms of continuously differentiable functions. In particular, we show that the system trajectories of a state-dependent impulsive dynamical system converge to an invariant set contained in a union of level surfaces characterized by the continuous-time system dynamics and the resetting system dynamics. Note that for addressing the stability of sets of an impulsive dynamical system the usual set stability definitions are valid [8].

Theorem 2.1: Consider the impulsive dynamical system (6) and (7), and assume Assumptions 1–3 hold. Assume $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a positively invariant set with respect to (6) and (7), assume that if $x_0 \in \mathcal{Z}$, then $x_0 + f_{\rm d}(x_0) \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, and assume that there exists a continuously differentiable function $V : \mathcal{D}_{\rm ci} \to \mathbb{R}$ such that

$$V'(x)f_{\rm c}(x) \leq 0, \quad x \in \mathcal{D}_{\rm ci}, \quad x \notin \mathcal{Z}, \tag{11}$$

$$V(x + f_{\rm d}(x)) \leq V(x), \quad x \in \mathcal{D}_{\rm ci}, \quad x \in \mathcal{Z}. \tag{12}$$

Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_{\rm ci} : x \notin \mathcal{Z}, V'(x)f_{\rm c}(x) = 0\} \cup \{x \in \mathcal{D}_{\rm ci} : x \in \mathcal{Z}, V(x + f_{\rm d}(x)) = V(x)\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $x_0 \in \mathcal{D}_{\rm ci}$, then $x(t) \to \mathcal{M}$ as $t \to \infty$. Furthermore, if $\mathcal{D}_0 \subset \overset{\circ}{\mathcal{D}}_{\rm ci}$, $V(x) = 0$, $x \in \mathcal{D}_0$, $V(x) > 0$, $x_0 \in \mathcal{D}_{\rm ci}\backslash\mathcal{D}_0$, and the set $\mathcal{R}$ contains no invariant set other than the set $\mathcal{D}_0$, then the set $\mathcal{D}_0$ is asymptotically stable with respect to (6) and (7), and $\mathcal{D}_{\rm ci}$ is a subset of the domain of attraction of (6) and (7).

Proof. The proof of this result is similar to the proof of Theorem 2.3 and Corollary 2.1 given in [2] and, hence, is omitted. □

Remark 2.3: Setting $\mathcal{D} = \mathbb{R}^n$ and requiring $V(x) \to \infty$ as $\|x\| \to \infty$ in Theorem 2.1, it follows that the set $\mathcal{D}_0$ is globally asymptotically stable. A similar remark holds for Theorem 2.2 below.

Theorem 2.2: Consider the impulsive dynamical system (6) and (7), and assume Assumptions 1–3 hold. Assume $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a positively invariant set with respect to (6) and (7) such that $\mathcal{D}_0 \subset \overset{\circ}{\mathcal{D}}_{\rm ci}$, assume that if $x_0 \in \mathcal{Z}$, then $x_0 + f_{\rm d}(x_0) \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, and assume that for every $x_0 \in \mathcal{D}_{\rm ci}\backslash\mathcal{D}_0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, where $x(t)$, $t \geq 0$, denotes the solution to (6) and (7) with the initial condition x_0. Furthermore, assume there exists a continuously differentiable function $V : \mathcal{D}_{\rm ci} \to \mathbb{R}$ such that $V(x) = 0$, $x \in \mathcal{D}_0$, $V(x) > 0$, $x_0 \in \mathcal{D}_{\rm ci}\backslash\mathcal{D}_0$,

$$V(x + f_{\rm d}(x)) < V(x), \quad x \in \mathcal{D}_{\rm ci}, \quad x \in \mathcal{Z}, \tag{13}$$

and (11) is satisfied. Then the set $\mathcal{D}_0 \subset \mathcal{D}_{\rm ci}$ is asymptotically stable with respect to (6) and (7) and $\mathcal{D}_{\rm ci}$ is a subset of the domain of attraction.

Proof. It follows from (13) that $\mathcal{R} = \{x \in \mathcal{D}_{\rm ci} : x \notin \mathcal{Z}, V'(x)f_{\rm c}(x) = 0\}$. Since for every $x_0 \in \mathcal{D}_{\rm ci}\backslash\mathcal{D}_0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, it follows that the largest invariant set contained in $\mathcal{R}$ is $\mathcal{D}_0$. Now, the result is a direct consequence of Theorem 2.1. □

III. Hybrid Stabilization of Sets

In this section, we present a hybrid controller design framework for stabilization of sets. Specifically, we consider nonlinear dynamical systems $\mathcal{G}_{\rm p}$ of the form given by (1) and (2). Furthermore, we consider hybrid resetting dynamic controllers $\mathcal{G}_{\rm c}$ of the form

$$\dot{x}_{\rm c}(t) = f_{\rm cc}(x_{\rm c}(t), y(t)), \quad x_{\rm c}(0) = x_{\rm c0}, \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \tag{14}$$

$$\Delta x_{\rm c}(t) = \eta(y(t)) - x_{\rm c}(t), \quad (x_{\rm c}(t), y(t)) \in \mathcal{Z}_{\rm c}, \tag{15}$$

$$y_{\rm c}(t) = h_{\rm cc}(x_{\rm c}(t), y(t)), \tag{16}$$

where $x_{\rm c}(t) \in \mathcal{D}_{\rm c} \subseteq \mathbb{R}^{n_{\rm c}}$, $\mathcal{D}_{\rm c}$ is an open set, $y(t) \in \mathbb{R}^l$, $y_{\rm c}(t) \in \mathbb{R}^m$, $f_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{n_{\rm c}}$ is smooth on $\mathcal{D}_{\rm c} \times \mathbb{R}^l$, $\eta : \mathbb{R}^l \to \mathcal{D}_{\rm c}$ is continuous, and $h_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^m$ is smooth.

Consider the negative feedback interconnection of $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ given by $y = u_{\rm c}$ and $u = -y_{\rm c}$. In this case, the closed-loop system $\mathcal{G}$ is given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \quad t \geq 0, \tag{17}$$

$$\Delta x(t) = f_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{18}$$

where $t \geq 0$, $x(t) \triangleq [x_{\rm p}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$, $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{Z}_{\rm c}\}$,

$$f_{\rm c}(x) = \begin{bmatrix} f_{\rm p}(x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \\ f_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \end{bmatrix}, \tag{19}$$

$$f_{\rm d}(x) = \begin{bmatrix} 0 \\ \eta(h_{\rm p}(x_{\rm p})) - x_{\rm c} \end{bmatrix}. \tag{20}$$

The objective is to design the hybrid resetting controller (14)–(16) in such a way that the set $\mathcal{D}_0 = \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : x_{\rm p} \in \mathcal{D}_{\rm p0}\}$, where $\mathcal{D}_{\rm p0} \subset \mathcal{D}_{\rm p}$, is asymptotically stable with respect to the closed-loop system (17) and (18). In order to do this, we associate with the plant a generalized energy function $V_{\rm p} : \mathcal{D}_{\rm p} \to \overline{\mathbb{R}}_+$ such that $V_{\rm p}(x_{\rm p}) = 0$, $x_{\rm p} \in \mathcal{D}_{\rm p0}$, and $V_{\rm p}(x_{\rm p}) > 0$, $x_{\rm p} \in \mathcal{D}_{\rm p}\backslash\mathcal{D}_{\rm p0}$. Furthermore, we associate with the controller a generalized energy function $V_{\rm c} : \mathcal{D}_{\rm c} \times$

$\mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_\mathrm{c}(x_\mathrm{c}, y) \geq 0$, $x_\mathrm{c} \in \mathcal{D}_\mathrm{c}$, $y \in \mathbb{R}^l$, and $V_\mathrm{c}(x_\mathrm{c}, y) = 0$ if and only if $x_\mathrm{c} = \eta(y)$. Finally, we associate with the closed-loop system the generalized energy function $V(x) \triangleq V_\mathrm{p}(x_\mathrm{p}) + V_\mathrm{c}(x_\mathrm{c}, h_\mathrm{p}(x_\mathrm{p}))$.

Next, we construct the resetting set for the closed-loop system $\mathcal{G}$ in the following way:

$$\begin{aligned}\mathcal{Z} = \{(x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c} : L_{f_\mathrm{c}} V_\mathrm{c}(x_\mathrm{c}, h_\mathrm{p}(x_\mathrm{p})) = 0 \\ \text{and } V_\mathrm{c}(x_\mathrm{c}, h_\mathrm{p}(x_\mathrm{p})) > 0\}.\end{aligned} \tag{21}$$

The resetting set $\mathcal{Z}$ is thus defined to be the set of all points in the closed-loop state space that correspond to the instant when the controller is at the verge of decreasing its generalized energy function $V_\mathrm{c}(\cdot)$. By resetting the controller states, the generalized energy function $V_\mathrm{p}(\cdot)$ can never increase after the first resetting event. Furthermore, if the closed-loop system generalized energy function $V(\cdot)$ is conserved between resetting events, then a decrease in $V_\mathrm{p}(\cdot)$ is accompanied by a corresponding increase in $V_\mathrm{c}(\cdot)$. Hence, this approach allows the generalized plant energy to flow to the controller, where it increases the emulated generalized controller energy but does not allow the emulated generalized controller energy to flow back to the plant after the first resetting event.

This energy dissipating hybrid controller effectively enforces a one-way generalized energy transfer between the plant and the controller after the first resetting event. For practical implementation, knowledge of x_c and y is sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$. That is, the full state x_p need not be known in order to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$, neither is it needed for feedback control between resettings determined by (16).

The next theorem gives sufficient conditions for asymptotic stability of the set $\mathcal{D}_0 \subset \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c}$ with respect to the closed-loop system $\mathcal{G}$ using state-dependent hybrid controllers.

Theorem 3.1: Consider the closed-loop impulsive dynamical system $\mathcal{G}$ given by (17) and (18), and assume that $\mathcal{D}_\mathrm{ci} \subset \mathcal{D}$ is a positively invariant set with respect to $\mathcal{G}$ such that $\mathcal{D}_0 \subset \overset{\circ}{\mathcal{D}}_\mathrm{ci}$, where $\mathcal{D}_0 = \{(x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c} : x_\mathrm{p} \in \mathcal{D}_\mathrm{p0}\}$ and $\mathcal{D}_\mathrm{p0} \subset \mathcal{D}_\mathrm{p}$. Assume that there exists a continuously differentiable function $V_\mathrm{p} : \mathcal{D}_\mathrm{p} \to \overline{\mathbb{R}}_+$ such that $V_\mathrm{p}(x_\mathrm{p}) = 0$, $x_\mathrm{p} \in \mathcal{D}_\mathrm{p0}$, and $V_\mathrm{p}(x_\mathrm{p}) > 0$, $x_\mathrm{p} \in \mathcal{D}_\mathrm{p} \backslash \mathcal{D}_\mathrm{p0}$, and assume there exists a smooth (i.e., infinitely differentiable) function $V_\mathrm{c} : \mathcal{D}_\mathrm{c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_\mathrm{c}(x_\mathrm{c}, y) \geq 0$, $x_\mathrm{c} \in \mathcal{D}_\mathrm{c}$, $y \in \mathbb{R}^l$, and $V_\mathrm{c}(x_\mathrm{c}, y) = 0$ if and only if $x_\mathrm{c} = \eta(y)$. Furthermore, assume that every $x_0 \in \overline{\mathcal{Z}}$ is k-transversal to (17) and

$$\dot{V}_\mathrm{p}(x_\mathrm{p}(t)) + \dot{V}_\mathrm{c}(x_\mathrm{c}(t), y(t)) = 0, \quad x(t) \notin \mathcal{Z}, \tag{22}$$

where $y = u_\mathrm{c} = h_\mathrm{p}(x_\mathrm{p})$ and $\mathcal{Z}$ is given by (21). Then, the set $\mathcal{D}_0 \subset \mathcal{D}_\mathrm{ci}$ is asymptotically stable with respect to the closed-loop system $\mathcal{G}$. Finally, if $\mathcal{D}_\mathrm{p} = \mathbb{R}^{n_\mathrm{p}}$, $\mathcal{D}_\mathrm{c} = \mathbb{R}^{n_\mathrm{c}}$, and $V(\cdot)$ is radially unbounded, then the set $\mathcal{D}_0 \subset \mathcal{D}_\mathrm{ci}$ is globally asymptotically stable with respect to $\mathcal{G}$.

To demonstrate the utility of Theorem 3.1, let the set $\mathcal{D}_\mathrm{p0}$ be given by the zero level set of the function $Q_\mathrm{p} : \mathcal{D}_\mathrm{p} \to \mathbb{R}^{s_\mathrm{p}}$ and let $V_\mathrm{p} : \mathcal{D}_\mathrm{p} \to \overline{\mathbb{R}}_+$ be given by

$$V_\mathrm{p}(x_\mathrm{p}) = Q^\mathrm{T}(x_\mathrm{p}) P Q(x_\mathrm{p}), \quad x_\mathrm{p} \in \mathcal{D}_\mathrm{p}, \tag{23}$$

where $P \in \mathbb{R}^{s_\mathrm{p} \times s_\mathrm{p}}$ and $P > 0$. Furthermore, let $V_\mathrm{c} : \mathcal{D}_\mathrm{c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ be given by

$$\begin{aligned}V_\mathrm{c}(x_\mathrm{c}, h_\mathrm{p}(x_\mathrm{p})) = (x_\mathrm{c} - \eta(h_\mathrm{p}(x_\mathrm{p})))^\mathrm{T} P_\mathrm{c} (x_\mathrm{c} - \eta(h_\mathrm{p}(x_\mathrm{p}))), \\ (x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c},\end{aligned} \tag{24}$$

where $P_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c} \times n_\mathrm{c}}$ and $P_\mathrm{c} > 0$. In this case, the functions $f_\mathrm{cc}(\cdot,\cdot)$, $h_\mathrm{cc}(\cdot,\cdot)$, and $\eta(\cdot)$ can be selected using (22) in Theorem 3.1. These constructions are shown for the specific problems of consensus and formation control for multiagent systems in the next sections.

IV. Specialization to Linear Dynamical Systems

In this section, we specialize the results of Section III to the class of linear dynamical systems given by

$$\dot{x}_\mathrm{p}(t) = A x_\mathrm{p}(t) + B u(t), \quad x_\mathrm{p}(0) = x_\mathrm{p0}, \quad t \geq 0, \tag{25}$$

$$y(t) = C x_\mathrm{p}(t), \tag{26}$$

where $x_\mathrm{p}(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{l \times n}$. Here, for simplicity of exposition, we assume $n_\mathrm{p} = n_\mathrm{c} = n$ and $C = I_n$. The case where $C \neq I_n$ can be addressed using an identical analysis as shown below with F_2, H_2, and M in (28)–(30) replaced by $F_2 C, H_2 C$, and MC, respectively. For the system (25) and (26) we construct a hybrid feedback controller of the form (14)–(16) that asymptotically stabilizes the set $\mathcal{D}_0$ given by

$$\mathcal{D}_0 = \{(x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c} : x_\mathrm{p} \in \mathcal{D}_\mathrm{p0}\}, \tag{27}$$

where $\mathcal{D}_\mathrm{p0} = \{x_\mathrm{p} \in \mathcal{D}_\mathrm{p} : T x_\mathrm{p} = 0\}$ and $T \in \mathbb{R}^{s_\mathrm{p} \times n}$. Specifically, we set

$$f_\mathrm{cc}(x_\mathrm{c}, x_\mathrm{p}) = F_1 x_\mathrm{c} + F_2 x_\mathrm{p}, \tag{28}$$

$$h_\mathrm{cc}(x_\mathrm{c}, x_\mathrm{p}) = -H_1 x_\mathrm{c} - H_2 x_\mathrm{p}, \tag{29}$$

$$\eta(x_\mathrm{p}) = M x_\mathrm{p}, \tag{30}$$

where $F_1 \in \mathbb{R}^{n \times n}$, $F_2 \in \mathbb{R}^{n \times n}$, $H_1 \in \mathbb{R}^{m \times n}$, $H_2 \in \mathbb{R}^{m \times n}$, and $M \in \mathbb{R}^{n \times n}$. Thus, the closed-loop system (25), (26), and (14)–(16) with the negative feedback interconnection $u = -y_\mathrm{c}$ is given by

$$\begin{aligned}\dot{x}_\mathrm{p}(t) = (A + BH_2) x_\mathrm{p}(t) + BH_1 x_\mathrm{c}(t), \\ (x_\mathrm{p}(t), x_\mathrm{c}(t)) \notin \mathcal{Z},\end{aligned} \tag{31}$$

$$\dot{x}_\mathrm{c}(t) = F_1 x_\mathrm{c}(t) + F_2 x_\mathrm{p}(t), \quad (x_\mathrm{p}(t), x_\mathrm{c}(t)) \notin \mathcal{Z}, \tag{32}$$

$$\Delta x_\mathrm{c}(t) = M x_\mathrm{p}(t) - x_\mathrm{c}(t), \quad (x_\mathrm{p}(t), x_\mathrm{c}(t)) \in \mathcal{Z}, \tag{33}$$

where $\mathcal{Z}$ is given by (21).

Next, define the generalized energy functions

$$V_\mathrm{p}(x_\mathrm{p}) = \frac{1}{2} x_\mathrm{p}^\mathrm{T} T^\mathrm{T} T x_\mathrm{p}, \quad x_\mathrm{p} \in \mathcal{D}_\mathrm{p}, \tag{34}$$

$$\begin{aligned}V_\mathrm{c}(x_\mathrm{c}, x_\mathrm{p}) = \frac{1}{2} (x_\mathrm{c} - M x_\mathrm{p})^\mathrm{T} P_\mathrm{c} (x_\mathrm{c} - M x_\mathrm{p}), \\ (x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c},\end{aligned} \tag{35}$$

where $P_\mathrm{c} \in \mathbb{R}^{n \times n}$ and $P_\mathrm{c} > 0$. Note that $V_\mathrm{p}(x_\mathrm{p}) = 0$, $x_\mathrm{p} \in \mathcal{D}_\mathrm{p0}$, and $V_\mathrm{p}(x_\mathrm{p}) > 0$, $x_\mathrm{p} \in \mathcal{D}_\mathrm{p} \backslash \mathcal{D}_\mathrm{p0}$. Furthermore, note that $V_\mathrm{c}(x_\mathrm{c}, x_\mathrm{p}) \geq 0$, $(x_\mathrm{p}, x_\mathrm{c}) \in \mathcal{D}_\mathrm{p} \times \mathcal{D}_\mathrm{c}$, and $V_\mathrm{c}(x_\mathrm{c}, x_\mathrm{p}) = 0$ if and only if $x_\mathrm{c} = \eta(x_\mathrm{p})$. For the closed-loop system (31)–(33), condition (22) in Theorem 3.1 gives

$$\begin{aligned}&\dot{V}_\mathrm{p}(x_\mathrm{p}(t)) + \dot{V}_\mathrm{c}(x_\mathrm{c}(t), x_\mathrm{p}(t)) \\ &= x_\mathrm{p}^\mathrm{T}(t)(T^\mathrm{T} T B H_1 + F_2^\mathrm{T} P_\mathrm{c} - A^\mathrm{T} M^\mathrm{T} P_\mathrm{c} - H_2^\mathrm{T} B^\mathrm{T} M^\mathrm{T} P_\mathrm{c} \\ &\quad - M^\mathrm{T} P_\mathrm{c} F_1 + M^\mathrm{T} P_\mathrm{c} M B H_1) x_\mathrm{c}(t) \\ &\quad + x_\mathrm{p}^\mathrm{T}(t)(T^\mathrm{T} T A + T^\mathrm{T} T B H_2 - M^\mathrm{T} P_\mathrm{c} F_2 \\ &\quad + M^\mathrm{T} P_\mathrm{c} M A + M^\mathrm{T} P_\mathrm{c} M B H_2) x_\mathrm{p}(t) \\ &\quad + x_\mathrm{c}^\mathrm{T}(t)(P_\mathrm{c} F_1 - P_\mathrm{c} M B H_1) x_\mathrm{c}(t) \\ &= 0, \quad (x_\mathrm{p}(t), x_\mathrm{c}(t)) \notin \mathcal{Z}.\end{aligned} \tag{36}$$

Since $x_{\rm p}$ and $x_{\rm c}$ are independent state variables, (36) holds if and only if there exist skew-symmetric matrices $A_{\rm p} \in \mathbb{R}^{n\times n}$ and $A_{\rm c} \in \mathbb{R}^{n\times n}$ such that

$$T^{\rm T}TBH_1 + F_2^{\rm T}P_{\rm c} - A^{\rm T}M^{\rm T}P_{\rm c} - H_2^{\rm T}B^{\rm T}M^{\rm T}P_{\rm c} - M^{\rm T}P_{\rm c}F_1 + M^{\rm T}P_{\rm c}MBH_1 = 0, \quad (37)$$

$$T^{\rm T}TA + T^{\rm T}TBH_2 - M^{\rm T}P_{\rm c}F_2 + M^{\rm T}P_{\rm c}MA + M^{\rm T}P_{\rm c}MBH_2 = A_{\rm p}, \quad (38)$$

$$P_{\rm c}F_1 - P_{\rm c}MBH_1 = A_{\rm c}. \quad (39)$$

The skew-symmetric matrices $A_{\rm p} \in \mathbb{R}^{n\times n}$ and $A_{\rm c} \in \mathbb{R}^{n\times n}$ are free design parameters. Furthermore, if the matrices $H_1 \in \mathbb{R}^{m\times n}$ and $H_2 \in \mathbb{R}^{m\times n}$ are fixed, then it follows from (37)–(39) that

$$F_1 = P_{\rm c}^{-1}A_{\rm c} + MBH_1, \quad (40)$$

$$F_2 = MA + MBH_2 - P_{\rm c}^{-1}A_{\rm c}M - P_{\rm c}^{-1}H_1^{\rm T}B^{\rm T}T^{\rm T}T, \quad (41)$$

where $M \in \mathbb{R}^{n\times n}$ satisfies

$$T^{\rm T}TA + T^{\rm T}TBH_2 + M^{\rm T}A_{\rm c}M + M^{\rm T}H_1^{\rm T}B^{\rm T}T^{\rm T}T = A_{\rm p}.$$

Note that if $A_{\rm c}$ is skew-symmetric, then $M^{\rm T}A_{\rm c}M$ is also skew-symmetric. In this case, we can set $A_{\rm p} = \tilde{A}_{\rm p} + M^{\rm T}A_{\rm c}M$, where $\tilde{A}_{\rm p} \in \mathbb{R}^{n\times n}$ is an arbitrary skew-symmetric matrix, so that

$$NM = L, \quad (42)$$

where

$$N \triangleq T^{\rm T}TBH_1, \quad (43)$$

$$L \triangleq -\tilde{A}_{\rm p} - A^{\rm T}T^{\rm T}T - H_2^{\rm T}B^{\rm T}T^{\rm T}T. \quad (44)$$

Recall that a solution M to the matrix equation (42) exists if and only if [17, Fact 6.4.43, p. 421]

$$NN^{\dagger}L = L, \quad (45)$$

where $N^{\dagger} \in \mathbb{R}^{n\times n}$ is the Moore-Penrose generalized inverse of $N \in \mathbb{R}^{n\times n}$. If (45) is satisfied, then every solution to (42) is given by

$$M = N^{\dagger}L + Y - N^{\dagger}NY, \quad (46)$$

where $Y \in \mathbb{R}^{n\times n}$ is an arbitrary matrix; and if $Y = 0$, then tr $M^{\rm T}M$ is minimized. Thus, the existence of a hybrid controller that asymptotically stabilizes the set $\mathcal{D}_0$ given by (27) is characterized by a matrix condition (45). Finally, if

$$T^{\rm T}TBH_1 \neq 0, \quad (47)$$

then the k-transversality condition (10) is satisfied. To see this, note that (47) implies that $\dot{V}_{\rm p}(x_{\rm p}) \not\equiv 0$, which, using (22), implies that $\dot{V}_{\rm c}(x_{\rm c}, x_{\rm p}) \not\equiv 0$. This shows that k-transversality condition, with $k = 1$, holds for the closed-loop system (31)–(33). Note that (47) is guaranteed by (45).

V. Hybrid Control Design for Consensus in Multiagent Networks

In this section, we specialize the results of Section IV to design hybrid consensus controllers for multiagent networks of single integrator systems. Specifically, the consensus problem involves the design of a dynamic protocol algorithm that guarantees system state equipartition [4], [6], that is, $\lim_{t\to\infty} x_{{\rm p}i}(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$, where $x_{{\rm p}i}(t)$ denotes the ith component of the system state vector $x_{\rm p}(t)$. In particular, consider q continuous-time integrator agents with dynamics

$$\dot{x}_{{\rm p}i}(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, q, \quad (48)$$

$$y_i(t) = x_{{\rm p}i}(t), \quad (49)$$

where, for each $i \in \{1, \ldots, q\}$, $x_{{\rm p}i}(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes information control input for all $t \geq 0$. In this case, the set $\mathcal{D}_{\rm p0} = \{x_{\rm p} \in \mathbb{R}^q : x_{\rm p1} = \cdots = x_{{\rm p}q}\}$, where $x_{\rm p} \triangleq [x_{\rm p1}, \ldots, x_{{\rm p}q}]^{\rm T}$, characterizes the state of consensus in the multiagent network.

In the following analysis, we construct a hybrid feedback controller (14)–(16) that asymptotically stabilizes a more general form of the classical consensus steady state for the multiagent network characterized by

$$\mathcal{D}_0 = \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : x_{\rm p} \in \mathcal{D}_{\rm p0}\}, \quad (50)$$

where $\mathcal{D}_{\rm p0} = \{x_{\rm p} \in \mathcal{D}_{\rm p} : Tx_{\rm p} = 0\}$ and $T \in \mathbb{R}^{s_{\rm p}\times q}$. Clearly, $\mathcal{D}_{\rm p0}$ characterizes the equipartitioned consensus state of a multiagent network with

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{(q-1)\times q}. \quad (51)$$

In order to stabilize $\mathcal{D}_0$ given by (50), consider the hybrid feedback controller (14)–(16) with

$$f_{\rm cc}(x_{\rm c}, x_{\rm p}) = F(x_{\rm c} - x_{\rm p}), \quad (52)$$

$$h_{\rm cc}(x_{\rm c}, x_{\rm p}) = -H(x_{\rm c} - x_{\rm p}), \quad (53)$$

$$\eta(x_{\rm p}) = Mx_{\rm p}, \quad (54)$$

where $F \in \mathbb{R}^{q\times q}$, $H \in \mathbb{R}^{q\times q}$, and $M \in \mathbb{R}^{q\times q}$. In this case, Equations (37)–(39) developed for a general class of linear dynamical systems specialize to

$$P_{\rm c}F - P_{\rm c}MH = A_{\rm c}, \quad (55)$$

$$-T^{\rm T}TH + M^{\rm T}A_{\rm c} = A_{\rm p}, \quad (56)$$

$$A_{\rm p} = A_{\rm c}, \quad (57)$$

for the system (48) and (49). Note that (55)–(57) can be further simplified to give

$$F - MH = P_{\rm c}^{-1}A_{\rm c}, \quad (58)$$

$$-T^{\rm T}TH + M^{\rm T}A_{\rm c} = A_{\rm c}. \quad (59)$$

Note that if q is even, then we can always choose a skew-symmetric matrix $A_{\rm c} \in \mathbb{R}^{q\times q}$ such that $A_{\rm c}^{-1}$ exists. In this case, it follows from (58) and (59) that

$$M = A_{\rm c}^{-1}(A_{\rm c} - H^{\rm T}T^{\rm T}T), \quad (60)$$

$$F = P_{\rm c}^{-1}A_{\rm c} + A_{\rm c}^{-1}(A_{\rm c} - H^{\rm T}T^{\rm T}T)H. \quad (61)$$

For the following numerical example, we consider four agents with the dynamics given by (48) and (49) and the objective being to stabilize the equipartitioned consensus state with $T \in \mathbb{R}^{3\times 4}$ given by (51). For our design we set

$$A_{\rm c} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0.5 \\ 0 & 1 & -0.5 & 0 \end{bmatrix}, \quad (62)$$

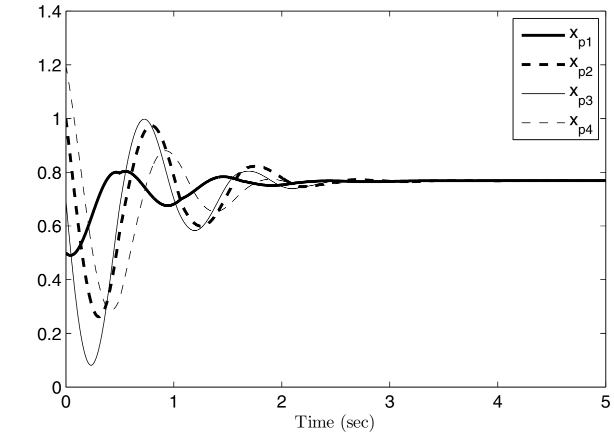

Fig. 1. Plant states $x_{\rm p}$ versus time.

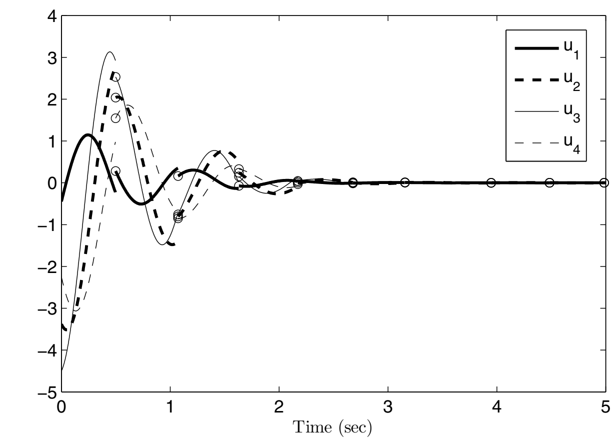

Fig. 2. Control inputs versus time.

$H = 1.5I_4$, and $P_{\rm c} = 0.75I_4$ so that $M \in \mathbb{R}^{4\times 4}$ and $F \in \mathbb{R}^{4\times 4}$ are computed using (60) and (61). Note that with the above choice of $T \in \mathbb{R}^{3\times 4}$ and $H \in \mathbb{R}^{4\times 4}$, condition (47) is satisfied. For the initial conditions $x_{\rm p}(0) = [0.5, 1, 0.7, 1.2]^{\rm T}$ and $x_{\rm c}(0) = Mx_{\rm p}(0)$, Figure 1 shows the system states history versus time, whereas Figure 2 shows the control input history versus time. Finally, Figure 3 shows the time history of the generalized energy functions $V_{\rm p}(x_{\rm p}(t))$ and $V_{\rm c}(x_{\rm c}(t), x_{\rm p}(t))$ versus time. It can be seen from Figure 2 that the control inputs u_i, $i = 1, \ldots, 4$, are discontinuous functions of time.

VI. Conclusion

In this paper, we have developed a general energy-based hybrid control framework for formation control protocols of general dynamical systems using hybrid stabilization of sets. The proposed framework is used to develop a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations which include consensus control of multiagent systems. Specifically, a specified formation is characterized by a hyperplane in the state space and a hybrid feedback architecture is designed that achieves set stabilization for the desired formation thereby addressing consensus control protocols for general nonlinear dynamical models.

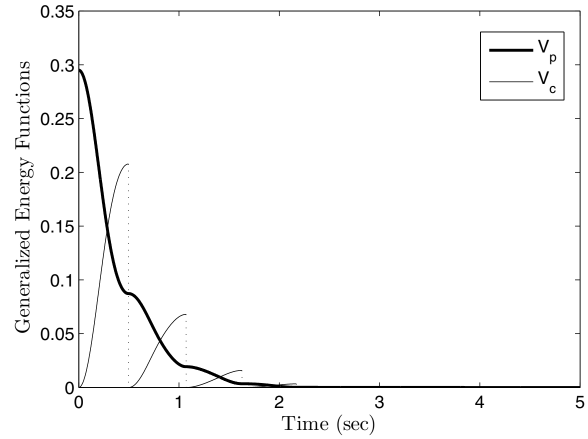

Fig. 3. Generalized energy functions $V_{\rm p}$ and $V_{\rm c}$ versus time.

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