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Spatial Proportional-Integral-Derivative penalization of distributed consensus filters for spatially distributed processes

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***Abstract*— The main thrust of this work is on the penalization of the pairwise state estimates used to enforce consensus in spatially distributed filters. It is assumed that a spatially distributed process has a network of in-domain sensors with spatially distributed filters corresponding to each sensor in the network. To better improve the agreement of the distributed filters, the spatial gradient of the pairwise difference of state estimates is used as a means to penalize their disagreement. Additionally, a proportional penalization and an integral penalization for the pairwise differences are also examined in order to lay down the foundation for a spatial proportional-integral-derivative penalization of the spatially distributed filters. Addressing the partial connectivity issue, a condition that resembles the Lagrangian potential for infinite dimensional systems is given in terms of the inner product of the state errors and their pairwise differences. In a forward looking approach, the extension to a more general class of partial differential equations, written as evolution equations in an appropriate Hilbert space, are examined and the conditions regarding the network connectivity are expressed as conditions on the inner product of the consensus operator and the pairwise difference of the state estimation errors.**

***Index Terms*— Distributed parameter systems; consensus filters; gradient consensus penalty.**

I. Introduction

With the ever increasing interest in the collaborative estimation via distributed learning, as realized through distributed filters with consensus, there comes the question of how to enhance agreement and learning. More specifically, in distributed filters with interaction that takes the form of a penalty on the pairwise difference of the state estimates, the question is what action you confer upon these pairwise differences.

This paper addresses the question of how to penalize the pairwise difference for a class of partial differential equations. It is assumed that a network of agents, each of which is equipped with a sensing device and processing and communication capabilities, can provide its own filter for the spatially distributed process. The interesting observation about spatially distributed systems, such as the current partial differential equation under study, is that the action on the pairwise difference of the state estimates acquires an intuitive sense. Specifically, each spatially distributed filter communicates (broadcasts and receives) the state estimates from its neighbors and penalizes the spatial gradient of the pairwise difference. The closest action from the finite dimensional literature is when the pairwise difference is given as an input to a transfer function.

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The contribution of this work is as follows: it proposes spatially distributed filters for a spatially distributed system (partial differential equation) that penalize the spatial gradient of the pairwise difference of the state estimates. The consensus filters follow in their basic structure the ones first presented in [1]. This penalization is essentially a *derivative consensus control*. It also examines the penalization of the pairwise difference, thereby implementing a *proportional consensus control* and to possibly address a "spatial memory", it also considers the spatial integral of the pairwise difference as another approach for penalization. The latter is a form of an *integral consensus control*. Each of the above actions can be considered on its own, or combined with the other two and when all are combined, result in *spatial Proportional-Integral-Derivative consensus controllers* for distributed filters. The closest counterpart for finite dimensional systems, either for consensus on estimates of synchronization, implemented a temporal integral action to penalize the past history of disagreement, [2], [3], [4].

The issue of limited connectivity is also addressed in this work and conditions for the type of network connectivity are given implicitly.

II. Problem formulation

A representative example of a spatially distributed process is the one-dimensional diffusion equation

$$\begin{aligned}\partial_t x(t,\xi) = \partial_\xi(\alpha(\xi)\partial_\xi x(t,\xi)) - \beta(\xi)\partial_\xi x(t,\xi) \\ -\gamma(\xi)x(t,\xi) + b_2(\xi)u(t),\end{aligned} \tag{1}$$

equipped with the appropriate boundary and initial conditions. For the ensuing numerical studies, we assume Dirichlet boundary conditions $x(t,0) = x(t,L) = k$, $k \in \mathbb{R}$, and the initial conditions are $x(0,\xi) = x_0(\xi)$, $\xi \in [0,L]$. Well-posedness of the above PDE system can be inferred by imposing the ellipticity condition [5] on the operator

$$\mathcal{L}\varphi \triangleq -\partial_\xi(\alpha(\xi)\partial_\xi\varphi(\xi)) + \beta(\xi)\partial_\xi\varphi(\xi) + \gamma(\xi)\varphi(\xi)$$

and square integrability of the control signal u. It is assumed that m measurements are available to be used by the m distributed filters. A model for the m observations is given in terms of the spatially weighted state

$$y_i(t) = \int_0^L c_i(\xi)x(t,\xi)\,\mathrm{d}\xi, \quad i = 1,\ldots,m. \tag{2}$$

The functions $c_i(\xi)$ have support in the spatial domain and describe the weighting of the state $x(t,\xi)$ over their support.

III. Spatial PID penalty control of consensus terms

We consider three different consensus filters and obtain energy balance results for each case. When all three are defined, then a spatial PID penalization of the consensus filters can be constructed. To simplify notation, we use x_{ij} to denote the difference $x_i - x_j$.

A. Derivative penalization of consensus filters

The m consensus filters in this case are given by

$$\begin{aligned}\partial_t \widehat{x}_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi \widehat{x}_i(t,\xi)\Big) - \beta(\xi)\partial_\xi \widehat{x}_i(t,\xi)\\ &\quad -\gamma(\xi)\widehat{x}_i(t,\xi) + \kappa_i(\xi)(y_i(t) - \widehat{y}_i(t))\\ &\quad +b_2(\xi)u(t) - \delta \sum_{j=1,j\neq i}^{m} \partial_\xi \widehat{x}_{ij}(t,\xi), \qquad (3)\\ \widehat{x}_i(t,0) &= \widehat{x}_i(t,L) = k,\\ \widehat{x}_i(0,\xi) &= \widehat{x}_{0i} \neq x_0(\xi), \quad i = 1,\dots,m\end{aligned}$$

where the filter gain has the specific form $\kappa_i(\xi) = \gamma_i c_i(\xi)$, $\gamma_i > 0$, $i = 1,\dots,m$. Defining the state estimation errors $e_i(t,\xi) = x(t,\xi) - \widehat{x}_i(t,\xi)$, $i = 1,\dots,m$ and the output estimation errors $\varepsilon_i(t) = y_i(t) - \widehat{y}_i(t) = \int_0^L c_i(\xi)e_i(t,\xi)\,\mathrm{d}\xi$, we then arrive at

$$\begin{aligned}\partial_t e_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi e_i(t,\xi)\Big) - \beta(\xi)\partial_\xi e_i(t,\xi)\\ &\quad -\gamma(\xi)e_i(t,\xi) - \kappa_i(\xi)\varepsilon_i(t) - \delta\sum_{j=1,j\neq i}^{m}\partial_\xi e_{ij}(t,\xi), \qquad (4)\\ e_i(t,0) &= e_i(t,L) = 0,\ e(0,\xi) = e_0(\xi),\ i = 1,\dots,m\end{aligned}$$

Lemma 1: Using the derivative penalty on the consensus terms, we have the following bounds

$$\sum_{i=1}^{m}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi + \sum_{i=1}^{m}\ell_i\int_0^t\int_0^L e_i^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau + \sum_{i=1}^{m}\gamma_i\int_0^t \varepsilon_i^2(\tau)\,\mathrm{d}\tau \leq \sum_{i=1}^{m}\frac{1}{2}\int_0^L e_i^2(0,\xi)\,\mathrm{d}\xi$$

for some $\ell_i, \gamma_i > 0$, $i = 1,\dots,m$.

Prior to the proof of Lemma 1, we consider the following.

Technical Lemma 1: When the aggregate estimation errors are considered, the "cross" terms satisfy

$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \delta e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi = 0$$

Proof: The above can easily be shown by induction. We show this when $m = 3$ and the case for any m immediately follows. For simplicity, we drop the constant δ in the calculations below as it will not affect the result:

$$\begin{aligned}&\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3}\int_0^L e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi =\\ &\int_0^L e_1(t,\xi)\partial_\xi e_2(t,\xi)\,\mathrm{d}\xi + \int_0^L e_1(t,\xi)\partial_\xi e_3(t,\xi)\,\mathrm{d}\xi\\ &+\int_0^L e_2(t,\xi)\partial_\xi e_1(t,\xi)\,\mathrm{d}\xi + \int_0^L e_2(t,\xi)\partial_\xi e_3(t,\xi)\,\mathrm{d}\xi\\ &+\int_0^L e_3(t,\xi)\partial_\xi e_1(t,\xi)\,\mathrm{d}\xi + \int_0^L e_3(t,\xi)\partial_\xi e_2(t,\xi)\,\mathrm{d}\xi\end{aligned}$$

Using integration by parts, we have for the ij term

$$\int_0^L e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi = e_i(t,\xi)e_j(t,\xi)\Big|_0^L - \int_0^L e_j(t,\xi)\partial_\xi e_i(t,\xi)\,\mathrm{d}\xi$$

Applying the boundary conditions in (4), we then arrive at

$$\int_0^L e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi = -\int_0^L e_j(t,\xi)\partial_\xi e_i(t,\xi)\,\mathrm{d}\xi.$$

Applying the above to the six terms, we obtain

$$\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3}\int_0^L e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi = 0.$$

Using the assumption of full connectivity, the above can be extended to any number of agents m. ■

Equipped with the above result, we now proceed with the proof of the main lemma (Lemma 1).

Proof: [Lemma 1] Using the $L_2(\Omega)$ norm of the error $e_i(t,\xi)$, we have from (4)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi = \int_0^L e_i(t,\xi)\partial_t e_i(t,\xi)\,\mathrm{d}\xi\\ &= \int_0^L e_i(t,\xi)\partial_\xi\Big(\alpha(\xi)\partial_\xi e_i(t,\xi)\Big)\,\mathrm{d}\xi\\ &-\int_0^L \beta(\xi)e_i(t,\xi)\partial_\xi e_i(t,\xi)\,\mathrm{d}\xi - \int_0^L \gamma(\xi)e_i^2(t,\xi)\,\mathrm{d}\xi\\ &-\gamma_i\varepsilon_i(t)\int_0^L e_i(t,\xi)c_i(\xi)\,\mathrm{d}\xi\\ &-\delta\int_0^L e_i(t,\xi)\sum_{j=1,j\neq i}^{m}\partial_\xi e_{ij}(t,\xi)\,\mathrm{d}\xi\\ &= -\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi - \gamma_i\varepsilon_i^2(t)\\ &-\delta\sum_{j=1,j\neq i}^{m}\int_0^L e_i(t,\xi)\partial_\xi e_i(t,\xi) - e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi\end{aligned}$$

At this stage, we consider the terms involving $e_i(t,\xi)$ only

$$\begin{aligned}
&-\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi-\gamma_i\varepsilon_i^2(t)\\
&-\delta\sum_{j=1,j\neq i}^{m}\int_0^L e_i(t,\xi)\partial_\xi e_i(t,\xi)\,\mathrm{d}\xi\\
&=-\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi-\gamma_i\varepsilon_i^2(t)\\
&-m\delta\int_0^L e_i(t,\xi)\partial_\xi e_i(t,\xi)\,\mathrm{d}\xi\\
&=-\int_0^L e_i(t,\xi)\Big[\mathcal{L}e_i(t,\xi)+m\delta\partial_\xi e_i(t,\xi)\Big]\,\mathrm{d}\xi-\gamma_i\varepsilon_i^2(t)
\end{aligned}$$

One has freedom to choose the consensus derivative gain δ such that the operator

$$\mathcal{L}_d\varphi\triangleq-\partial_\xi\Big(\alpha(\xi)\partial_\xi\varphi\Big)+(\beta(\xi)+m\delta)\partial_\xi\varphi+\gamma(\xi)\varphi$$

is elliptic. When a general class of PDEs is considered as evolution equations in a Hilbert space, then additional flexibility can be obtained by the choice of the filter gain function $\kappa_i(\xi)$. Next, we consider the "cross" terms

$$\delta\sum_{j=1,j\neq i}^{m}\int_0^L e_i(t,\xi)\partial_\xi e_j(t,\xi)\,\mathrm{d}\xi.$$

Combining the results on the ellipticity of the operator and the vanishing of the "cross" terms, one then obtains

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi=-\sum_{i=1}^{m}\int_0^L e_i(t,\xi)\mathcal{L}_d e_i(t,\xi)\,\mathrm{d}\xi\\
&-\sum_{i=1}^{m}\gamma_i\varepsilon_i^2(t)\leq-\sum_{i=1}^{m}\ell_i\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi-\sum_{i=1}^{m}\gamma_i\varepsilon_i^2(t)
\end{aligned}$$

Time integration from 0 to t provides the requisite result. ∎

Remark 1: An alternative to the above derivative penalty can be considered. One may opt to exclude the "cross" terms in the closed loop operator $\mathcal{L}_d$. Instead the cross terms can be bundled together as follows

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \delta e_i(t,\xi)\partial_\xi\Big(e_i(t,\xi)-e_j(t,\xi)\Big)\,\mathrm{d}\xi=\\
&\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\delta\int_0^L e_{ij}(t,\xi)\partial_\xi e_{ij}(t,\xi)\,\mathrm{d}\xi
\end{aligned}$$

Using Technical Lemma 1, one can similarly show

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\delta\int_0^L e_{ij}(t,\xi)\partial_\xi e_{ij}(t,\xi)\,\mathrm{d}\xi=0$$

In this case, the bound from Lemma 1 uses the operator $\mathcal{L}$ instead of $\mathcal{L}_d$ and it now becomes

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi=-\sum_{i=1}^{m}\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi\\
&-\sum_{i=1}^{m}\gamma_i\varepsilon_i^2(t)
\end{aligned}$$

B. Proportional penalization of consensus filters

The consensus filters with proportional penalty are

$$\begin{aligned}
\partial_t\widehat{x}_i(t,\xi)&=\partial_\xi\Big(\alpha(\xi)\partial_\xi\widehat{x}_i(t,\xi)\Big)-\beta(\xi)\partial_\xi\widehat{x}_i(t,\xi)\\
&-\gamma(\xi)\widehat{x}_i(t,\xi)+\kappa_i(\xi)(y(t)-\widehat{y}_i(t))\\
&+b_2(\xi)u(t)-\pi(\xi)\sum_{j=1,j\neq i}^{m}\widehat{x}_{ij}(t,\xi)
\end{aligned}\tag{5}$$

$$\widehat{x}_i(t,0)=\widehat{x}_i(t,L)=k,\ \widehat{x}_i(0,\xi)=\widehat{x}_{0i},\ i=1,\ldots,m$$

where $\pi(\xi)>0$ is the proportional consensus gain. The associated error systems are given by

$$\begin{aligned}
\partial_t e_i(t,\xi)&=\partial_\xi\Big(\alpha(\xi)\partial_\xi e_i(t,\xi)\Big)+\beta(\xi)\partial_\xi e_i(t,\xi)\\
&-\gamma(\xi)e_i(t,\xi)-\kappa_i(\xi)\varepsilon_i(t)-\pi(\xi)\sum_{j=1,j\neq i}^{m}e_{ij}(t,\xi)
\end{aligned}\tag{6}$$

$$e_i(t,0)=e_i(t,L)=0,\ e_i(0,\xi)=e_{0i},\ i=1,\ldots,m$$

In the proportional case, we have the following result.

Lemma 2: Using the proportional penalty on the consensus terms, we have the following bounds

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi+\sum_{i=1}^{m}\mu_i\int_0^t\int_0^L e_i^2(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\
&\leq\sum_{i=1}^{m}\frac{1}{2}\int_0^L e_i^2(0,\xi)\,\mathrm{d}\xi
\end{aligned}$$

for some $\mu_i>0$, $i=1,\ldots,m$.

To help with the proof, we similarly have a technical lemma to account for the "cross" terms.

Technical Lemma 2: The "cross" terms in (5) satisfy

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L\pi(\xi)e_i(t,\xi)e_{ij}(t,\xi)\,\mathrm{d}\xi=\\
&\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L\pi(\xi)e_{ij}^2(t,\xi)\,\mathrm{d}\xi
\end{aligned}$$

Proof: [Lemma 2] We use again the following

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi=\int_0^L e_i(t,\xi)\partial_t e_i(t,\xi)\,\mathrm{d}\xi\\
&=\int_0^L e_i(t,\xi)\Big(\mathcal{L}e_i(t,\xi)\Big)\,\mathrm{d}\xi-\varepsilon_i(t)\int_0^L\kappa_i(\xi)e_i(t,\xi)\,\mathrm{d}\xi\\
&-\sum_{j=1,j\neq i}^{m}\int_0^L\pi(\xi)e_i(t,\xi)e_{ij}(t,\xi)\,\mathrm{d}\xi
\end{aligned}$$

Using the technical lemma 2 along with the fact that the operator $\mathcal{L}$ is elliptic, then we have

$$\begin{aligned}
&\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi=-\sum_{i=1}^{m}\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi\\
&-\gamma_i\varepsilon_i^2(t)-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L\pi(\xi)e_{ij}^2(t,\xi)\,\mathrm{d}\xi\\
&\leq-\sum_{i=1}^{m}\mu_i\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi.
\end{aligned}$$

∎

C. Integral penalization of consensus filters

The last type of penalization is a spatial average penalty of the pairwise difference of the distributed filters, given by

$$\begin{aligned}\partial_t\widehat{x}_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi\widehat{x}_i(t,\xi)\Big) - \beta(\xi)\partial_\xi\widehat{x}_i(t,\xi)\\ &-\gamma(\xi)\widehat{x}_i(t,\xi) + \kappa_i(\xi)(y(t)-\widehat{y}_i(t)) + b_2(\xi)u(t)\\ &-\iota(\xi)\sum_{j=1,j\neq i}^{m}\int_0^L \widehat{x}_{ij}(t,\xi)\,\mathrm{d}\xi\\ \widehat{x}_i(t,0) &= \widehat{x}_i(t,L) = k,\, \widehat{x}_i(0,\xi) = \widehat{x}_{0i},\, i=1,\dots,m\end{aligned} \tag{7}$$

where $\iota(\xi) > 0$ is the integral weight. The state errors satisfy

$$\begin{aligned}\partial_t e_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi e_i(t,\xi)\Big) - \beta(\xi)\partial_\xi e_i(t,\xi)\\ &-\gamma(\xi)e_i(t,\xi) - \kappa_i(\xi)\varepsilon_i(t) - \iota(\xi)\sum_{j=1,j\neq i}^{m}\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\\ e_i(t,0) &= e_i(t,L) = 0,\, e_i(0,\xi) = e_{0i} \neq 0,\, i=1,\dots,m\end{aligned}$$

With the above equations, we have the following bounds

Lemma 3: The aggregate state errors with the integral penalization of the consensus filters satisfy

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{m}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi + \sum_{i=1}^{m}\int_0^t\int_0^L e_i(\tau,\xi)\mathcal{L}e_i(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\\ &+\gamma_i\int_0^t \varepsilon_i^2(\tau)\,\mathrm{d}\tau + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\iota_0\int_0^t\Big(\int_0^L e_{ij}(\tau,\xi)\,\mathrm{d}\xi\Big)^2\mathrm{d}\tau\\ &\qquad\le \sum_{i=1}^{m}\frac{1}{2}\int_0^L e_i^2(0,\xi)\,\mathrm{d}\xi\end{aligned}$$

In a similar fashion as in the proportional case, we have the following technical lemma.

Technical Lemma 3: When the integral consensus gain $\iota(\xi) = \iota_0 > 0$, then the integral "cross" terms satisfy

$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \iota(\xi)e_i(t,\xi)\,\mathrm{d}\xi\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\iota_0\Big(\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\Big)^2$$

Proof: [Technical Lemma 3] This requires some care due to the integral weight $\iota(\xi)$. We have

$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \iota(\xi)e_i(t,\xi)\,\mathrm{d}\xi\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \iota(\xi)e_{ij}(t,\xi)\,\mathrm{d}\xi \times \int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi$$

Using the assumption $\iota(\xi) = \iota_0 > 0$, we have

$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\int_0^L \iota(\xi)e_i(t,\xi)\,\mathrm{d}\xi\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\iota_0\Big(\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\Big)^2.$$

∎

Proof: [Lemma 3] Examining the time derivative of the $L_2(\Omega)$ norm of the local errors, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi = \int_0^L e_i(t,\xi)\partial_t e_i(t,\xi)\,\mathrm{d}\xi\\ &= \int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi - \varepsilon_i(t)\int_0^L \kappa_i(\xi)e_i(t,\xi)\,\mathrm{d}\xi\\ &-\sum_{j=1,j\neq i}^{m}\int_0^L \iota(\xi)e_i(t,\xi)\,\mathrm{d}\xi\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\end{aligned}$$

Using the ellipticity of the operator $\mathcal{L}$ and the result of technical lemma 3, we have

$$\begin{aligned}&\sum_{i=1}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_0^L e_i^2(t,\xi)\,\mathrm{d}\xi = -\sum_{i=1}^{m}\int_0^L e_i(t,\xi)\mathcal{L}e_i(t,\xi)\,\mathrm{d}\xi\\ &-\gamma_i\varepsilon_i^2(t) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\iota_0\Big(\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\Big)^2.\end{aligned}$$

Time integration provides the requisite result. ∎

IV. Relaxation of all-to-all connectivity: special cases

While the all-to-all connectivity facilitates the stability analysis, it still imposes stringent conditions on the network connectivity. A way to address this is to relax the full connectivity condition for a type of partial connectivity. In each of the three consensus controllers, an implicit condition on the network connectivity will be presented. In each case, the associated consensus filters will have an almost identical form to (3), (5) and (7).

A. Derivative penalization with limited connectivity

The consensus filters in this case are given by

$$\begin{aligned}\partial_t\widehat{x}_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi\widehat{x}_i(t,\xi)\Big) - \beta(\xi)\partial_\xi\widehat{x}_i(t,\xi)\\ &-\gamma(\xi)\widehat{x}_i(t,\xi) + \kappa_i(\xi)(y(t)-\widehat{y}_i(t))\\ &+b_2(\xi)u(t) - \delta(\xi)\sum_{j\in N_i}\partial_\xi\widehat{x}_{ij}(t,\xi)\\ \widehat{x}_i(t,0) &= \widehat{x}_i(t,L) = k,\, \widehat{x}_i(0,\xi) = \widehat{x}_{0i},\, i=1,\dots,m\end{aligned} \tag{8}$$

and the associated state estimation errors by

$$\begin{aligned}\partial_t e_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi e_i(t,\xi)\Big) - \beta(\xi)\partial_\xi e_i(t,\xi)\\ &-\gamma(\xi)e_i(t,\xi) - \kappa_i(\xi)\varepsilon_i(t) - \delta(\xi)\sum_{j\in N_i}\partial_\xi e_{ij}(t,\xi)\\ e_i(t,0) &= e_i(t,L) = 0,\, e_i(0,\xi) = e_{0i},\, i=1,\dots,m\end{aligned}$$

The condition will be given in terms of the requirement in Technical Lemma 1

$$\sum_{i=1}^{m}\sum_{j\in N_i}\int_0^L \delta(\xi)\varphi_i(\xi)\partial_\xi\varphi_{ij}(\xi)\,\mathrm{d}\xi \ge 0.$$

This implicit condition on the network connectivity, when satisfied, allows for more general boundary conditions. When an undirected graph is assumed, then the above condition reduces to

$$\sum_{i=1}^{m}\sum_{j\in N_i}\int_0^L \delta(\xi)\varphi_{ij}(\xi)\partial_\xi\varphi_{ij}(\xi)\,\mathrm{d}\xi \geq 0.$$

B. Proportional penalization with limited connectivity

Similar to (5), the filters are given by

$$\begin{aligned}\partial_t\widehat{x}_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi\widehat{x}_i(t,\xi)\Big) - \beta(\xi)\partial_\xi\widehat{x}_i(t,\xi)\\ &\quad -\gamma(\xi)\widehat{x}_i(t,\xi) + \kappa_i(\xi)(y(t)-\widehat{y}_i(t)) + b_2(\xi)u(t)\\ &\quad -\pi(\xi)\sum_{j\in N_i}\widehat{x}_{ij}(t,\xi)\\ \widehat{x}_i(t,0) &= \widehat{x}_i(t,L) = k,\ \widehat{x}_i(0,\xi) = \widehat{x}_{0i},\ i=1,\ldots,m\end{aligned}$$

and the condition based on Technical Lemma 2 becomes

$$\sum_{i=1}^{m}\sum_{j\in N_i}\int_0^L \pi(\xi)\varphi_i(\xi)\varphi_{ij}(\xi)\,\mathrm{d}\xi \geq 0.$$

which for an undirected graph and $\pi(\xi) = \pi_0 > 0$ becomes

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j\in N_i}\int_0^L \pi_0\varphi_{ij}^2(\xi)\,\mathrm{d}\xi \geq 0$$

and which is satisfied automatically as it considers the finite sum of the $L_2(\Omega)$ norms of pairwise errors. It is also equal to the Laplacian potential. Unlike the derivative penalization, the kronecker product of the graph Laplacian with the identity operator simplifies the above condition to the Laplacian potential, a condition that cannot be guaranteed when one takes the Kronecker product of the graph Laplacian with the differential operator.

C. Integral penalization with limited connectivity

The integral penalization shares similarities with the proportional one, in the sense that one can express the equivalent of Technical Lemma 3 using the Laplacian potential [6].

The filters employing integral penalization and with limited connectivity are

$$\begin{aligned}\partial_t\widehat{x}_i(t,\xi) &= \partial_\xi\Big(\alpha(\xi)\partial_\xi\widehat{x}_i(t,\xi)\Big) - \beta(\xi)\partial_\xi\widehat{x}_i(t,\xi)\\ &\quad -\gamma(\xi)\widehat{x}_i(t,\xi) + \kappa_i(\xi)(y(t)-\widehat{y}_i(t)) + b_2(\xi)u(t)\\ &\quad -\iota(\xi)\sum_{j\in N_i}\int_0^L\widehat{x}_{ij}(t,\xi)\,\mathrm{d}\xi\\ \widehat{x}_i(t,0) &= \widehat{x}_i(t,L) = k,\ \widehat{x}_i(0,\xi) = \widehat{x}_{0i},\ i=1,\ldots,m\end{aligned}$$

The condition similar to that in Technical Lemma 3 is

$$\sum_{i=1}^{m}\sum_{j\in N_i}\int_0^L \iota(\xi)e_i(t,\xi)\,\mathrm{d}\xi\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi \geq 0.$$

For an undirected graph with $\iota(\xi) = \iota_0$, it is equivalent to

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j\in N_i}\iota_0\Big(\int_0^L e_{ij}(t,\xi)\,\mathrm{d}\xi\Big)^2 \geq 0$$

which is automatically satisfied, since it involves the finite sum of the squares of integrals of the pairwise errors.

V. Forward looking: extending to more general infinite dimensional systems

We consider classes of infinite dimensional systems, which include the parabolic PDEs in (1) and which are written as evolution equations in a Hilbert space

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad y_i(t) = \mathcal{C}_i x(t) \quad i=1,\ldots,m,$$

The distributed filters without any consensus terms are given by the standard state observer equations

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= \mathcal{A}\widehat{x}_i(t) + \mathcal{G}_i\Big(y_i(t) - \mathcal{C}_i\widehat{x}_i(t)\Big) + \mathcal{B}u(t)\\ \widehat{x}_i(0) &\neq x(0), \qquad i=1,\ldots,m,\end{aligned}$$

where $\mathcal{G}_i$ are the filter operators with the property that the "closed-loop" operators $\mathcal{A} - \mathcal{G}_i\mathcal{C}_i$ generate exponentially stable semigroups. The consensus filters follow from [1]

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= \mathcal{A}\widehat{x}_i(t) + \mathcal{G}_i\Big(y_i(t) - \mathcal{C}_i\widehat{x}_i(t)\Big) + \mathcal{B}u(t)\\ &\quad +\mathcal{P}_i\sum_{j\in N_i}\widehat{x}_{ij}(t), \quad \widehat{x}_i(0) \neq x(0),\ \ i=1,\ldots,m,\end{aligned}$$

The consensus operators $\mathcal{P}_i$ may include the identity operator (proportional), the spatial average operator (integral) and the spatial gradient operator (derivative). Combining the conditions of the three Technical Lemmas as applied to partial connectivity, this partial connectivity condition is

$$\sum_{i=1}^{m}\sum_{j\in N_i}\langle e_i, \mathcal{P}_i e_{ij}\rangle \leq 0. \tag{9}$$

This implicit connectivity condition seeks to find consensus operators $\mathcal{P}_i$ and the set of neighbors of the jth agent, denoted by N_i, such that (9) is satisfied. This condition for undirected graphs simplifies to

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j\in N_i}\langle e_{ij}, \mathcal{P}_i e_{ij}\rangle \leq 0. \tag{10}$$

VI. Numerical results

Equation (1) was simulated using Galerkin-based approximating scheme using 40 linear functions over the spatial interval $[0,1]$, having $\alpha = 0.1, \beta = \gamma = 0$. Five $m = 5$ pointwise sensors placed at $0.23, 0.39, 0.47, 0.66$ and 0.81 were considered to provide process information to the five distributed filters. Using "colocated" filter gains with values $\gamma_i = 0.0270, 0.0315, 0.0338, 0.0360, 0.0382$, the derivative consensus controllers in (3) were simulated over $[0,3]s$.

The information exchange between the m agents is modeled by an undirected connected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, with $\mathcal{V} = \{1,\ldots,m\}$ the set of nodes and with $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ the set of edges. The edge $(i,j) \in \mathcal{V}$ denotes that an agent i obtains information from agent j. The set of neighbors of node i is given by $N_i = \{j \in \mathcal{V}\,|\,(i,j) \in \mathcal{E}\}$, [7]. The connectivity of the 5 filters is represented by the undirected graph of Figure 1 A metric to assess the level of agreement of the 5 filters that is independent of the network topology, is the deviation of each $\widehat{x}_i(t,\xi)$ from the mean, given by

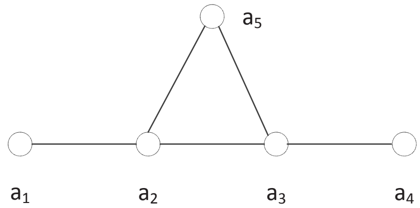

Fig. 1. Undirected graph on 5 vertices.

case	error norm
without	1.1854
with low	1.0515
with med	0.9729
with high	0.8365

TABLE I
$L_2([0,3])$ NORM OF THE AVERAGED CUMULATIVE SPATIAL NORM.

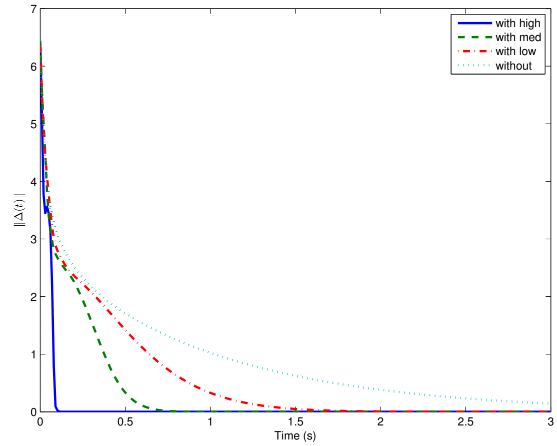

Fig. 2. Evolution of $\|\Delta(t)\|$.

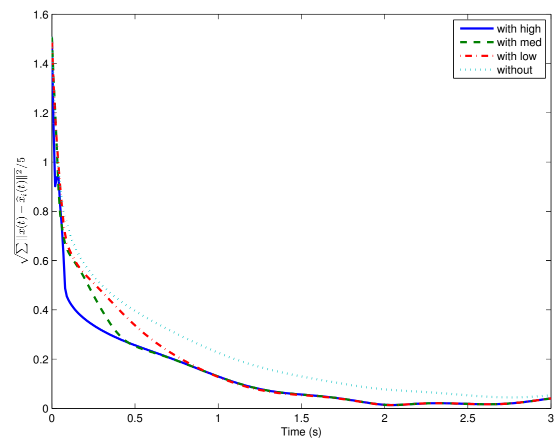

Fig. 3. Evolution of averaged cumulative norm .

$\Delta_i(t,\xi) = \widehat{x}_i(t,\xi) - \frac{1}{m}\sum_{i=1}^m \widehat{x}_i(t,\xi)$, $i = 1,\ldots,5$. The spatial $L_2(0,1)$ norm of all deviations $\Delta_i(t,\xi)$ given by

$$\|\Delta(t)\|^2 = \sum_{i=1}^m \int_0^1 \delta_i^2(t,\xi)\,\mathrm{d}\xi.$$

provides the cumulative deviation over Ω. Its time evolution is depicted in Figure 2 for the range of the consensus weights $\delta = 50, 100, 500$ depicting low, medium and high level of derivative penalization in (8). As expected, the higher δ is, the faster the convergence of the deviation $\Delta(t)$ to zero.

The evolution of the averaged cumulative norm of the state estimation errors, given by

$$\frac{1}{m}\sum_{i=1}^m \|x(t) - \widehat{x}_i(t)\|^2 = \frac{1}{m}\sum_{i=1}^m \int_0^1 (x(t,\xi) - \widehat{x}_i(t,\xi))^2\,\mathrm{d}\xi$$

is plotted in Figure 3, whereas its $L_2(0,t)$ rms norm,

$$\sqrt{\int_0^t \frac{1}{m}\sum_{i=1}^m \|x(t) - \widehat{x}_i(t)\|^2\,\mathrm{d}t}$$

is given in table I. In both the figure and the table it is observed that the estimation error has an improved convergence for higher values of the derivative consensus gain δ.

VII. CONCLUSIONS

The penalization of the spatial gradient of the pairwise disagreement of distributed filters for a class of PDEs was proposed. The scheme that penalizes not only the pairwise disagreement of the pairwise differences but their spatial gradient and perhaps the spatial averages, thereby implementing a spatial proportional-integral-derivative consensus controller constitutes a paradigm shift in the manner that consensus filters for spatially distributed systems are viewed.

Such an approach would provide a new algorithm for the finite dimensional case where one may penalize not simply the (temporal) integral of the pairwise differences of distributed filters, but a function of these differences via a transfer function. In this case, the pairwise differences would be viewed as the input to a transfer function. Such a framework for both finite and infinite dimensional systems is currently under consideration by the author.

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