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Gain-scheduled synthesis with dynamic stable strictly positive real multipliers: A complete solution

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***Abstract*— In this paper we continue our efforts to arrive at a solution of the gain-scheduling controller design problem with dynamic multipliers. Although a solution for D-scalings is available, this scenario leads to conservatism if scheduling on real parameters. This motivates to consider the very same problem with D/G-scalings, for which we recently proposed a design framework with generalized positive real multipliers. The present paper closes a gap in this previous approach and provides necessary and sufficient convex conditions for the existence of gain-scheduling controllers if the multipliers are stable and positive real. Examples illustrate the possibility for a substantial reduction of conservatism with the novel solution.**

***Index Terms*— Gain-scheduled control, dynamic multipliers, positive real synthesis, D/G-scalings, LMIs**

I. INTRODUCTION

We consider the gain-scheduling problem for systems and controllers with a linear fractional representation as initialized in [1], [2] and as depicted in Fig. 1. Stability and performance is guaranteed through multipliers that are chosen according to the characteristics of the uncertainty. For notational simplicity we only address systems with one real repeated uncertainty δI_p and without performance channel. Here $\delta \in \mathbb{R}$ is assumed to be passive which just means $\delta \geq 0$. The extension of our results to structured blocks $\mathrm{diag}(\delta_1 I_{p_1}, \ldots, \delta_m I_{p_m})$ with $\delta_1, \ldots, \delta_m \geq 0$ and to problems with an H_∞-performance constraint is routine.

Through a linear fractional transformation, it is well known that this problem is equivalent to one with parametric uncertainty for the new parameter $\theta = (1-\delta)/(1+\delta)$ satisfying $|\theta| \leq 1$ [3]. If δ is time-varying, an LMI-solution to the gain-scheduling problem with constant (frequency-independent) D-scalings has been obtained in [1], [2], while a reduction of conservatism was achieved in [3], [4] by using constant D/G-scalings. A convex solution for the respective problem with dynamic (frequency-dependent) D-scalings has been recently given in [5]. (See also [6], [7] for non-convex solutions).

Frequency dependence in multipliers is of crucial importance in order to reduce conservatism if the scheduling parameters are time-invariant (or slowly time-varying [8]). As such we aim at a viable alternative to existing synthesis approaches that are based on parameter-dependent Lyapunov functions, as initialized in [9], [10] and with many recent references and applications appearing in [11]. The ultimate goal is to arrive at a convex solution to the synthesis problem for general IQCs [12], which paves the way for scheduling on a large class of systems in the feedback path, including e.g. delays [13] and nonlinearities. We also addressed in detail the relevance of all these techniques for reducing the conservatism in distributed controller design for spatially distributed systems in [14].

The technical contribution of this paper is a complete solution to the gain-scheduling synthesis problem with dynamic multipliers that are stable and strictly positive real (as defined precisely below). In [15] we considered the very same problem for generalized (possibly unstable) strictly positive real multipliers; the emphasis in the previous paper was on a synthesis framework with unstable weights, and only lead to sufficient conditions for the existence of gain-scheduled controllers. In the present paper we provide necessary and sufficient conditions, which closes a gap in [15] but requires the stability of the multipliers. We illustrate the benefits of our results by academic numerical examples.

The paper is structured as follows. In Section II we recapitulate know results from robustness analysis and synthesis. In Section III we formulate sufficient LMI conditions for the existence of gain-scheduling controllers, and show for a particular multiplier sequence the necessity of these conditions, together with a numerical example which illustrates the benefit of the new approach in this paper. The proof of Theorem 5 is deferred to the appendix.

Notation. For a (real-rational proper) transfer matrix G define $G^*(s) = G(-s)^T$ which implies $G^*(\lambda) = G(\lambda)^*$ for all $\lambda \in \mathbb{C}^0 := i\mathbb{R} \cup \{\infty\}$. If G has no poles in $\mathbb{C}^0$, $G \prec 0$ means that G satisfies the frequency domain inequality (FDI) $G^*(\lambda) = G(\lambda) \prec 0$ for all $\lambda \in \mathbb{C}^0$; G is generalized strictly positive real if $\mathrm{He}(G) := G + G^* \succ 0$; if, in addition, G is stable, it is said to be strictly positive real (SPR). For $Q = Q^T$ we will often consider FDIs of the form $G^*QG \prec 0$. If (A, B, C, D) is a state-space realization of G, we express this fact by $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $G_{\rm ss} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = (A, B, C, D)$; the Kalman-Yakubovich-Popov (KYP) Lemma allows to relate the FDI $G^*QG \prec 0$ to the following linear matrix inequality (LMI) in the variable X:

$$\mathscr{L}(X, Q, G_{\rm ss}) := \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \\ C & D \end{pmatrix} \prec 0. \quad (1)$$

If (1) holds, we say that it certifies $G^*QG \prec 0$ or that X is

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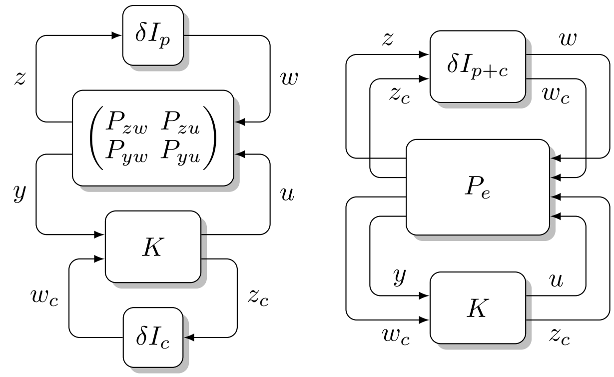

Fig. 1. Analysis configuration for gain-scheduling synthesis

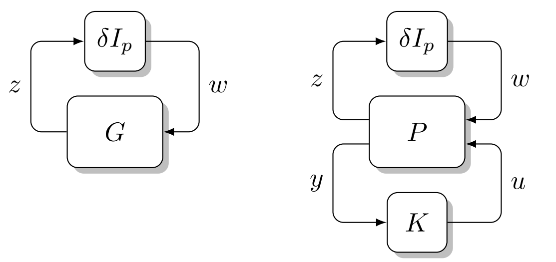

Fig. 2. Loops for analysis and robust synthesis

a certificate for this FDI; note that $\mathscr{L}(X, Q, G_{\rm ss})$ displays the certificate X, the middle matrix Q and the employed state-space realization $G_{\rm ss}$ of G. The Kronecker product of A, B is denoted by $A \otimes B$ and blocks in matrices that can be inferred by symmetry or are irrelevant are indicated by $\star$.

II. Recap of known facts

The present section serves to recapitulate some well-known facts from robustness analysis and synthesis.

A. Robust stability analysis

Consider the loop on the left in Fig. 2 for a transfer matrix G and for real $\delta \geq 0$. Robust stability is guaranteed if G is stable and if there exists some $\psi \in RH_\infty^{p\times p}$ with

$$\text{He}(\psi) \succ 0 \quad \text{and} \quad \text{He}(\psi G) \prec 0. \tag{2}$$

This follows from the passivity theorem due to $\text{He}(\psi\delta) \geq 0$ for all $\delta \geq 0$, just because ψ is SPR. With $\hat{Q} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ we express (2) as the so-called primal analysis FDIs

$$-\begin{pmatrix} \psi \\ I \end{pmatrix}^* \hat{Q} \begin{pmatrix} \psi \\ I \end{pmatrix} \prec 0, \quad \begin{pmatrix} \psi G \\ I \end{pmatrix}^* \hat{Q} \begin{pmatrix} \psi G \\ I \end{pmatrix} \prec 0.$$

Let $G_{\rm ss} = (A, B, C, D)$ and $\psi_{\rm ss} = (A_\psi, B_\psi, C_\psi, D_\psi)$ with $\dim(A) = n$, $\dim(A_\psi) = n_\psi$ respectively. With

$$\begin{pmatrix} \psi \\ I \end{pmatrix}_{\rm ss} = \left(\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \\ 0 & I \end{array}\right), \quad \begin{pmatrix} \psi G \\ I \end{pmatrix}_{\rm ss} = \left(\begin{array}{cc|c} A_\psi & B_\psi C & B_\psi D \\ 0 & A & B \\ \hline C_\psi & D_\psi C & D_\psi D \\ 0 & 0 & I \end{array}\right),$$

the following result is standard.

Lemma 1: A_ψ, A are stable and the FDIs (2) hold iff there exist $R \succ 0$, $X \succ 0$ with

$$\mathscr{L}\left(R, -\hat{Q}, \begin{pmatrix} \psi \\ I \end{pmatrix}_{\rm ss}\right) \prec 0, \tag{3}$$

$$\mathscr{L}\left(X, \hat{Q}, \begin{pmatrix} \psi G \\ I \end{pmatrix}_{\rm ss}\right) \prec 0. \tag{4}$$

For the inverse $\phi = \psi^{-1}$, (2) are easily seen to be equivalent to the so-called dual analysis FDIs

$$-\begin{pmatrix} -\phi^* \\ I \end{pmatrix}^* \hat{Q} \begin{pmatrix} -\phi^* \\ I \end{pmatrix} \succ 0, \quad \begin{pmatrix} \phi^* \\ -G^* \end{pmatrix}^* \hat{Q} \begin{pmatrix} \phi^* \\ -G^* \end{pmatrix} \succ 0. \tag{5}$$

If $\phi_{\rm ss} = (A_\phi, B_\phi, C_\phi, D_\phi)$ with $A_\phi \in \mathbb{R}^{n_\phi \times n_\phi}$ and for

$$\begin{pmatrix} -\phi^* \\ I \end{pmatrix}_{\rm ss} = \left(\begin{array}{c|c} -A_\phi^T & -C_\phi^T \\ \hline -B_\phi^T & -D_\phi^T \\ 0 & I \end{array}\right), \quad \begin{pmatrix} \phi^* \\ -G^* \end{pmatrix}_{\rm ss} = \left(\begin{array}{cc|c} -A_\phi^T & 0 & C_\phi^T \\ 0 & -A^T & -C^T \\ \hline -B_\phi^T & 0 & D_\phi^T \\ 0 & -B^T & -D^T \end{array}\right),$$

we arrive at the following dual LMI characterization of (5).

Lemma 2: A_ϕ, A are stable and the FDIs (5) hold iff there exist $S \succ 0$, $Y \succ 0$ with

$$\mathscr{L}\left(S, -\hat{Q}, \begin{pmatrix} -\phi^* \\ I \end{pmatrix}_{\rm ss}\right) \succ 0, \tag{6}$$

$$\mathscr{L}\left(Y, \hat{Q}, \begin{pmatrix} \phi^* \\ -G^* \end{pmatrix}_{\rm ss}\right) \succ 0. \tag{7}$$

Clearly $\phi = \psi^{-1}$ is implied by the tight coupling

$$\begin{pmatrix} A_\phi & B_\phi \\ C_\phi & D_\phi \end{pmatrix} = \begin{pmatrix} A_\psi - B_\psi D_\psi^{-1} C_\psi & B_\psi D_\psi^{-1} \\ -D_\psi^{-1} C_\psi & D_\psi^{-1} \end{pmatrix} \tag{8}$$

of the *realizations* of ψ and ϕ. If (8) holds, let us recall the following easily verified relation between the primal and dual LMIs [16]: R satisfies (3) iff $S = R^{-1}$ satisfies (6); similarly X satisfies (4) iff $Y = X^{-1}$ satisfies (7). Whenever relevant, X and Y are assumed to be partitioned according to the A-matrix of the outer factor realizations.

B. Robust synthesis

Now we consider the configuration on the right in Fig. 2. The uncontrolled nominal system is defined by the open-loop transfer matrix P with state-space description

$$P = \left[\begin{array}{c|cc} A & B & B_u \\ \hline C & D & D_u \\ C_y & D_y & 0 \end{array}\right] \text{ where } A \in \mathbb{R}^{n\times n} \text{ and } P_{\rm zw} := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Robust synthesis means to design an LTI controller K which internally stabilizes P and which renders the loop in Fig. 2 robustly stable; the transfer matrix resulting from interconnecting P with K is denoted by G. Again, standard arguments [17] allow to eliminate the controller parameters from the analysis LMIs in order to arrive at existence conditions for a robust controller. For their formulation we introduce the full column rank annihilator matrices

$$\text{im}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \ker\left(C_y \ D_y\right), \ \text{im}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \ker\left(B_u^T \ D_u^T\right)$$

and their extensions

$$U := \begin{pmatrix} I_{n_\psi} & 0 \\ 0 & U_1 \\ 0 & U_2 \end{pmatrix}, \quad V := \begin{pmatrix} I_{n_\phi} & 0 \\ 0 & V_1 \\ 0 & V_2 \end{pmatrix}.$$

Theorem 3: There exists a controller K which stabilizes P and such that (2) holds for $\psi \in RH_\infty^{p\times p}$ iff there exist symmetric R, S with (3), (6) and symmetric X, Y with

$$U^T \mathscr{L}\left(X, \hat{Q}, \begin{pmatrix} \psi P_{zw} \\ I \end{pmatrix}_{\rm ss}\right) U \prec 0, \tag{9a}$$

$$V^T \mathscr{L}\left(Y, \hat{Q}, \begin{pmatrix} \phi^* \\ -P_{zw}^* \end{pmatrix}_{\rm ss}\right) V \succ 0 \tag{9b}$$

that are coupled as

$$\left(\begin{array}{cc|cc} X_{11} & X_{12} & T & 0 \\ X_{21} & X_{22} & 0 & I \\ \hline T^T & 0 & Y_{11} & Y_{12} \\ 0 & I & Y_{21} & Y_{22} \end{array}\right) \succ 0, \quad \begin{pmatrix} R & T \\ T^T & S \end{pmatrix} \succ 0 \tag{10}$$

with $T = I$ and the multiplier realizations related by (8).

Note that $\dim(X_{11}) = n_\psi$ and $\dim(Y_{11}) = n_\phi$. For *fixed* multipliers ψ, ϕ related by (8), the conditions involve convex constraints in order to decide on the existence of a stabilizing controller which achieves (2); feasibility allows us to construct a corresponding controller exactly as in [17].

However, genuine robust synthesis requires the additional search for suitable multipliers ψ and ϕ to render these conditions satisfied - the tight coupling (8) destroys convexity.

We stress that the formulation in Theorem 3 is redundant: Under the constraint (8) and if A_ψ, A_ϕ are stable, the second inequality in (10) can be dropped since it is automatically feasible. Indeed, if R satisfies (3), then $S = R^{-1} + \epsilon I$ is a solution of (6) for some sufficiently small $\epsilon > 0$ and we have $\begin{pmatrix} R & I \\ I & S \end{pmatrix} \succ 0$. The key novel contribution of this paper is to show that (10) *without the tight constraint (8)* is the correct multiplier coupling for gain-scheduling synthesis; this leads to a *convex* synthesis algorithm.

III. Gain-scheduling synthesis

A. Existence conditions for gain-scheduled controllers

Let us now consider the gain-scheduling configuration on the left of Fig. 1, with a controller that consists of an LTI component K scheduled with δI_c; this so-called scheduling block is determined through the number of repetitions c which could possibly be different from p. The goal is to determine c and K which internally stabilizes P and such that the scheduled controller renders the loop robustly stable.

As going back to the seminal work of [1], [2], this problem is approached by viewing it as a robust synthesis problem for the configuration to the right in Fig. 1; P_e extends P as

$$\begin{pmatrix} z \\ z_c \\ \hline w_c \\ y \end{pmatrix} = P_e \begin{pmatrix} w \\ w_c \\ \hline z_c \\ u \end{pmatrix} = \left(\begin{array}{cc|cc} P_{zw} & 0 & 0 & P_{zu} \\ 0 & 0 & I_c & 0 \\ \hline 0 & I_c & 0 & 0 \\ P_{yw} & 0 & 0 & P_{yu} \end{array}\right) \begin{pmatrix} w \\ w_c \\ \hline z_c \\ u \end{pmatrix}.$$

The interconnection of P_e and K is just given by the star-product of P and K which is denoted as $P \star K$ [18]. Corresponding to the uncertainty $\mathrm{diag}(\delta I_p, \delta I_c) = \delta I_{p+c}$, we use the $(p+c)$-dimensional primal and dual extended multipliers

$$\psi_e = \left[\begin{array}{c|cc} A_\psi & B_\psi & B_\psi^2 \\ \hline C_\psi & D_\psi & D_\psi^{12} \\ C_\psi^2 & D_\psi^{21} & D_\psi^{22} \end{array}\right], \ \phi_e = \left[\begin{array}{c|cc} A_\phi & B_\phi & B_\phi^2 \\ \hline C_\phi & D_\phi & D_\phi^{12} \\ C_\phi^2 & D_\phi^{21} & D_\phi^{22} \end{array}\right] \tag{11}$$

and just apply Theorem 3 for P_e. Due to its particular structure, it is somewhat cumbersome but well-established to check that the synthesis LMIs actually simplify to precisely those in (9), just for the left-upper $p \times p$ blocks of (11). The SPR conditions that correspond to (3) and (6) for the extended multipliers in (11) can be simplified to (3) and (6) for the left-upper blocks, merely by canceling the last c rows/columns of these LMIs. In this fashion one arrives at the very same LMIs (3), (6), (9)-(10) as for robust synthesis, but still with $T = I$ and with the extended multipliers tightly coupled through $\phi_e = \psi_e^{-1}$ in terms of equality of the corresponding realizations.

Dropping the coupling and allowing for general T leads to conditions that guarantee the existence of a gain-scheduled controller; this insight is the first contribution of this paper.

Theorem 4: Let there exist symmetric R, S and X, Y as well as any T that satisfy (3), (6) and (9) and that are coupled as (10). Then there exists some $\psi_e \in RH_\infty^{(p+c)\times(p+c)}$ and a controller K which internally stabilizes P and such that He$(\psi_e) \succ 0$ as well as He$(\psi_e(P \star K)) \prec 0$ hold true.

Note that Theorem 4 couples the multipliers ψ and ϕ merely in terms of the certificates R and S in (3) and (6) without any further constraints; we even allow for $n_\psi \neq n_\phi$ which implies that T is not square. The modification of the standard coupling of X, Y and the additional one between R, S, both related through some variable T, is new. It is important to note that this formulation renders the synthesis conditions invariant under (independent) coordinate changes in the multiplier realizations.

Proof: Suppose that (3), (6) and (9)-(10) hold for $\psi_{\rm ss} = (A_\psi, B_\psi, C_\psi, D_\psi)$ and $\phi_{\rm ss} = (A_\phi, B_\phi, C_\phi, D_\phi)$. The first step is to extend ψ, ϕ to SPR multipliers (11) whose realizations are tightly coupled as in (8). First suppose $m := n_\psi = n_\phi$. Then T is square and by a small perturbation (if necessary) we can assume $\det(T) \neq 0$. After a change of the state-coordinates of the ψ-realization we obtain $T = I$. With $R_{12} = R_{21} = R_{22} := R - S^{-1}$ let us now define the extension

$$R_e = \begin{pmatrix} R & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \succ 0 \text{ with } R_e^{-1} = \begin{pmatrix} S & S_{12} \\ S_{21} & S_{22} \end{pmatrix} =: S_e \succ 0.$$

With a to-be-chosen $A_2 \in \mathbb{R}^{m\times m}$ and $A_{12} := -R^{-1}R_{12}A_2$, take the following uncontrollable realization of ψ:

$$\left(\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right) := \left(\begin{array}{cc|c} A_\psi & A_{12} & B_\psi \\ 0 & A_2 & 0 \\ \hline C_\psi & 0 & D_\psi \end{array}\right).$$

If $R_s := R_{22} - R_{21}R^{-1}R_{12}$ we then infer

$$\text{He}\begin{pmatrix} R_eA_e & R_eB_e \\ -C_e & -D_e \end{pmatrix} = \text{He}\left(\begin{array}{cc|c} RA_\psi & 0 & RB_\psi \\ R_{21}A_\psi & R_sA_2 & R_{21}B_\psi \\ \hline -C_\psi & 0 & -D_\psi \end{array}\right). \tag{12}$$

Canceling the second block row/column leads to the matrix $\mathrm{He}\begin{pmatrix} RA_\psi & RB_\psi \\ -C_\psi & -D_\psi \end{pmatrix}$ which is negative definite, because this is just the left-hand side of (3). Since A_2 enters (12) only in the $(2,2)$ position as $A_2^T R_s + R_s A_2$ and since $R_s \succ 0$, it is possible to choose a stable A_2 (such as e.g. $A_2 = -\kappa I$ for sufficiently large $\kappa > 0$) which renders (12) negative definite; then we also have $A_2^T R_s + R_s A_2 \prec 0$.

Analogously, (6) reads as $\mathrm{He}\begin{pmatrix} A_\phi S & -B_\phi \\ C_\phi S & -D_\phi \end{pmatrix} \prec 0$. Hence there is a stable $\tilde{A}_2 \in \mathbb{R}^{n_\phi \times n_\phi}$ in the unobservable realization

$$\left(\begin{array}{c|c} \tilde{A}_e & \tilde{B}_e \\ \hline \tilde{C}_e & \tilde{D}_e \end{array}\right) := \left(\begin{array}{cc|c} A_\phi & 0 & B_\phi \\ -\tilde{A}_2 S_{21} S^{-1} & \tilde{A}_2 & 0 \\ \hline C_\phi & 0 & D_\phi \end{array}\right)$$

of ϕ such that $\mathrm{He}\begin{pmatrix} \tilde{A}_e S_e & -\tilde{B}_e \\ \tilde{C}_e S_e & -\tilde{D}_e \end{pmatrix} \prec 0$; then we also have $\tilde{A}_2 S_s + S_s \tilde{A}_2^T \prec 0$ for $S_s = S_{22} - S_{21} S^{-1} S_{12} \succ 0$.

By congruence and with $S_e^{-1} = R_e$ we arrive at

$$\mathrm{He}\begin{pmatrix} R_e A_e & R_e B_e \\ -C_e & -D_e \end{pmatrix} \prec 0 \quad \text{and} \quad \mathrm{He}\begin{pmatrix} R_e \tilde{A}_e & R_e \tilde{B}_e \\ -\tilde{C}_e & -\tilde{D}_e \end{pmatrix} \prec 0.$$

We have found realizations of $-\psi$ and $-\phi$ that admit a *common* certificate R_e for strict negative realness. This allows us to apply (up to sign-changes) the technique in [15, Section V] in order to obtain an extended SPR multiplier

$$\psi_e = \left[\begin{array}{c|cc} A_e & B_e & \star \\ \hline C_e & D_e & \star \\ \star & \star & \star \end{array}\right] \quad \text{with} \quad \phi_e := \psi_e^{-1} = \left[\begin{array}{c|cc} \tilde{A}_e & \tilde{B}_e & \star \\ \hline \tilde{C}_e & \tilde{D}_e & \star \\ \star & \star & \star \end{array}\right],$$

both of dimension $p + (2m + p)$. We stress that, in this construction, strict negative realness of $-\psi_e$, $-\phi_e$ is certified by R_e, S_e as in (3), (6) respectively, and that the extended realizations are coupled as in (8). Since $R_e = S_e^{-1}$, a slight perturbation implies that R_e, S_e also satisfy the second inequality in (10) for $T = I$.

By assumption, (9) admit solutions X, Y for the original realizations of ψ, ϕ; these are coupled as in the first inequality of (10) with $T = I$. The second main step of the proof is show that these three LMIs formulated for the realizations (A_e, B_e, C_e, D_e), $(\tilde{A}_e, \tilde{B}_e, \tilde{C}_e, \tilde{D}_e)$ of ψ, ϕ and referred to as $(9)_e$, $(10)_e$, do have solutions as well. For this purpose note that $(9a)_e$ involves the outer factor

$$\left(\begin{array}{ccc|c} A_\psi & A_{12} & B_\psi C & B_\psi D \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A & B \\ \hline C_\psi & 0 & D_\psi C & D_\psi D \\ 0 & 0 & 0 & I \end{array}\right) \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_1 \\ \hline 0 & 0 & U_2 \end{pmatrix}$$

in which, up to A_2, the 2nd block row vanishes. Similarly, $(9b)_e$ involves $(-\tilde{A}_e^T, \tilde{C}_e^T, -\tilde{B}_e^T, \tilde{D}_e^T)$ and has the very same structural property. Let us choose

$$X_\alpha := \begin{pmatrix} X_{11} & 0 & X_{12} \\ 0 & \alpha R_s & 0 \\ X_{21} & 0 & X_{22} \end{pmatrix} \quad \text{and} \quad Y_\alpha := \begin{pmatrix} Y_{11} & 0 & Y_{12} \\ 0 & \alpha S_s & 0 \\ Y_{21} & 0 & Y_{22} \end{pmatrix}.$$

Then $\alpha > 0$ only affects the $(2,2)$-blocks of $(9)_e$ in which $\alpha(A_2^T R_s + R_s A_2) \prec 0$ and $-\alpha(\tilde{A}_2 S_s + S_s \tilde{A}_2^T) \succ 0$ appear respectively. Hence, by (9) and the projection lemma, X_α, Y_α solve $(9)_e$ for all large α. In the same vein and by $R_s \succ 0$, $S_s \succ 0$, (10) implies $\begin{pmatrix} X_\alpha & I \\ I & Y_\alpha \end{pmatrix} \succ 0$ for all large α. We have shown that $(9)_e$ and $(10)_e$ are feasible. As seen above, these inequalities are identical to those for P_e and the extended multipliers ψ_e, ϕ_e with the constructed realizations. Hence applying Theorem 3 concludes the proof for $n_\psi = n_\phi$.

In case of $n_\psi \neq n_\phi$, one can introduce uncontrollable stable dynamics in the realization of ψ (if $n_\psi < n_\phi$) or of ϕ (if $n_\phi < n_\psi$) in order to enforce equality of the dimensions of the state-matrices. Solutions of the corresponding LMIs can then be constructed by diagonal augmentation, along the same lines as demonstrated in the previous step. ■

If solutions of the LMIs in Theorem 4 have been found, the proof provides a constructive procedure for determining a multiplier ψ_e and, through Theorem 3, some K such that robust stability of the interconnection in Fig. 1 is assured. Note that K has degree $n + 2\max\{n_\psi, n_\phi\}$ and that the dimension of the scheduling block is $c = p + 2\max\{n_\psi, n_\phi\}$.

Remark. An alternative approach to the gain-scheduled synthesis problem was given in [15], with LMIs formulated in terms of ψ and ϕ^{-1} and involving a direct coupling through a common SPR certificate (requiring $n_\psi = n_\phi$). As a benefit, this allows for unstable ψ, ϕ and the multiplier dilation can be performed without extra dynamics, which leads to controllers of degree $n + n_\psi$ with $c = p + n_\psi$. However, the conditions in [15] cannot be shown to be necessary! This gap is closed here as seen next.

B. Multiplier parametrization and necessity

Choose $\alpha > 0$ and define the basis matrix $b_\nu(s)$ through

$$\left[\begin{array}{c|c} A_\nu^p & B_\nu^p \\ \hline C_\nu^p & D_\nu^p \end{array}\right] := \left[\begin{array}{c|c} I_p \otimes \begin{pmatrix} -\alpha & \cdots & -2\alpha \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\alpha \end{pmatrix} & I_p \otimes \begin{pmatrix} -\sqrt{2\alpha} \\ \vdots \\ -\sqrt{2\alpha} \end{pmatrix} \\ \hline I_p \otimes \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & \sqrt{2\alpha} \\ \vdots & \iddots & \vdots \\ \sqrt{2\alpha} & \cdots & \sqrt{2\alpha} \end{pmatrix} & I_p \otimes \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \end{array}\right]; \quad (13)$$

the precise structure of A_ν^p follows from the fact that the realization is input-balanced (i.e. $A_\nu^p + (A_\nu^p)^T + B_\nu^p (B_\nu^p)^T = 0$). Note that $b_\nu(s) = \mathrm{col}\left(I_p \ \frac{s-\alpha}{s+\alpha} I_p \ \cdots \ \frac{(s-\alpha)^\nu}{(s+\alpha)^\nu} I_p\right)$ is stable, has dimension $p(\nu+1) \times p$ and McMillan degree $p\nu$. Given any $\psi \in RH_\infty^{p \times p}$, the choice of b_ν is motivated by the fact that there exists a sequence $M_\nu \in \mathbb{R}^{p \times p(\nu+1)}$ with $M_\nu b_\nu \to \psi$ (exponentially) for $\nu \to \infty$ in the H_∞-norm [19].

This motivates to parameterize ψ and ϕ in Theorem 4 as

$$M \left[\begin{array}{c|c} A_\nu^p & B_\nu^p \\ \hline C_\nu^p & D_\nu^p \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} (A_\mu^p)^T & (C_\mu^p)^T \\ \hline (B_\mu^p)^T & (D_\mu^p)^T \end{array}\right] N \qquad (14)$$

with free $M \in \mathbb{R}^{p \times p(\nu+1)}$ and $N \in \mathbb{R}^{p(\mu+1) \times p}$. By inspection, M and N enter (3), (6) and (9) *affinely*. Therefore, it is a convex feasibility problem to search for M, N which

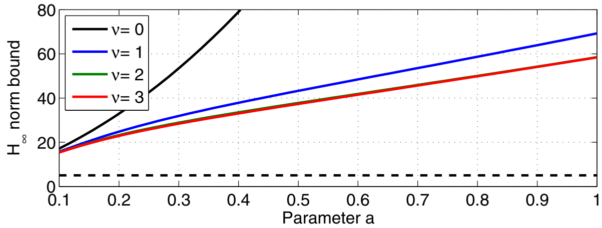

Fig. 3. Algorithm [15] with stable multipliers

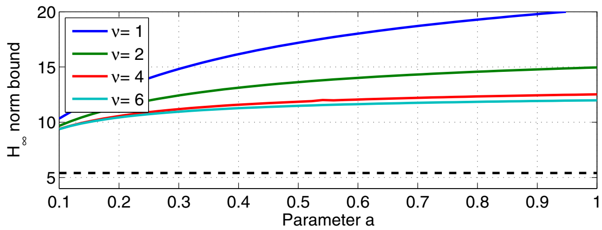

Fig. 4. New algorithm with stable multipliers

render the conditions in Theorem 4 satisfied. The above mentioned convergence properties even allow to prove that the conditions given in Theorem 4 are also *necessary* for the existence of a gain-scheduling controller. This leads to the following second main result, with a proof in Appendix A.

Theorem 5: There exists some $\psi_e \in RH_\infty^{(p+c)\times(p+c)}$ and a controller K which internally stabilizes P and such that $He(\psi_e) \succ 0$ as well as $He(\psi_e(P\star K)) \prec 0$ hold iff there exist $\nu, \mu \in \mathbb{N}_0$ and symmetric R, S, X, Y as well as matrices M, N, T that satisfy (3), (6), (9)-(10) with realization (14).

The extension to generalized (unstable) strict SPR multipliers is open. As argued in [15], this would lead to a solution of the gain-scheduling problem for D/G-scalings. a complete analogue of our results for D-scalings in [5].

C. A numerical example

Let us consider the simple interconnection

$$\begin{pmatrix} z \\ e \\ y \end{pmatrix} = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ \hline a & 0 & -0.1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.1 \\ 0 & 1 & -1 & 0.1 & 0 \end{array}\right] \begin{pmatrix} w \\ d \\ u \end{pmatrix}, \quad w = \delta z, \quad \delta \geq 0$$

with $a \in [0.1, 1]$; the parameter a serves to create a whole family of examples for the illustration of our results. Without giving details here, the technique proposed in this paper can be easily extended in order to design gain-scheduled controllers that achieve optimal bounds for the H_∞-norm of $d \to e$ within the respective multiplier classes.

A guaranteed lower bound for all multiplier classes is given by $\gamma_{\text{lb}} = \max_{\delta\geq 0} \gamma(\delta)$, where $\gamma(\delta)$ is the achievable H_∞-norm for fixed $\delta \geq 0$. We computed $\gamma_{\text{lb}} \approx 5.41$ and depict γ_{lb} in all subsequent figures by a dashed flat line.

For gain-scheduled controllers, the computed bounds on the H_∞-norm as a function of the parameter $a \in [0.1, 1]$ and

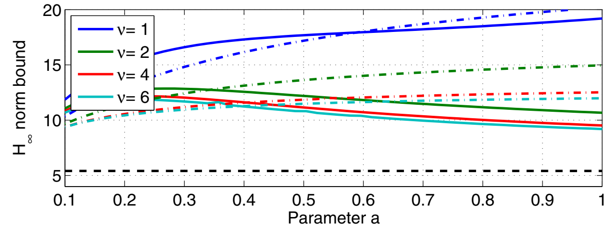

Fig. 5. New algorithm with stable multipliers (dash-dotted) and algorithm [15] with unstable multipliers (full)

with the technique from [15] for the "old" multiplier coupling are shown in Fig. 3; here ψ, ϕ^{-1} are parameterized as in Section III-B with $\alpha = 5$, $\alpha = 1$ respectively. The benefit of dynamic multipliers ($\nu > 0$) over static ones ($\nu = 0$) is substantial, but a further increase of ν beyond $\nu = 3$ does not lead to a significant improvement.

In contrast, the "new" multiplier coupling of Theorem 4 leads for ψ, ϕ parameterized with $\alpha = 5$, $\alpha = 1$ to the bounds in Figure 4, which reveals another big reduction of conservatism. By Theorem 5, the lowest curve gives close-to-optimal achievable bounds in the class of all SPR multipliers.

Let us also compare these results with those for the "old" technique and unstable multipliers, now parameterized with col $\left(\frac{(s-\alpha)^{-\nu}}{(s+\alpha)^{-\nu}} \;\cdots\; I_p \;\frac{s-\alpha}{s+\alpha}I_p \;\cdots\; \frac{(s-\alpha)^{\nu}}{(s+\alpha)^{\nu}}\right)$. The curves in Fig. 5 indicate an additional advantage (for some a's) if we were able to extend the new approach to unstable multipliers. The potentials of such an extension are as well visible in the rather large gap between the achieved bounds and γ_{lb}; this difference should be close to zero for one-parameter problems and a non-conservative synthesis technique.

IV. Conclusions

In this paper we have given necessary and sufficient convex conditions for the existence of gain-scheduled controllers with dynamic stable SPR. Numerical examples illustrate the substantial improvement of dynamic over static multipliers and over earlier derived sufficient existence conditions.

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APPENDIX

A. Sketch of proof of Theorem 5

The statement "if" is Theorem 3 such that it suffices to prove "only if". With a sufficiently large μ and a suitable coefficient matrix, we approximate $\psi_e^{-1} \in RH_\infty^{(p+c)\times(p+c)}$ by

$$\phi_e := \left[\begin{array}{c|c} (A_\mu^{p+c})^T & (B_\mu^{p+c})^T \\ \hline (C_\mu^{p+c})^T & (D_\mu^{p+c})^T \end{array}\right] \begin{pmatrix} N & N_3 \\ N_2 & N_4 \end{pmatrix},\ N \in \mathbb{R}^{p(\nu+1)\times p}$$

so closely such that $\mathrm{He}(\phi_e^{-1}) \succ 0$ and $\mathrm{He}(\phi_e^{-1}(P \star K)) \prec 0$ hold. The realization of ϕ_e^{-1} obtained by the standard state-space formulas (see (8)) is denoted and partitioned as

$$\phi_e^{-1} = \left[\begin{array}{c|cc} A_\psi & B_\psi & B_\psi^2 \\ \hline C_\psi & D_\psi & D_\psi^{12} \\ C_\psi^2 & D_\psi^{21} & D_\psi^{22} \end{array}\right] \in RH_\infty^{(p+c)\times(p+c)}.$$

By Theorem 3 applied to P_e and ϕ_e^{-1}, ϕ_e, there exist solutions of the corresponding LMIs (3), (6), (9)-(10). Note again that (3), (6) and (9) only involve the left-upper $p \times p$ blocks of ϕ_e^{-1} and ϕ_e and hence the realizations

$$\left(\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} (A_\mu^p)^T & 0 & (B_\mu^p)^T N \\ 0 & (A_\mu^c)^T & (B_\mu^c)^T N_2 \\ \hline (C_\mu^p)^T & 0 & (D_\mu^p)^T N \end{array}\right)$$

respectively. Due to unobservable modes of that on the right, canceling the second block row/olumn in (6), (9b) implies that the elements in the second block row/column of S, Y drop out; the corresponding rows/columns can also be canceled in (10). One arrives at inequalities (3), (6), (9) for

$$\psi := \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} (A_\mu^p)^T & (B_\mu^p)^T N \\ \hline (C_\mu^p)^T & (D_\mu^p)^T N \end{array}\right]$$

and the coupling condition (10) for $T := \begin{pmatrix} I_{\nu p} \\ 0 \end{pmatrix}$.

For $K_\nu \succ 0$ (to be constructed below), we trivially have

$$\left(\begin{array}{ccc|cc} X_{11} & 0 & X_{12} & T & 0 \\ 0 & K_\nu & 0 & 0 & 0 \\ X_{21} & 0 & X_{22} & 0 & I \\ \hline T^T & 0 & 0 & Y_{11} & Y_{12} \\ 0 & 0 & I & Y_{21} & Y_{22} \end{array}\right) \succ 0, \quad \left(\begin{array}{cc|c} R & 0 & T \\ 0 & K_\nu & 0 \\ \hline T^T & 0 & S \end{array}\right) \succ 0. \tag{15}$$

From this point on we make heavy use of the results in [19]. W.l.o.g. we can assume that ψ is invertible on $\mathbb{C}^0$ and that its realization is minimal (by slightly perturbing its realization matrices without violating the LMIs). Now choose a sequence M_ν with

$$\psi_\nu := \left[\begin{array}{c|c} A_\nu^p & B_\nu^p \\ \hline M_\nu C_\nu^p & M_\nu D_\nu^p \end{array}\right] \xrightarrow{\text{exp}} \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right]. \tag{16}$$

Fix $\epsilon > 0$ such that (3) and (9a) persist to hold for $-\hat{Q} + \epsilon I$ replacing $-\hat{Q}$ and for $\hat{Q} + \epsilon I$ replacing $\hat{Q}$, respectively. By [19, Theorem 2], there exist $K \succ 0$, $L = L^T$ and sequences $K_\nu \succ 0$, $\begin{pmatrix} T_\nu \\ S_\nu \end{pmatrix}$ (invertible) with the following property: Set

$$\begin{pmatrix} Z_{11}^{\nu,\alpha} & Z_{12}^{\nu,\alpha} \\ Z_{21}^{\nu,\alpha} & Z_{22}^{\alpha,\beta} \end{pmatrix} := \begin{pmatrix} T_\nu & 0 \\ S_\nu & 0 \\ 0 & I \end{pmatrix}^T \left(\begin{array}{cc|c} \alpha K & 0 & -\alpha K \\ 0 & K_\nu & 0 \\ \hline -\alpha K & 0 & \alpha K + \beta L \end{array}\right) \begin{pmatrix} T_\nu & 0 \\ S_\nu & 0 \\ 0 & I \end{pmatrix};$$

for any $\beta \in (0, 1)$ and any sufficiently large $\alpha > 0$, one can then choose ν sufficiently large such that

$$\mathscr{L}\left(\begin{pmatrix} Z_{11}^{\nu,\alpha} & Z_{12}^{\nu,\alpha} \\ Z_{21}^{\nu,\alpha} & Z_{22}^{\alpha,\beta} \end{pmatrix}, \begin{pmatrix} Q_2 & 0 \\ 0 & -Q_1 \end{pmatrix}, \left(\begin{array}{cc|c} A_\nu^p & 0 & B_\nu^p \\ 0 & A_\psi & B_\psi \\ \hline M_\nu C_\nu^p & 0 & M_\nu D_\nu^p \\ 0 & C_\psi & D_\psi \end{array}\right)\right) \prec 0$$

holds for $Q_2 := -\hat{Q}$, $Q_1 := -\hat{Q} + \epsilon I$ and $Q_2 := \hat{Q}$, $Q_1 := \hat{Q} + \epsilon I$ (with $Q = Q_2 - Q_1 = -\epsilon I \prec 0$ in both cases). The gluing lemma [19, Corollary 4] then implies that

$$\tilde{R}^{\nu,\alpha,\beta} := Z_{11}^{\nu,\alpha} - Z_{12}^{\nu,\alpha}(Z_{22}^{\alpha,\beta} + R)^{-1} Z_{11}^{\nu,\alpha},$$

$$\tilde{X}^{\nu,\alpha,\beta} := \begin{pmatrix} Z_{11}^{\nu,\alpha} & 0 \\ 0 & X_{22} \end{pmatrix} - \begin{pmatrix} Z_{12}^{\nu,\alpha} \\ X_{21} \end{pmatrix} (Z_{22}^{\alpha,\beta} + X_{11})^{-1} (Z_{11}^{\nu,\alpha}\ X_{12})$$

satisfy (3) and (9a) for ψ_ν as realized in (16)!

Since $K \succ 0$, it is easy to check that

$$\lim_{\alpha\to\infty,\,\beta\to 0} \tilde{X}^{\nu,\alpha,\beta} = \begin{pmatrix} T_\nu & 0 \\ S_\nu & 0 \\ 0 & I \end{pmatrix}^T \left(\begin{array}{cc|c} X_{11} & 0 & X_{12} \\ 0 & K_\nu & 0 \\ \hline X_{21} & 0 & X_{22} \end{array}\right) \begin{pmatrix} T_\nu & 0 \\ S_\nu & 0 \\ 0 & I \end{pmatrix},$$

$$\lim_{\alpha\to\infty,\,\beta\to 0} \tilde{R}^{\nu,\alpha,\beta} = \begin{pmatrix} T_\nu \\ S_\nu \end{pmatrix}^T \begin{pmatrix} R & 0 \\ 0 & K_\nu \end{pmatrix} \begin{pmatrix} T_\nu \\ S_\nu \end{pmatrix}.$$

Thus, by (15), we can fix $\beta > 0$ (small) and α (large) with

$$\left(\begin{array}{cc|cc} \tilde{X}_{11}^{\nu,\alpha,\beta} & \tilde{X}_{12}^{\nu,\alpha,\beta} & T_\nu^T T & 0 \\ \tilde{X}_{21}^{\nu,\alpha,\beta} & \tilde{X}_{22}^{\nu,\alpha,\beta} & 0 & I \\ \hline T^T T_\nu & 0 & Y_{11} & Y_{12} \\ 0 & I & Y_{21} & Y_{22} \end{array}\right) \succ 0, \quad \begin{pmatrix} \tilde{R}^{\nu,\alpha,\beta} & T_\nu^T T \\ T^T T_\nu & S \end{pmatrix} \succ 0,$$

which is the desired coupling condition. In summary, for all large ν, $\tilde{R}^{\nu,\alpha,\beta}$, S and $\tilde{X}^{\nu,\alpha,\beta}$, Y together with N, M_ν, $T_\nu^T T$ satisfy all the LMIs as claimed in Theorem 5.