

# Average consensus in symmetric nonlinearly coupled delayed networks

Anton V. Proskurnikov

**Abstract**—The paper addresses consensus under nonlinear couplings and bounded delays for multi-agent systems, where the agents have the single-integrator dynamics. The network topology is undirected and may alter as time progresses. The couplings are uncertain and satisfy a conventional sector condition with known sector slopes. The delays are uncertain, time-varying and obey known upper bounds. The network satisfies a symmetry condition that resembles the Newton’s Third Law. Explicit analytical conditions for the robust consensus are offered that employ only the known upper bounds for the delays and the sector slopes.

## I. INTRODUCTION.

Different types of regular coordinated behavior in multi-agent systems, provided by local interactions between the agents have attracted enormous attention of the research community recent decades. One of the simplest yet important samples of such a behavior is *consensus* or synchronization in the network, which underlies many natural phenomena (such as flocking and swarming) and is a cornerstone of many engineering designs (e.g. control of multi-vehicle formations and smart power grids).

Despite the overall progress (the reader is referred to [8], [17], [18] for an excellent survey), some issues of the consensus theory still remain unexplored even for the agents with the simplest first-order dynamics. Among them the problem of robustness against uncertainties in interactions between the agents, including imprecise measurements and time delays. Delays are inescapable in real-world applications and may deteriorate the system performance. The effects of delay on consensus dynamics has been investigated for a few special situations only [12].

A deeply investigated situation is where each agent has access to its current output, and delays affect only data communicated from neighbors [1], [11], [14]. In this case consensus is robust against arbitrarily large time-varying delays provided that they remain bounded. This can be proved by using the techniques elaborated for non-delayed systems and based on the fact that the agents’ convex hull shrinks as time progresses [1], [7], which hull should be replaced in the delayed case by the convex hull of all states observed during a sufficiently large time interval. However the contracting property does not remain valid dealing self-delays, which may arise, for instance, from delayed self-actuation [21] or delayed effect of the neighbors that is generated from relative measurements. In general, large enough delays of such type may cause instability; which makes it a real concern to find the delay margin below which the consensus is maintained. For linear time-invariant networks tight margins are usually established using Nyquist-type criteria [5], [12], [22] and other frequency-domain techniques

[2], [13], [21]. For switching networks only lower bound for delay margins (giving thus sufficient but not necessary conditions for consensus) are known and main progress is due to Lyapunov-Krasovskii approach which leads to high-dimensional LMIs and imposes some restrictions on delays (e.g. require them to be smooth with derivative not exceeding 1). Another serious restriction is the linearity of couplings between the agents. However nonlinear couplings are intrinsically inherent in many applications [17]. For example, synchronization between oscillators is often arranged via their periodic coupling [17], [20]; range- and rate-restricted communication is another typical source of nonlinearities in couplings [6], [7].

Unlike the previous research, this paper deals with nonlinear couplings, switched topology, and time-varying delays, including self-delays, assumptions about which are inspired by the situation where the self-delay is incorporated into the delayed coupling and based on relative measurements response from neighbors. We limit ourselves to networks of first-order agents with undirected topologies and symmetric delayed time-varying couplings. Both delays and couplings may be uncertain; the known data comes to the delays upper bounds and a sector containing the graphs of the couplings. The main results of the paper are explicit analytic criteria for not only consensus achievement but also its robustness against this uncertainty. These criteria concern both continuous and discrete time systems.

Some results of this note were partly reported in [16] for the particular case of constant delays and under much stronger assumptions about the interaction graph.

## II. PRELIMINARIES AND THE PROBLEM SETUP.

We first recall some concepts from the graph theory. A *graph* is a pair  $G = (V, E)$  constituted by the finite set of *nodes*  $V$  and the set of *arcs*  $E \subset V \times V$ . The graph is said to be *undirected* if  $(v, w) \in E \Leftrightarrow (w, v) \in E \forall v, w \in V$ ; a sequence of its nodes  $v_1, v_2, \dots, v_k$  with  $(v_i, v_{i+1}) \in E \forall i$  is called the *path* between  $v_1$  and  $v_k$ ; the graph is said to be *connected* if a path exists between any two nodes.

We consider the networked systems governed by the equations of the form:

$$\dot{x}_j(t) = \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(t, x_k(t - \tau_{jk}(t)) - x_j(t - \tau_{jk}(t))). \quad (1)$$

Here  $t \geq 0$ ,  $j = 1, \dots, N$ ,  $x_j(t) \in \mathbb{R}^n$  stands for the state of the  $j$ -th node, the maps  $\varphi_{jk}(t, x)$  are called *couplings*,  $\tau_{jk}(t) \geq 0$  are time-varying delays, and  $a_{jk}(t) \geq 0$  are

weighting coefficients. They define the time-varying interaction graph: the  $k$ -th node influences the  $j$ -th one at time  $t$  if and only if  $a_{jk}(t) > 0$ . When dealing with discontinuous at  $t = 0$  solutions of (1), we assume them right-continuous at  $t = 0$  for the definiteness. We assume also that the initial functions  $x_j(t)$ ,  $t < 0$ , are bounded.

The objective is to disclose conditions under which the system comes to consensus in the following sense.

*Definition 1:* The networked system (1) reaches the consensus if  $|x_k(t) - x_j(t)| \xrightarrow{t \rightarrow \infty} 0$  for any initial data and  $j, k$ , and the average consensus if for any initial data,

$$x_j(t) \xrightarrow{t \rightarrow \infty} \frac{1}{N} (x_1(0) + x_2(0) + \dots + x_N(0)) \quad \forall j. \quad (2)$$

In general, consensus does not imply the average consensus or even existence of the limits  $\lim_{t \rightarrow +\infty} x_j(t)$ .

### III. MAIN ASSUMPTIONS.

Our starting assumption concerns the network topology.

*Assumption 1:* The matrix  $A(t) = (a_{jk}(t))$  is symmetric and Lebesgue measurable. There exist  $\varepsilon > 0$  and  $T > 0$  such that the graph  $(V_N, \mathcal{E}_t)$  with the set of nodes  $V_N = \{1, \dots, N\}$  and that of arcs  $\mathcal{E}_t = \{(j, k) : \int_t^{t+T} a_{jk}(s) ds > \varepsilon\}$  is connected for all  $t \geq 0$ .

The first claim implies that the interaction graph is undirected. The second property is often referred to as the *uniform connectivity* of the network and is commonly adopted in the literature along with its analogs for directed graphs [7], [11], [14]. It prohibits disintegration of the network into separated clusters and is acknowledged as nearly necessary for consensus. For the constant weight matrix  $A(t) \equiv A = A^T$ , Assumption 1 means that the graph  $G = (V_N, E)$ ,  $E := \{(j, k) : a_{jk} > 0\}$  is connected.

The following *symmetry assumption* imposes a relationship that is similar in spirit to the Newton's Third Law.

*Assumption 2:* For any  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $j, k \in V_N$ ,  $j \neq k$ , one has  $\varphi_{jk}(t, x) = -\varphi_{kj}(t, -x)$  and  $\tau_{jk}(t) = \tau_{kj}(t)$ .

Under this assumption, consensus implies the average consensus (2). Indeed, summing up (1) over  $j = 1, \dots, N$  yields  $\dot{s} \equiv 0$  for  $s(t) := N^{-1} \sum_j x_j(t)$ . So  $s(t) \equiv s(0)$  and  $x_k(t) - x_j(t) \rightarrow 0 \forall j, k \implies x_j(t) \rightarrow s(0) \forall j$ .

We are concerned with the situation where the couplings  $\varphi_{jk}$  are not completely known. The available knowledge about them basically comes to a certain *quadratic inequality*. Specifically for a known constant  $\gamma > 0$ , they belong  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  to the set  $\mathfrak{S}(\gamma)$  of continuous maps  $\varphi : [0; +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\varphi(t, 0) \equiv 0$  and

$$\gamma x^T \varphi(t, x) \geq |\varphi(t, x)|^2 \quad \forall t \geq 0, x \neq 0; \quad (3)$$

$$\inf_{t \geq 0, |x| > \delta} |\varphi(t, x)| > 0 \quad \forall \delta > 0. \quad (4)$$

In the scalar case ( $n = 1$ ), (3) and (4) express the conventional sector condition: the graph of the function  $\varphi(t, \cdot)$  lies between the lines  $y = \gamma x$  and  $y = 0$  (intersecting the latter line at the origin only). So for  $n \geq 2$ , the inclusion  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  may be viewed as a multi-variable analog of the sector condition.

The coupling weights  $a_{jk}$  and delays  $\tau_{jk}$  are Lebesgue measurable and uncertain, with the known bounds

$$a_{jk}(t) \leq \bar{a}_{jk}, \tau_{jk} \leq \bar{\tau}_{jk}, d_j(t) \leq \bar{d}_j \quad \forall t \geq 0, \quad (5)$$

where  $d_j(t) := \sum_{k=1}^N a_{jk}(t)$ . Consensus criterion should be given in terms of these bounds  $\bar{a}_{jk}$ ,  $\bar{\tau}_{jk}$ ,  $\bar{d}_j \leq \sum_k \bar{a}_{jk}$  and the "sector slope"  $\gamma$ , but not the couplings, weights, and delays themselves. Such a criterion in fact ensures the *robust consensus* in the sense that consensus holds for all uncertainties satisfying the above requirements.

### IV. MAIN RESULTS.

*Theorem 1:* Let (5) and Assumptions 1, 2 hold and  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  for some  $\gamma > 0$ . The network (1) reaches the average consensus whenever

$$\frac{1}{2\gamma} - \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk} \bar{\tau}_{jk}^2 \right)^{1/2} > 0 \quad \forall j. \quad (6)$$

The proof, rather technical, may be found in [15]

Inequality (6) is evidently implied by a coarser condition

$$\bar{\tau}_{jk} \leq \left( 2\gamma \sum_{m=1}^N \bar{a}_{jm} \right)^{-1} \quad \forall j, k, \quad (7)$$

which does not employ  $\bar{d}_j$  and is in touch with the previous literature [2], [21], [22]. Unlike this paper, it deals with completely known and trivial couplings  $\varphi_{jk}(x) = x$  (which implies  $\gamma = 1$  in our setting) and fixed topology  $a_{jk}(t) \equiv \bar{a}_{jk}$ . For constant delays of special types, Remark 4 in [21, Sect. IV] claims that (7) implies consensus even if the interaction graph is directed with oriented spanning tree, whereas Assumption 2 does not necessarily hold. Under Assumption 2, tight estimates of the maximal tolerable delay level were obtained in [2] in the cases of both time-variant and constant delays. In the latter case, the estimate takes the form  $\bar{\tau}_{jk} \leq \pi / (2\lambda_{max}(L)) \quad \forall j, k$ , where  $\lambda_{max}(L)$  is the maximal eigenvalue of the Laplacian matrix  $L$  of the weighted graph given by  $(a_{jk})$  which is defined as

$$L := \begin{bmatrix} \sum_{j=1}^N a_{1j} & -a_{12} & \dots & -a_{1N} \\ -a_{21} & \sum_{j=1}^N a_{2j} & \dots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \dots & \sum_{j=1}^N a_{Nj} \end{bmatrix}.$$

The right-hand side of this inequality exceeds that of (7) since  $\pi/2 > 1$  and  $\lambda_{max} < 2 \sum_{k=1}^N a_{jk}$  by the Gershgorin theorem [8]. So in the case of fixed interaction topology and trivial couplings, (7) is only sufficient but not necessary for the consensus. In [12], [22], the assumptions of [2] about the delays were relaxed (e.g., the self-delay was allowed to differ from the delay in the data from the neighbors). It should be also emphasized that the techniques from [2], [21], [22] are not applicable to switching networks and nonlinear couplings, which are among the main concerns of this paper.

Theorem 1 can be extended on discrete-time networks:

$$x_j(t+1) = x_j(t) + \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(x_k(t - \tau_{jk}(t)) - x_j(t - \tau_{jk}(t))), \quad (8)$$

where  $t \geq 0$  and  $\tau_{jk}(t) \geq 0$  take only integer values.

**Theorem 2:** Let the assumptions of Theorem 1 hold and

$$\frac{1}{2\gamma} - \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk}(1 + \bar{\tau}_{jk})^2 \right)^{1/2} > 0 \quad \forall j. \quad (9)$$

Then the network (8) reaches the average consensus.

**Proof.** Following the approach from [10], a continuous-time system is introduced so that sample sequences of its solutions are identical to solutions of the difference equation (8). Specifically, this is the system (1), where the weights, couplings, and delays are obtained from those in (8) via their extension from the integer grid on all  $t \geq 0$ :  $a_{jk}(t) := a_{jk}(\lfloor t \rfloor)$ ,  $\varphi_{jk}(t, \cdot) := \varphi_{jk}(\lfloor t \rfloor, \cdot)$ ,  $\tau_{jk}(t) := t - \lfloor t \rfloor + \tau_{jk}(\lfloor t \rfloor)$ , where  $\lfloor t \rfloor$  is the integer floor of  $t$ . It is easy to see that the solutions  $x_j^+(\cdot)$  and  $x_j(\cdot)$  of (1) and (8), respectively, are related by

$$x_j^+(t) = x_j(\lfloor t \rfloor) + (t - \lfloor t \rfloor) \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(\lfloor t \rfloor, y_{jk}(\lfloor t \rfloor)),$$

where  $y_{jk}(t) := x_k(t - \tau_{jk}(t)) - x_j(t - \tau_{jk}(t))$ ,  $t = 0, 1, \dots$ . It remains to note that  $\sup_{t \in [0, \infty)} \tau_{jk}(t) \leq 1 + \sup_{t=0,1,2,\dots} \tau_{jk}(t) \leq 1 + \bar{\tau}_{jk}$  and apply Theorem 1.  $\square$

## V. EXTENSIONS: DISTRIBUTED DELAYS AND TRAFFIC FLOW MODELS.

In this section we discuss several extensions of Theorem 1. The first of them addresses the case when delays are not discrete but distributed. Let  $(\mu_{jk}^t)$  (where  $j, k = 1, \dots, N$ ,  $j \neq k$  and  $t \geq 0$ ) be a family of probability measures defined on Borel subsets of  $\mathbb{R}_+ := [0; +\infty]$ . For a function  $y : \mathbb{R} \rightarrow \mathbb{R}^n$  which is bounded on any interval  $(-\infty; T]$ ,  $T < \infty$ , we define the function  $T_{jk}y : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  by:

$$(M_{jk}y)(t) := \int_0^\infty y(t-s) \mu_{jk}^t(ds) \quad (10)$$

A natural extension of the model (1) is the following:

$$\dot{x}_j(t) = \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(t, M_{jk}y_{jk}(t)), \quad y_{jk} := x_k - x_j. \quad (11)$$

Introducing Dirac  $\delta$ -measure by  $\delta(\{0\}) := 1$  and  $\delta((0; \infty)) := 0$ , one easily finds that a model with discrete delays (1) is nothing but a (11) with  $\mu_{jk}^t(\cdot) = \delta(\cdot - \tau_{jk}(t))$ .

Now we are in position to give an extension of Theorem 1. It appears that a bound  $\bar{\tau}_{jk}$  in (6) may be replaced with a maximal standard deviation of the measures  $\mu_{jk}^t$ .

**Theorem 3:** Let Assumptions 1, 2 hold and  $\varphi_{jk} \in \mathfrak{S}(\gamma)$ . Suppose that  $a_{jk}(t) \leq \bar{a}_{jk}$ ,  $d_j(t) \leq \bar{d}_j$  and  $\int_0^\infty s^2 \mu_{jk}^t(ds) \leq \bar{\rho}_{jk}^2$ . The network (11) reaches the average consensus if

$$\frac{1}{2\gamma} - \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk} \bar{\rho}_{jk}^2 \right)^{1/2} > 0 \quad \forall j. \quad (12)$$

The distributed delays naturally arise in many physical and technical applications. For instance, such a delay model appears to be essential in *microscopic traffic flow models* [9], [19], reflecting some properties of the human memory.

Despite more than 60 years of intensive research, study of the vehicular traffic dynamics still represents a real challenge for the research community and is vital in the face of considerable economic and ecological losses due to traffic congestions. Microscopic traffic flow models are closely related to models of self-propelled particle ensembles [4] and commonly adopted as simple but instructive tools for traffic analysis. Since the pioneering work [3], the *delay* in drivers reaction has been recognized as a factor essentially affecting the overall flow dynamics, see e.g. [9], [19] and references therein for a historical survey.

A simple model of such kind [3], [4], [19] deals with  $N$  vehicles, indexed 1 through  $N$ , following along a common circular single lane road. Each vehicle tries to equalize its velocity with that of its predecessor:

$$\dot{v}_j(t) = K(v_{j \oplus 1}(t - \tau) - v_j(t - \tau)).$$

Here  $v_j(t)$  is the velocity of the  $j$ -th vehicle,  $\tau$  is the delay in the driver's action,  $\oplus$  is the summation modulus  $N$ , and  $K$  stands for the driver's "sensitivity" to alterations of the relative velocity of the vehicle in front of him.

A key issue addressed via this model is that of stability of the constant-velocity equilibria  $v_1 \equiv \dots \equiv v_N \equiv \text{const}$ . The respective results of [9] deal with fixed observation topology and assume that a driver may watch not only its predecessor but also other vehicles and the distributed delays in driver reactions are equal and do not depend on time. Now we extend the results of [9], [19] on the more realistic case where the delays in driver reactions depend on time, and the observation topology alters over time (a driver loses or acquires sight of the companions depending on the relief and weather conditions). However, unlike the mentioned papers we assume the network to be symmetrical.

Specifically, we assume that the driver of the  $j$ -th vehicle adjusts the velocity  $v_j$  based on the relative velocities of  $p \leq N - 1$  preceding and  $p$  following vehicles:

$$\dot{v}_j(t) = K \sum_{m=-p}^p a^m(t) \int_0^\infty (v_{j \oplus m}(t-s) - v_j(t-s)) \mu^m(ds). \quad (13)$$

Here  $a^0(t) := 0$  and  $\mu^m$  are delay distributions. This model takes into account that the reaction on the nearest neighbors may be faster then on distant ones ( $\mu^m$  depends on  $m$ ). This "order-based" determinism leads to the assumption that the response to the  $m$ -th predecessor and the  $m$ -th follower are equally sharp and fast:  $a^m = a^{-m}$  and  $\mu^m = \mu^{-m} \forall m$ .

Applying Theorem 3 gives rise to the following.

**Theorem 4:** Let  $a^m(t) = a^{-m}(t) \geq 0$  and  $\mu^m(t) = \mu^{-m}(t) \geq 0$ . Suppose that  $T, \varepsilon > 0$  exist such that  $\int_t^{t+T} a^1(t) dt > \varepsilon$  for any  $t \geq 0$ . Let  $\bar{a}^m := \sup_{t \geq 0} a^m(t)$ ,  $\bar{\rho}^m := \sup_{t \geq 0} (\int_0^\infty s^2 \mu^m(ds))^{1/2}$  satisfy the inequality

$$\max_{m=-p, \dots, p} \bar{\rho}^m \leq \left( 2 \sum_{m=-p}^p \bar{a}^m \right)^{-1}.$$

Then the system (13) achieves the average consensus, i.e.,  $v_j(t) \rightarrow N^{-1}(v_1(0) + \dots + v_N(0))$  as  $t \rightarrow \infty$  for all  $j$ .

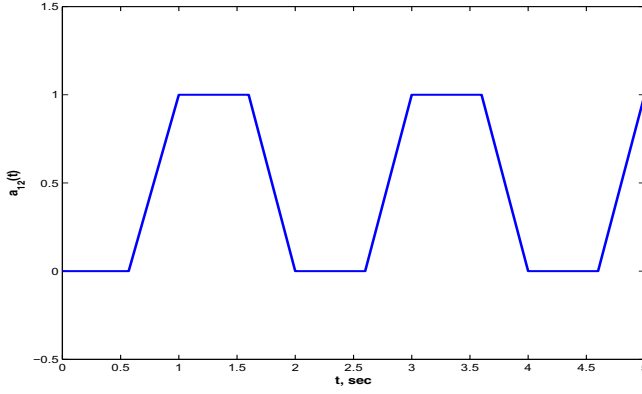


Fig. 1. Weight function  $a_{12}$ .

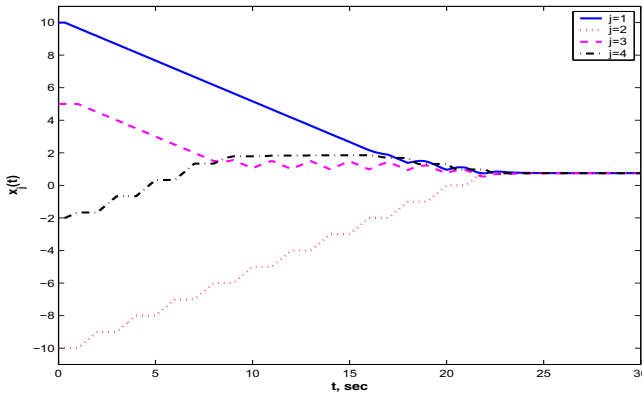


Fig. 2. Continuous-time dynamics with saturation couplings.

By Theorem 4, the vehicles travel with asymptotically equal velocities and traffic jams are impossible provided that the drivers react with small enough delays.

## VI. SIMULATIONS.

To illustrate Theorems 1 and 2, two simulation tests were performed for  $N = 4$  scalar ( $n = 1$ ) agents. The first test concerns the continuous-time network (1) with saturated couplings:  $\varphi_{jk}(x) = 0.5(|x + 0.5| - |x - 0.5|)$  ( $\gamma = 1$ ) and periodic weights  $a_{12}(t) = a_{21}(t)$  (shown on Fig.1),  $a_{23}(t) = a_{32}(t) := a_{12}(t + 1)$ ,  $a_{34}(t) = a_{43}(t) := a_{12}(t + 0.5)$ ,  $a_{14}(t) = a_{41}(t) := a_{12}(t + 1.5)$  (thus  $\bar{a}_{12} = \bar{a}_{23} = \bar{a}_{34} = \bar{a}_{14} = 1$  and  $d_j = 2\forall j$ ), and the other weights  $a_{jk}$  being zero. The delays are as follows  $\tau_{12} = \tau_{21} = \tau_{34} = \tau_{43} = 0.1$  and  $\tau_{23} = \tau_{32} = \tau_{14} = \tau_{41} = 0.339$ , thus (6) holds. Figure 2 displays the result of the simulation with the initial data  $x(0) = [10, -10, 5, 2]^T$  and  $x(t) \equiv 0$  for  $t < 0$ .

The second test illustrates Theorem 2 and deals with the network (8) with  $\varphi_{jk}(t, x) = x(1.01 + \cos x)$  ( $\gamma = 2.01$ ),  $\tau_{jk} = 2$ ,  $a_{12} = a_{23} = a_{34} = a_{41} = a_{21} = a_{32} = a_{43} = a_{14} = K$  (thus  $d_j = 2K\forall j$ ), with  $K = 0.041$  and the other  $a_{jk}$  being zero. Figure 3 shows the corresponding simulation result. All simulation tests have confirmed that the consensus is achieved.

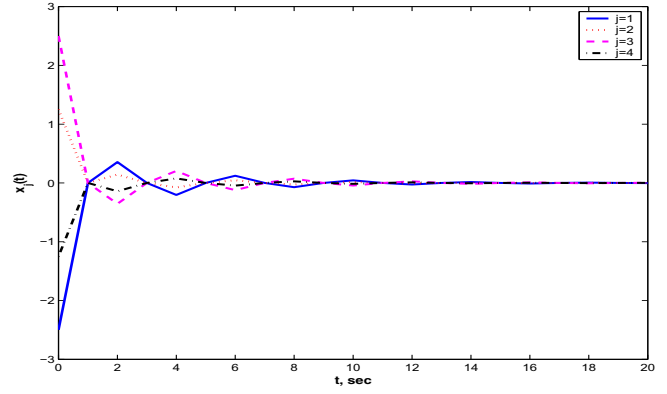


Fig. 3. Discrete-time dynamics,  $\varphi_{jk}(x) = x(1.01 + \cos x)$ .

## VII. CONCLUSION

The state consensus among first-order agents was addressed for network with switching undirected topology and nonlinear uncertain couplings. The network satisfies the uniform connectivity assumption, a symmetry condition similar in flavor to the Newton Third Law, and a sector condition with known slopes. Interaction between the agents is corrupted by uncertain time-varying delays with known upper bounds. A new criterion for robust consensus is obtained in terms of these bounds and the sector slope, which is confirmed by simulations. Its extensions on the leader-following formation control, reference-tracking consensus, and agents with more general dynamics is subjects of ongoing research.

## VIII. ACKNOWLEDGEMENTS.

The paper is partially supported by Russian Fund for Basic Research, grants 11-08-01218, 12-01-00808, 13-08-01014.

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