2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

A Convergence Result for the Unscented Kalman-Bucy Filter using Contraction Theory

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Abstract— **This paper applies contraction theory to establish necessary conditions for contraction, hence, exponential convergence of the unscented Kalman-Bucy Filter. It follows that regions of contraction can subsequently be defined, given such necessary conditions. Both state and measurement models are Itô-type stochastic differential equations. By employing a virtual/actual system framework, a special relation is established between sigma-point dynamics, and observed process states, with respect to contraction and convergence. The proposed theory is illustrated on an isothermal, non-linear CSTR process.**

I. INTRODUCTION

The Kalman Filter (KF) is a well-established and widely used method for state estimation due to the optimality, tractability and robustness. Furthermore, if the underlying process- and measurement model is linear, with additive Gaussian noise, then the KF is the optimal Bayesian filter [1]. If strict assumptions on linearity and Gaussian noise are violated, which is often the case, then an alternative is to approximate the optimal Bayesian filter [2]. Popular methods in how to perform approximation include, but not limited to, Extended KF (EKF) and Unscented KF (UKF) [1] and variants thereof [3]. Sequential Monte Carlo methods, or more commonly referred to as Particle Filtering (PF) [2] is an alternative popular method for approximating the optimal Bayesian filter.

The EKF is an approximate filter for non-linear systems, based on first order linearisation. The EKF, however, can produce unstable filters when local linearity is violated. Furthermore, obtaining Jacobians by system linearisation are non-trivial in most applications [1]. The UKF is a recursive minimum mean-square-error estimator based on an approximate optimal Gaussian KF framework, that addresses some of the shortcomings associated with the EKF [4]. The UKF employs carefully chosen sigma-points (samples), which are propagated through the true non-linear process and measurement models respectively. Consequently, estimates of the posterior mean and covariance are accurate up to the second order (third order if the prior estimates are truly Gaussian) of the Taylor series expansion of any non-linear system [5]. PF in contrast to UKF utilizes a large amount of random samples to estimate the posterior distributions. In essence, PF approximates a filtering density function using a large amount of specially weighted samples [2], hence contributing to a large computational cost.

Several results on stability and convergence for the EKF can be found [6], [7], [8]. Also, results on general convergence for PF under certain conditions are provided [2]. However, despite aforementioned benefits of the UKF over the EKF, limited results exists with respect to stability and convergence analysis for the UKF [9]. The stability analysis strategy proposed by [7], applied to the EKF, has been adopted for the analysis of the discrete-time UKF with intermittent observations [10] and continuous-time UKF [11] respectively.

In this work we employ contraction theory as means to establish contraction, and subsequently, convergence of the continuous-time UKF. Contraction theory analyses convergence differentially where the latter may imply generalizing the classical Krasovskii theorem, or, linear eigenvalue analysis [12]. Contraction theory for the deterministic case has been presented by [12], and extended for a stochastic case by [13]. Contraction analysis has also been applied in a wide field of applications; two-link robot [14], [15], continuous stirred-tank reactor [16], ocean vehicles [14], [17], Fitzhugh-Nagumo Oscillators [13], and ship maneuvering [15]. Contraction theory has been used in analysing contraction, hence exponential convergence of the deterministic EKF [15], and general output-feedback observers [13]. However, no formal procedure in formulating a contraction metric for the UKF has been presented by either [15], or [13]. Hence, we propose a method for constructing a contraction metric, and provide necessary conditions which guarantees contraction, and consequently exponential convergence of the continuous UKF. A special relation that exists between observed process states and sigma-points, with respect to contraction, is established.

Notation

The symbols $\mathbb{R}$ and $\mathbb{I}$ denote the sets of real and integer numbers respectively. We denote $\mathbb{R}^{n\times m}$ real valued matrices of n-rows and m-columns. The mean of a jointly distributed real-valued time-varying variable $x(t)$ is defined by the expectation operator $\mathbf{E}\{x(t)\}$. Covariance between jointly distributed real-valued time-varying variables $x(t)$ and $y(t)$ is expressed by the covariance operator $\mathbf{cov}\{x(t), y(t)\}$. The trace of matrix A is denoted $\mathbf{Tr}\{A\}$.

II. KALMAN FILTERING

We consider the time-varying, non-linear and continuous system

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), t\right) + \sigma_x w(t) \\ z(t) &= h\left(x(t), t\right) + \sigma_z v(t) \end{aligned} \tag{1}$$

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with state vector $x(t) \in \mathbb{R}^n$, process model $f : \mathbb{R} \mapsto \mathbb{R}^n$ and measurement model $h : \mathbb{R} \mapsto \mathbb{R}^m$. Consider the process noise, $w(t) \sim (0, Q_c(t))$, and measurement noise, $v(t) \sim (0, R_c(t))$, to be independent white noise, with spectral density matrices $Q_c(t)$, $R_c(t)$ respectively. Define $\sigma_x \in \mathbb{R}^{n\times n}$, $\sigma_z \in \mathbb{R}^{m\times m}$ the process and measurement noise intensity matrices. We assume that $z(t) = \frac{dy(t)}{dt}$ is a differential measurement. The process and measurement models (1) can equivalently be formulated as Itô-type stochastic differential equations

$$\begin{aligned} dx(t) &= f\left(x(t), t\right) dt + \sigma_x dW_w \\ dy(t) &= h\left(x(t), t\right) dt + \sigma_z dW_v \end{aligned} \tag{2}$$

in which $dW_w := w(t)dt$ and $dW_v := v(t)dt$ are independent, standard Wiener processes.

Assumption 1 (Bounded noise intensity):
We assume that the noise intensity matrices σ_x, σ_z are for all $t \geqslant t_0$ uniformly upper-, and lower bounded by some positive real constants respectively.

The formulation, and presentation of the discrete-time UKF is thoroughly documented [4], [5], [1], [18], and thus will not be covered in-depth here. Prior to presenting the continuous-time UKF, we adopt a matrix framework to formulate the respective sigma-points and weights in a compact manner. Weighted averages and covariances are subsequently expressed using the matrix framework.

A. Matrix Framework for compact UKF

We adopt the compact matrix formulation of the discrete-time UKF, earlier presented by [3]. Discrete-time sampling of the continuous-time system and measurement models (1) at time $t = k$ gives

$$\begin{aligned} x\left(k+1\right) &= f_d\left(x\left(k\right)\right) + w(k) \\ z\left(k\right) &= h_d\left(x\left(k\right)\right) + v(k) \end{aligned} \tag{3}$$

where $w(k) \sim \mathcal{N}\left(0, Q\right)$, and $v(k) \sim \mathcal{N}\left(0, R\right)$, $\forall k \in \mathbb{I}_{\geqslant 0}$. The intuition behind the UKF is that it is easier to approximate an arbitrary Gaussian distribution of system states, than some arbitrary non-linear function, or transformation [4]. Subsequently, the UKF employs deterministically drawn sigma-points, which once propagated through the process and measurement models, aid in estimating the sample mean and covariances of the actual process states.

Definition 1 (Matrix format of Sampled Sigma-points):
We assume the expected mean and covariance of the process (3) at time $k = 0$ to be

$$\hat{x}\left(0\right) = \mathbf{E}\{x\left(0\right)\}, \quad P = \mathbf{cov}\{x\left(0\right), x\left(0\right)\} \tag{4}$$

For $k \in \mathbb{I}_{\geq 1}$, we sample $2n+1$ sigma-points, $\hat{\mathcal{X}}_i \in \mathbb{R}^n$,

$$\begin{aligned} &\hat{\mathcal{X}}_0\left(k-1\right) = \hat{x}(k-1) \\ &\hat{\mathcal{X}}_i\left(k-1\right) = \hat{x}\left(k-1\right) + \left(\sqrt{n+\kappa}\sqrt{P\left(k-1\right)}\right)_i \\ &\hat{\mathcal{X}}_{n+i}\left(k-1\right) = \hat{x}\left(k-1\right) - \left(\sqrt{n+\kappa}\sqrt{P\left(k-1\right)}\right)_{n+i} \end{aligned} \tag{5}$$

in which $\kappa \in \mathbb{R}$ and $(\cdot)_i$ is the i-th column of the matrix $(\cdot)$, $\forall i \in \mathbb{I}_{1:n}$. Define $\hat{X} \in \mathbb{R}^{n\times 2n+1}$ a matrix representation for the $2n+1$ sigma-points

$$\hat{X}\left(k-1\right) = \left[\hat{\mathcal{X}}_0\left(k-1\right), \cdots, \hat{\mathcal{X}}_{2n}\left(k-1\right)\right], \; k \in \mathbb{I}_{\geq 1} \tag{6}$$

Definition 2 (Weighted Mean and Covariance): Define W_i the respective weight for the i^{th} sigma point

$$W_0 = \frac{\kappa}{n+\kappa}, \quad W_i = \frac{1}{2\left(n+\kappa\right)}, \quad \forall i \in \mathbb{I}_{1:2n} \tag{7}$$

in which κ being a secondary scaling parameter usually being set to zero. Define a vector of weights W_i, $\forall i \in \mathbb{I}_{0:2n}$ as $\bar{W} := \left[W_0, \cdots, W_{2n}\right]^T \in \mathbb{R}^{2n+1}$. Using $\bar{W}$, given any matrix $Y \in \mathbb{R}^{n\times 2n+1}$, the weighted average of the $2n+1$ elements in the respective n-rows are calculated as $Y\bar{W}$. Furthermore, we define matrix

$$\mathcal{W} = \left(I - \left[\bar{W}, \cdots, \bar{W}\right]\right) diag\left(\bar{W}\right) \left(I - \left[\bar{W}, \cdots, \bar{W}\right]\right)^T \tag{8}$$

in which $\mathcal{W} \in \mathbb{R}^{2n+1\times 2n+1}$. It follows that the variance of the $2n+1$ elements in the respective n-rows of matrix Y is calculated as $Y\mathcal{W}Y^T$.

B. Continuous-time UKF

The derivation of the continuous-time UKF (henceforth called the Unscented Kalman-Bucy Filter (UKBF)) follows along similar lines of deriving the continuous-time KF, also referred to as the Kalman-Bucy filter. The Kalman-Bucy filter derivation starts by taking the limit for the discrete KF, in which time step Δt asymptotically approaches zero [18]. The UKBF is derived in similar fashion by applying the limit $\Delta t \to 0$ of the discrete-time UKF [3]. We adopt the matrix formulation of sigma-points and weights, previously defined in Definition 1 and 2, and derive the UKBF in the Appendix I. Algorithm 2.1 summarizes the implementation steps.

Algorithm 2.1 (UKBF):
Step 1: (Initialize) Initialize the UKBF

$$\hat{x}\left(0\right) = \mathbf{E}\{x\left(0\right)\}, \quad P(0) = \mathbf{cov}\{x\left(0\right), x\left(0\right)\} \tag{9}$$

Sample initial $2n+1$ sigma-points at time $t_0 = 0$

$$\hat{X}\left(0\right) = \left[\hat{\chi}_0\left(0\right), \cdots, \hat{\chi}_{2n}\left(0\right)\right]$$

Step 2: (Solve ODE with continuous measurements) The continuous UKBF observer gain $\mathcal{K}(t)$, and ordinary differential equations of the estimated mean value of process state, and covariance, is expressed for time $t \geqslant t_0$

$$\mathcal{K}(t) = \hat{X}(t)\mathcal{W}h^T(\hat{X}(t), t)[\sigma_z R_c(t)\sigma^T]^{-1} \tag{10a}$$

$$\dot{\hat{x}}(t) = f(\hat{X}(t), t)\bar{W} + \mathcal{K}(t)[z(t) - h(\hat{X}(t), t)\bar{W}] \tag{10b}$$

$$\dot{P}(t) = \begin{aligned} &\hat{X}(t)\mathcal{W}f^T(\hat{X}(t), t) + f(\hat{X}(t), t)\mathcal{W}\hat{X}^T(t) \\ &+\sigma_x Q_c(t)\sigma_x^T - \mathcal{K}(t)\sigma_z R_c(t)\sigma_z^T\mathcal{K}^T(t) \end{aligned} \tag{10c}$$

in which $z(t) = \lim\limits_{\Delta t\to 0} \frac{\Delta y}{\Delta t}$ follows from our differential measurement assumption.

Remark 2.1: Suppose the actual process (1), has a process state-expectation $\bar{x}(t) := \mathbf{E}\{x(t)\}$ and variance $P(t) := \mathbf{var}\{x(t), x(t)\}$. Then, similar to (5), one can sample $2n+1$ sigma-points from the distribution $x(t) \sim \mathcal{N}\left(\bar{x}(t), P(t)\right)$, and construct a matrix formulation of the $2n+1$ sigma-points which we define as the *actual* process sampled sigma-point matrix[1], $X(t) \in \mathbb{R}^{n \times 2n+1}$. It is clear that the weighted average of X, i.e., $X(t)\bar{W}$, will simply result in the actual expected mean of process value, i.e., $\bar{x}(t)$ since $\mathbf{E}\{X(t)\bar{W}\} = \bar{x}(t)$. Similarly, if we take the weighted average[2] of the measurement model $h(X(t),t)$, i.e., $h(X(t),t)\bar{W}$, we also end up with the actual process measurement $z(t) = h(X(t),t)\bar{W}$.

Lemma 2.1 (Dynamic Relationship):
The estimated sigma-point dynamics $d\hat{X}(t)/dt$ is linearly related to the estimated expected mean process dynamics $d\hat{x}(t)/dt$ as $\frac{d\hat{X}(t)}{dt}\bar{W} = \frac{d\hat{x}(t)}{dt}$.
Proof. Extend the continuous-time matrix formulation of the $2n+1$ *sigma-points,* $\hat{X}(t) = [\hat{\mathcal{X}}_0(t), \cdots, \hat{\mathcal{X}}_{2n}(t)]$, *and take the expected mean thereof, i.e.,*

$$\boldsymbol{E}\Big\{\hat{X}(t)\Big\} = \boldsymbol{E}\Big\{[\hat{x}(t), \cdots, \hat{x}(t)] + \sqrt{n+\kappa}[0, \sqrt{P(t)}, -\sqrt{P(t)}]\Big\} \quad (11)$$

In light of Definition 2 we can express (11) equivalently as $\hat{X}(t)\bar{W} = \hat{x}(t)$ *where the derivative of the later concludes the proof.* ■

The result of Lemma 2.1 enables us to express the expected mean of estimated process dynamics (10b) equivalently as

$$\frac{d\hat{X}(t)}{dt}\bar{W} = f(\hat{X}(t))\bar{W} + \mathcal{K}(t)[z(t) - h(\hat{X}(t),t)\bar{W}] \quad (12)$$

Definition 3 (Sigma-point dynamics):
In light of (12), we define the sigma-point dynamics as

$$\dot{\hat{X}}(t) = f(\hat{X}(t),t) + \mathcal{K}(t)[Z(t) - h(\hat{X}(t),t)] \quad (13)$$

Observe that (12) is obtained by multiplying (13) by $\bar{W}$ and using the relation $z(t) = Z(t)\bar{W}$ in which $Z(t) = h(X(t),t)$.

Remark 2.2: Lemma 2.1 and Definition 3 provides us with an equivalence result between the particular sigma-point dynamics and corresponding estimated process state dynamics. Hence, by Lemma 2.1 any conclusions drawn for the sigma-point dynamics, with respect to convergence/divergence and contraction, can directly be applied to the estimated process state dynamics.

III. CONTRACTION ANALYSIS OF UKBF

We are interested in applying contraction theory to analyse the contraction, hence exponential convergence properties of the UKBF, (Algorithm 2.1) when used as an observer for system (1). The equivalent Itô-type stochastic differential interpretation of the estimated expected mean value of process state, $\dot{\hat{x}}(t)$, in the UKBF Algorithm 2.1, is defined as

$$d\hat{x}(t) = (f(\hat{X}(t),t) + \mathcal{K}(t)[Z(t) - h(\hat{X}(t),t)])\bar{W}dt \quad (14a)$$
$$+ \mathcal{K}(t)\sigma_z dW_v \quad (14b)$$

Remark 3.1: The stochastic differential interpretation (14) follows from definition $z(t)$ in (1), and relations $v(t)dt = dW_v$, $z(t) = Z(t)\bar{W} + \sigma_z v(t)$, $Z(t) = h(X(t),t)$.

The Itô process (14) consists of a deterministic (14a) and stochastic part (14b). Using Lemma 2.1, we present a result on deterministic contraction for the sigma-point dynamics (13). We employ a virtual/actual system framework [15] in conjunction with contraction analysis theory [12].

A. Deterministic contraction analysis

Definition 4 (Virtual-Actual framework):
Define the following virtual-actual system framework

$$\dot{\chi}(t) = f(\chi(t),t) + \mathcal{K}(t)[Z(t) - h(\chi(t),t)] =: \tilde{f} \quad (15)$$

Remark 3.2: The sigma-point dynamics, (13), and the deterministic part (14a) of the actual process model (1) are particular solutions of the virtual system (15), given $\chi(t)$ is being replaced with with either $\hat{X}(t)$, or $\hat{x}(t)$ respectively.

Using contraction theory, to analyse the contraction properties of the virtual system (15), requires the definition of a virtual displacement operator, $\delta_{\tilde{f}}\chi(t)$, [12], which is an infinitesimal displacement at fixed time for system $\tilde{f}$. Assuming that (15) is continuously differentiable, the first variation of (15) is

$$\delta_{\tilde{f}}\dot{\chi}(t) = \frac{\partial\tilde{f}}{\partial\chi}\delta_{\tilde{f}}\chi(t) = (A - \mathcal{K}C)\,\delta_{\tilde{f}}\chi(t) \quad (16)$$

where $A := \frac{\partial f(\chi(t),t)}{\partial\chi(t)}\Big|_{\chi(t)=\hat{\chi}(t)}$; $C := \frac{\partial h(\chi(t),t)}{\partial\chi(t)}\Big|_{\chi(t)=\hat{\chi}(t)}$.

Assumption 2 (Observability): Assume that for the continuous, linear, time-varying system $\dot{x}(t) = \tilde{A}x(t)$ there exists a stable equilibrium point at $x(t) = 0, t \geqslant 0$ where $\tilde{A} := A - \mathcal{K}C$.

Remark 3.3: Assumption 2 can be seen as an implicit observability assumption and may only be valid for a small, local region. However, the result on contraction, hence, exponential convergence for the UKBF supersedes this assumption, where Assumption 2 is incorporated to establish results on a finite region of contraction.

Assumption 3 (Bounds on $\mathcal{K}(t)$ and $\tilde{A}(t)$): Assume that $\mathcal{K}(t)$ and $\tilde{A}(t)$ are for all $t \geqslant 0$ uniformly bounded.

Consider the following matrix definition,

$$\Psi := diag\left(\begin{bmatrix} \Psi_A, & \Psi_B + diag\left([|\lambda|\, I, 0, |\lambda|\, I]\right) \end{bmatrix}\right) \quad (17)$$

in which

$$\Psi_A := \begin{bmatrix} I & I \\ I & I \end{bmatrix},\ \Psi_B := \begin{bmatrix} R_c(t) & -I & 0 \\ -I & I & 0 \\ 0 & 0 & Q_c(t) \end{bmatrix} \quad (18)$$

and λ the minimum eigenvalue of matrix Ψ_B. It is clear that Ψ is positive semi-definite since Ψ is a Hermitian matrix with all eigenvalues not negative. Also define

$$g := \begin{bmatrix} \tilde{A}^T(t), & P(t) - \hat{\mathcal{X}}(t)\mathcal{W}\hat{\mathcal{X}}^T(t), & \mathcal{K}(t), & \hat{\mathcal{X}}(t)\mathcal{W}Z^T(t), & I \end{bmatrix}^T \quad (19)$$

Definition 5 (Matrix $Q_A(t)$): Define a positive definite matrix $Q_A(t) := g\Psi g^T + \sigma I$ in which σ being some positive constant .

Theorem 3.1 (Continuous Lyapunov equation):
Suppose Assumption 2 holds. Let $Q_A(t)$ be defined as in Definition 5. Then, there exists a continuously differentiable,

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[1] It is important to take note that $X(t)$ is presented here for the sake of analysis. In practise, the distribution of the actual process states need to be estimated, and is not known a-priori.

[2] see footnotes in Appendix for explanation of notation used

bounded, positive definite, symmetric matrix $P_A(t)$ that satisfy

$$-\dot{P}_A(t) = P_A(t)\tilde{A}(t) + \tilde{A}^T(t)P_A(t) + Q_A(t) \tag{20}$$

Proof. *For proof see [19].* ■

Let $\theta(t)$ be a square, transformation matrix, with subsequent state transformation $\delta_{\tilde{f}}\eta(t) = \theta(t)\delta_{\tilde{f}}\chi(t)$. Next, we define a uniformly positive definite, symmetric, continuously differential metric $M(t) := \theta^T(t)\theta(t)$, such that we can express the following transformed, virtual displacement distance

$$\delta_{\tilde{f}}^T\eta(t)\delta_{\tilde{f}}\eta(t) = \delta_{\tilde{f}}^T\chi(t)M(t)\delta_{\tilde{f}}\chi(t)$$

We are interested in finding a metric $M(t)$, and positive definite scalar β_M, such that given the velocity of the virtual displacement,

$$\delta_{\tilde{f}}{}^T\chi(t)\left(\tfrac{\partial \tilde{f}^T}{\partial \chi}M(t) + \dot{M}(t) + M(t)\tfrac{\partial \tilde{f}}{\partial \chi}\right)\delta_{\tilde{f}}\chi(t) \tag{21}$$

we can conclude exponential convergence to a nominal trajectory in the regions of

$$\tfrac{\partial \tilde{f}}{\partial \chi}^T M(t) + \dot{M}(t) + M(t)\tfrac{\partial \tilde{f}}{\partial \chi} \leqslant -\beta_M M(t),\ \beta_M > 0 \tag{22}$$

Definition 6 (Contraction metric):
We define a metric $M(t)$ as

$$M(t) = P_A(t) + P(t) - \hat{\chi}(t)\mathcal{W}\hat{\chi}^T(t) \tag{23}$$

with $P_A(t)$ being the solution to (20) (matrix $Q_A(t)$ and $\tilde{A}$ are defined according to Definition 5 and Assumption 2 respectively). We have $P(t)$ being the solution of (10c).

Lemma 3.1: Metric $M(t)$ is positive definite, symmetric and continuously differentiable.
Proof. *See Appendix II.* ■

Lemma 3.2: Suppose Assumptions 2 and 3 hold. For choice of contraction metric $M(t)$ (Definition 6) it follows that there exists a strictly positive definite scalar, β_M, such that the virtual system, (15), contracts, and exponential convergence to a nominal trajectory can be concluded in the regions of

$$\tilde{A}^T M(t) + \dot{M}(t) + M(t)\tilde{A} \leqslant -\beta_M M(t),\ \beta_M > 0 \tag{24}$$

Proof. *See Appendix III.* ■

Theorem 3.2 (Deterministic contraction): There exists a positive definite scalar β_m such that deterministic, estimated expected mean process dynamics, (14a) is contracting, with contraction rate β_M.

Proof. *Lemma 3.2 concludes contraction for the virtual system (15), hence exponential convergence to a single trajectory in the regions (24). The virtual system framework enables us to express the sigma-point dynamics (13) as a particular solution to the virtual system (15). It follows that the sigma-point dynamics, (13), inherits the contraction properties established with the virtual system, (15). From Lemma 2.1, we have stated the particular relation that holds between $\dot{\hat{x}}(t)$ and $\dot{\hat{X}}(t)$, where we therefore can conclude that the deterministic, estimated expected mean process dynamics, (14a) is contracting, with contraction rate β_M.* ■

B. Stochastic contraction of UKBF

Definition 7 (General stochastic process): Define a general Itô-type stochastic differential equation

$$d\xi(t) = F\left(\xi(t), t\right)dt + G\left(\xi(t), t\right)dW_\xi \tag{25}$$

where $F(\xi(t), t) : \mathbb{R} \mapsto \mathbb{R}^n$ being a function, $G(\xi(t), t) \in \mathbb{R}^{n\times n}$ a noise-intensity matrix, and dW_ξ a standard Wiener process.

We next state a stochastic contraction theorem for the general Itô process (25).

Theorem 3.3 (Stochastic contraction, [13]):
Given an Itô-type stochastic differential system (25), suppose the existence of a state-independent, uniformly positive definite metric $M(t)$ with lower bound $\beta > 0$, and a positive definite scalar β_M, which verifies

$$\frac{\partial F}{\partial \xi}^T M(t) + \dot{M}(t) + M(t)\frac{\partial F}{\partial \xi} \leqslant -\beta_M M(t),\ \beta_M > 0 \tag{26}$$

and

$$\mathbf{Tr}\left\{G^T(\xi)M(t)G(\xi)\right\} \tag{27}$$

being uniformly upper-bounded by some constant $c \geqslant 0$. Assume two given trajectories $\xi(t)$, $\hat{\xi}(t)$, with initial conditions, independent of noise, and probability distribution $p(\xi(0), \hat{\xi}(0))$. Then, for all $t \geqslant t_0$

$$\mathbf{E}\left\{\|\phi(t)\|^2\right\} \leq \frac{1}{\beta}\left(\frac{c}{\beta_M} + \mathbf{E}\left\{\phi(0)^T M\left(0\right)\phi\left(0\right)\right\}e^{-2\beta_M t}\right) \tag{28}$$

in which $\phi\left(t\right) = \xi\left(t\right) - \hat{\xi}\left(t\right)$.

Remark 3.4: Theorem 3.3 provides us with necessary conditions under which an Itô-type stochastic differential system (25) will contract with rate β_M and bound $\frac{c}{\beta\beta_M}$. For a noise free system, i.e., $c = 0$, we have that trajectories $\xi(t)$, $\hat{\xi}(t)$ will exponentially converge to each other as $t \to \infty$.

Lemma 3.3: Suppose Assumptions 1 and 3 hold. Then, there exists some positive real constant c which uniformly upper-bounds $\mathbf{Tr}\left\{\sigma_z{}^T\mathcal{K}^T(t)\mathcal{K}(t)\sigma_z\right\}$.
Proof. *It is clear by Assumptions 1 and 3.* ■

Theorem 3.4 (Contraction of UKBF): Given an actual, noisy process (1) and corresponding UKBF observer (10), suppose Assumptions 1, 2 and 3 hold. Then, there exists a contraction metric $M(t)$, a contraction rate $\beta_M > 0$, and a uniform upper bound $\frac{c}{\beta\beta_M}$ such that (14), and consequently UKBF Algorithm 2.1, will contract towards a bounded region with a nominal trajectory in the interior.
Proof. *Form Lemmas 2.1, 3.2 and Theorem 3.2 it follows that the deterministic part of the Itô-type stochastic differential equation (14) verifies condition (26). Lemma 3.3 verifies condition (27). Hence, it follows that all the conditions given in Theorem 3.3 for stochastic contraction are met, which concludes the stochastic contraction of (14), hence UKBF.* ■

Corollary 3.1 (Exponential convergence of UKBF): The estimated mean of process states $\hat{x}(t)$ exponentially converge to a bounded region with expected mean of nominal process state trajectory $\bar{x}(t)$ in its interior, i.e.,

$$\lim_{t\to\infty} \mathbf{E}\left\{\left\|\hat{x}(t)-\bar{x}(t)\right\|^2\right\} \le \frac{c}{\beta\beta_M} \tag{29}$$

Proof. A direct consequence of Theorem 3.4 in conjunction with Lemma 2.1 is that: (i.) Noisy system ($c > 0$): Exponentially asymptotic convergence of $\hat{x}(t)$ to a bounded region with $\bar{x}(t)$ in interior. (ii.) Noise-free system ($c = 0$): Exponentially asymptotic convergence of $\hat{x}(t)$ to $\bar{x}(t)$. ■

IV. CASE EXAMPLE

We consider a non-linear, irreversible chemical reaction in an isothermal CSTR [20], $A \to B$, $r = kc_A^n$, in which k is the rate constant and n is the reaction order. The material balance for this isothermal CSTR process is

$$\frac{dc_A(t)}{dt} = \frac{1}{\tau}\left(c_{Af}(t) - c_A(t)\right) - kc_A^n(t) + w(t)$$

with process measurement,

$$z(t) = -\frac{1}{2\tau}\left(c_{Af}(t) - c_A(t)\right) + \frac{1}{2}kc_A^n(t) + v(t)$$

in which $w(t)$ and $v(t)$ are additive process and measurement noise with a spectral densities of $Q_c = 0.4$ and $R_c = 0.1$ respectively. We have $c_A(t)$ the molar A concentration, $c_{Af}(t)$ a manipulated feed rate A concentration and parameters $\tau = 10, k = 1.2, n = 2$. We assume a sinusoidal manipulated feed rate, $c_{Af}(t) = 100 + 35sin(2t)$.

A. Results

The simulation was performed over a time period of 10s using a 4-th order stochastic Runge-Kutta method [21] with time steps $\Delta t = 0.01$. For initial conditions $\hat{x}(0) = \mathbf{E}\{x(0)\}$, $P(0) = \mathbf{cov}\{x(0), x(0)\}$, Figure 1.a depicts the convergence of UKBF state estimate to the actual process, where as Figure 1.b depicts the corresponding convergence of estimated sigma-points, validating Lemma 2.1 and remarks in Note 2.2. Figure 1.c illustrates the validity of the contraction condition (24) in Lemma 3.2 given the contraction rate choice of $\beta_M = 1$.

V. CONCLUSION

In this work necessary conditions for the existence of contraction for the UKBF was proposed when the underlying process and measurement models are Itô-type stochastic differential equations. A virtual/actual system framework in conjunction with contraction theory is employed, and it is subsequently shown that the expected convergence of the estimated process state and actual process state is upper bounded, where the bound is dependant on the adjacent noise of the system and contraction rate.

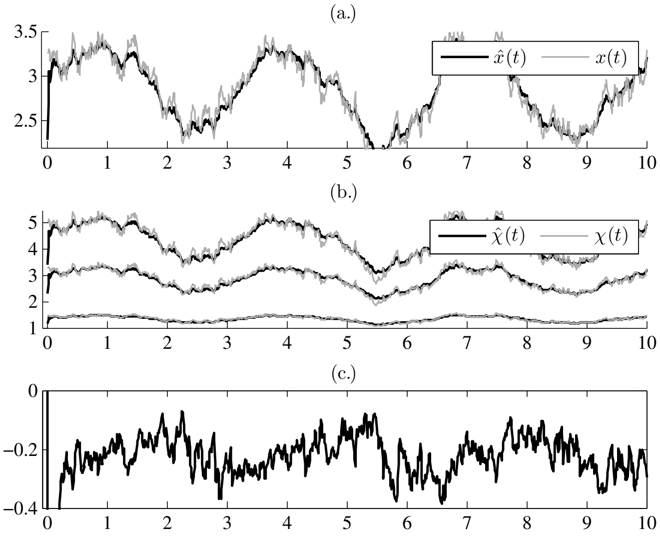

Fig. 1. Convergence of estimated process state and sigma-points, and condition of contraction.

REFERENCES

[1] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation," *Proceedings of IEEE*, vol. 92, no. 3, pp. 401–422, 2004.

[2] X. Hu, T. B. Schön, , and L. Ljung, "A general convergence result for particle filtering," *Signal Processing, IEEE Transactions on*, vol. 59, no. 7, pp. 3424–3429, 2011.

[3] S. Särkkä, "On unscented Kalman filtering for state estimation of continouos-time non-linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1631–1641, 2007.

[4] S. J. Julier and J. K. Uhlmann, "A new extension of the Kalman filter to nonlinear systems," *The Proceedings of AeroSense: The 11th International Symposium on Aerospace/Defense Sensing, Simulation and Controls.*, 1997.

[5] E. Wan and R. Van Der Merwe, "The unscented Kalman filter for nonlinear estimation," *Adaptive Systems for Signal Processing, Communications, and Control Symposium*, pp. 153–158, 2000.

[6] L. Ljung, "Asymptotic behaviour of the Extended Kalman Filter as a parameter estimator for linear systems," *IEEE Transactions on Automatic Control*, vol. 24, no. 1, pp. 36–50, 1979.

[7] M. Boutayeb, H. Rafaralahy, and M. Darouach, "Convergence analysis of the extended Kalman filter used as observer for nonlinear deterministic discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 42, no. 4, pp. 581–586, 1997.

[8] K. Reif, F. Sonnemann, and R. Unbehauen, "An EKF-based nonlinear observer with a prescribed degree of stability," *Automatica*, 1998.

[9] K. Xiong, H. Zhang, and C. Chan, "Performance evaluation of ukf-based nonlinear filtering," *Automatica*, vol. 42, pp. 261–270, 2006.

[10] L. Li and Y. Xia, "Stochastic stability of the unscented Kalman filter with intermittent observations," *Automatica*, vol. 48, pp. 978–981, 2012.

[11] J. Xu, S. Wang, and G. M. Dimirovski, "Stochastic stability of the continuous-time Unscented Kalman Filter," in *IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 5110–5115.

[12] W. Lohmiller and J. Slotine, "On contraction analysis for non-linear system," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[13] Q. Pham, N. Tabareau, and J. Slotine, "A contraction theory approach to stochastic incremental stability," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, 2009.

[14] W. Lohmiller and J.-J. E. Slotine, "Applications of metric observers for nonlinear systems," in *Proc. IEEE International Conference on Control Applications*.

[15] J. Jouffroy and T. Fossen, "A tutorial on incremental stability analysis using contraction theory," *Modelling, Identification and Control*, vol. 31, no. 3, pp. 93–106, 2010.

[16] W. Lohmiller and J.-J. E. Slotine, "Nonlinear process control using contraction theory," *AIChE Journal*, vol. 46, no. 3, pp. 588–596, 2000.

[17] J. Jouffroy and J. Lottin, "On the use of contraction theory for the design of nonlinear observers for ocean vehicles," in *Proc. American Control Conference*, Anchorage, Alaska, 2002, pp. 2647–2652.

[18] D. Simon, *Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches*. John Wiley and Sons, 2006.

[19] H. K. Khalil, *Nonlinear systems*. Prentice Hall, 2002.

[20] C. Lee and J. Bailey, "Modification of consecutive-competitive re-

action selectivity by periodic operation," *Ind. Eng. Chem. Process*, vol. 19, pp. pp. 160–166, 1980.

[21] N. Kasdin, "Runge-kutta algorithm for the numerical integration of stochastic differential equations," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 1, pp. 114–120, 1995.

APPENDIX I
DERIVATION OF UKBF

Proof. *We consider the continuous, system model (1). Discretize process model (1) with a time increment of* Δt *and assume that for* Δt *sufficiently small it holds that* $w(t)\Delta t \sim \mathcal{N}\left(0, Q_c(t)\Delta t\right)$ *and* $v\left(t\right)\Delta t \sim \mathcal{N}\left(0, R_c(t)\Delta t\right)$. *It follows that*

$$\begin{aligned} &x\left(t+\Delta t\right) = x\left(t\right) + f\left(x(t), t\right)\Delta t + \sigma_x(t) w\left(t\right)\Delta t + o\left(\Delta t\right) \\ &\Delta y = h\left(x(t+\Delta t), t+\Delta t\right)\Delta t + \sigma_z(t) v\left(t\right)\Delta t \end{aligned} \tag{30}$$

where $o\left(\Delta t\right)$ *is a discretization error where it holds that* $\frac{o(\Delta t)}{\Delta t} \to 0$ *and* $\Delta y = y\left(t+\Delta t\right) - y(t) \to \frac{dy(t)}{dt} = z(t)$ *when* $\Delta t \to 0$. *Consider the estimated state distribution* $x\left(t\right) \sim \mathcal{N}\left(\hat{x}\left(t\right), P\left(t\right)\right)$. *Define* $\hat{X}\left(t\right) = \left[\hat{\chi}_0\left(t\right), \; \cdots \; , \hat{\chi}_{2n}\left(t\right)\right]$ *a vector of continuous-time sigma-points deterministically sampled from the distribution of* $x(t)$ *in which* $\hat{\chi}_i(t)$, $\forall i \in \mathbb{I}_{0:2n}$ *is defined as in (5). For a* Δt *system evolution from* $x(t)$ *to* $x(t+\Delta t)$,[3]

$$\hat{X}(t+\Delta t) = \hat{X}(t) + f(\hat{X}(t), t)\Delta t + o(\Delta t) \tag{31}$$

express the predicted mean $\hat{x}^-(t+\Delta t) = \hat{X}(t+\Delta t)\bar{W}$ *and covariance as*

$$P^-\left(t+\Delta t\right) = \hat{X}\left(t+\Delta t\right)\mathcal{W}\hat{X}^T(t+\Delta t) \tag{32a}$$

$$+\sigma_x(t) Q_c(t) \sigma_x^T(t)\Delta t \tag{32b}$$

Next, substituting (31) into (32) and using the relations $\hat{x}^-\left(t\right) = \hat{X}\left(t\right)\bar{W}$ *and* $P\left(t\right) = \hat{X}\left(t\right)\mathcal{W}\hat{X}^T(t)$ *we have*

$$\begin{aligned} &\hat{x}^-(t+\Delta t) = \hat{x}^-(t) + f(\hat{X}(t), t)\bar{W}\Delta t + o(\Delta t) \\ &P^-(t+\Delta t) = P(t) + \hat{X}(t)\mathcal{W} f(\hat{X}^T(t), t)\Delta t \\ &\quad + f(\hat{X}(t), t)\mathcal{W}\hat{X}^T(t)\Delta t + \sigma_x(t) Q_c(t) \sigma_x^T(t)\Delta t \end{aligned} \tag{33}$$

assuming we are neglecting higher order terms. We can write the measurement update equations, given (33), as follows[4]

$$\begin{aligned} &\hat{X}^-\left(t+\Delta t\right) = \left[\hat{\chi}_0\left(t+\Delta t\right), \; \cdots \; , \hat{\chi}_{2n}\left(t+\Delta t\right)\right] \\ &\mathcal{Z}\left(t+\Delta t\right) = h(\hat{X}^-\left(t+\Delta t\right), t+\Delta t)\Delta t \\ &\hat{z}^-\left(t+\Delta t\right) = \mathcal{Z}\left(t+\Delta t\right)\bar{W}; \quad P_{zz}\left(t+\Delta t\right) = \\ &\quad \mathcal{Z}\left(t+\Delta t\right)\mathcal{W}\mathcal{Z}^T(t+\Delta t) + \sigma_z(t) R_c(t) \sigma_z^T(t)\Delta t \\ &P_{xz}\left(t+\Delta t\right) = \hat{X}^-\left(t+\Delta t\right)\mathcal{W}\mathcal{Z}^T(t+\Delta t) \end{aligned} \tag{34}$$

and the UKF update step as

$$\begin{aligned} &\mathcal{K}\left(t+\Delta t\right) = P_{xz}\left(t+\Delta t\right) P_{zz}^{-1}\left(t+\Delta t\right) \\ &\hat{x}\left(t+\Delta t\right) = \\ &\quad \hat{x}^-\left(t+\Delta t\right) + \mathcal{K}\left(t+\Delta t\right)\left[\Delta y - \hat{z}^-\left(t+\Delta t\right)\right] \\ &P\left(t+\Delta t\right) = \\ &\quad P^-\left(t+\Delta t\right) - \mathcal{K}\left(t+\Delta t\right) P_{zz}\left(t+\Delta t\right)\mathcal{K}^T(t+\Delta t) \end{aligned} \tag{35}$$

Next, substitute (34) into (35), and retain only first-order terms

$$\begin{aligned} &\mathcal{K}\left(t+\Delta t\right) = \hat{X}^-\left(t+\Delta t\right)\mathcal{W} h^T(\hat{X}^-\left(t+\Delta t\right), t+\Delta t) \\ &\quad \times[\sigma_z(t) R_c(t) \sigma_z^T(t)]^{-1} + \tfrac{o(\Delta t)}{\Delta t} \\ &\hat{x}\left(t+\Delta t\right) - \hat{x}^-\left(t\right) = f(\hat{X}\left(t\right), t)\bar{W}\Delta t \\ &\quad + \mathcal{K}\left(t+\Delta t\right)\left[\Delta y - h(\hat{X}^-\left(t+\Delta t\right), t+\Delta t)\bar{W}\Delta t\right] \end{aligned} \tag{36}$$

$$\begin{aligned} &P\left(t+\Delta t\right) - P\left(t\right) = \hat{X}\left(t\right)\mathcal{W} f(\hat{X}^T\left(t\right), t)\Delta t \\ &+ f(\hat{X}\left(t\right), t)\mathcal{W}\hat{X}^T(t)\Delta t + \sigma_x(t) Q_c(t) \sigma_x^T(t)\Delta t \\ &- \mathcal{K}\left(t+\Delta t\right)\sigma_z(t) R_c(t) \sigma_z^T(t)\left(t\right)\mathcal{K}^T(t+\Delta t)\Delta t \end{aligned} \tag{37}$$

Dividing by Δt *in the mean and covariance update expressions, and taking the time-limit of* $\Delta t \to 0$, *Algorithm 2.1 follows.* ■

APPENDIX II
PROPERTIES OF CONTRACTION METRIC M

Proof. *Consider discrete-time sigma-point system evolution (31). For a small step size* Δt, *we assume the Gaussian noise distribution* $w(t)\Delta t \sim \mathcal{N}(0, Q_c(t)\Delta t)$. *Define* $\bar{Q}_c(t) := Q_c(t)\Delta t$. *Next, consider the definition of the covariance of continuous-time white process noise*

$$\boldsymbol{E}\Big\{w\left(t\right) w^T\left(\tau\right)\Big\} := \sigma_x\left(t\right) Q_c\left(t\right) \sigma_x^T\left(t\right) = \sigma_x\left(t\right) \tfrac{\bar{Q}_c(t)}{\Delta t} \sigma_x^T\left(t\right) \tag{38}$$

The process covariance matrix (32) in conjunction with (38) implies

$$P^-\left(t+\Delta t\right) = \hat{X}\left(t+\Delta t\right)\mathcal{W}\hat{X}^T\left(t+\Delta t\right) \tag{39a}$$

$$+ \sigma_x(t) \bar{Q}_c(t) \sigma_x^T(t) \tag{39b}$$

Next, as we consider the limit $\Delta t \to 0$ *it follows that*

$$P\left(t\right) - \hat{X}\left(t\right)\mathcal{W}\hat{X}^T\left(t\right) = \sigma_x(t) \bar{Q}_c(t) \sigma_x^T(t) \geqslant 0 \tag{40}$$

Since matrix $P_A(t)$ *is a positive define matrix (see Theorem 3.1) it is clear that the contraction metric* M *is positive definite. Symmetry, and continuous differentiability is an obvious characteristic inherited from the individual components of the defined metric.* ■

Remark 2.1: It is important to note that the definition for covariance of continuous-time white process noise is strictly [18]

$$\mathbf{E}\Big\{w\left(t\right) w^T\left(\tau\right)\Big\} := Q_c \delta\left(t-\tau\right) \tag{41}$$

in which $\delta\left(t-\tau\right)$ is the continuous-time impulse response; a function of value ∞ at $t = \tau$ and 0 everywhere else. Continuous-time white process noise is not natural since it has infinite power. Hence, the assumption that for small Δt we have a Gaussian noise distribution with noise intensity σ_x allows us to express (39b) and consider the limit $\Delta \to 0$.

APPENDIX III
DETERMINISTIC CONTRACTION

Proof. *From the contraction metric Definition 23, expanding the left-hand side of relation (24) results in*[5]

$$\tilde{A}^T M + \dot{M} + M\tilde{A} = \tilde{A}^T P_A + \dot{P}_A + P_A\tilde{A} \tag{42a}$$

$$+ \tilde{A}^T(P - \hat{X}\mathcal{W}\hat{X}^T) + (P - \hat{X}\mathcal{W}\hat{X}^T)\tilde{A} \tag{42b}$$

$$+ Q_c + \mathcal{K} R_c \mathcal{K}^T - \mathcal{K} Z\mathcal{W}\hat{X}^T - \hat{X}\mathcal{W} Z^T\mathcal{K}^T \tag{42c}$$

where we have used (10a) to remove some of the $\mathcal{K} R_c \mathcal{K}^T$ *terms after simplification. From Theorem 3.1 we can replace the right-hand side of relation (42a) with* $-Q_A$. *From Definition 5, expanding* Q_A *for its respective terms in (42), and after some simplification, results in*

$$\tilde{A}^T M + \dot{M} + M\tilde{A} = \tag{43a}$$

$$- \tilde{A}^T\tilde{A} - (P - \hat{X}\mathcal{W}\hat{X}^T)(P - \hat{X}\mathcal{W}\hat{X}^T)^T \tag{43b}$$

$$- \hat{X}\mathcal{W} Z^T Z\mathcal{W}\hat{X}^T - \left|\lambda\right|\mathcal{K}\mathcal{K}^T - \left|\lambda\right| I - \sigma I \tag{43c}$$

$$\leqslant -\sigma I < 0 \tag{43d}$$

It is subsequently clear that there exists a constant $\beta_M > 0$ *such that* $\sigma I > \beta_M M$. ■

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[3] Given $\hat{X}(t) \in \mathbb{R}^{n\times 2n+1}$, and $f : \mathbb{R} \mapsto \mathbb{R}^n$, short-hand notation used in (31) implies the i-th column $\hat{X}_i(t+\Delta t)$ of the matrix $\hat{X}(t+\Delta t) \in \mathbb{R}^{n\times 2n+1}$ is formed as $\hat{X}_i\left(t+\Delta t\right) = f(\hat{X}_i\left(t\right), t)$.

[4] Given $\hat{X}^-(t+\Delta t) \in \mathbb{R}^{n\times 2n+1}$, and $h : \mathbb{R} \mapsto \mathbb{R}^m$, short-hand notation used in (34) implies the i-th column $\mathcal{Z}_i(t+\Delta t)$ of the matrix $\mathcal{Z}(t+\Delta t) \in \mathbb{R}^{m\times 2n+1}$ is formed as $\mathcal{Z}_i(t+\Delta t) = h(\hat{X}_i^-\left(t+\Delta t\right), t+\Delta t)$.

[5] We omit the time-varying variable t in this derivation to present of compact proof.