

Stability and Persistence Analysis of Large Scale Interconnected Positive Systems

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Abstract—This paper is concerned with the analysis of large-scale interconnected systems constructed from positive subsystems and a nonnegative interconnection matrix. We first show that the interconnected system is admissible and stable if and only if a Metzler matrix built from the coefficient matrices of the positive subsystems and the interconnection matrix is Hurwitz stable. By means of this key lemma, we further provide several results that characterize the admissibility and stability of interconnected systems in terms of the weighted L_1 -induced norm of each positive subsystem and the Frobenius eigenvalue of the interconnection matrix. Moreover, in the case where every subsystem is SISO, we provide explicit conditions under which the interconnected system has the property of persistence, i.e., the state of the interconnected system converges to a unique strictly positive vector (up to a strictly positive constant multiplicative factor) irrespective of nonnegative and nonzero initial states. We illustrate the effectiveness of the persistence results via formation control of multi-agent systems.

Keywords: positive system, interconnection, stability, persistence, multi-agent system, formation control.

I. INTRODUCTION

Recently, systems of interest in the field of engineering, biology, economics, etc., have become more complex and larger-scaled, and as such intensive research effort has been made for developing dedicated analysis and synthesis tools. The issue is how to derive sharpened analysis and synthesis conditions exploiting the properties of subsystems and interconnection structure [8], [12], [6]. In this paper, we are particularly interested in the case where the subsystems are positive. A linear dynamical system is said to be positive if its state and output are nonnegative for any nonnegative initial state and nonnegative input [5], [10]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel connections are all positive and this fact also motivates us to focus on the interconnected positive systems. Nowadays the study on linear positive system is active and remarkable results have been obtained along with convex optimization theory [15], [14], [16], [1].

For the analysis and synthesis of positive systems, we have recently proposed a novel technique using weighted L_1 -induced norm characterization [2], [4], [3]. The goal of this paper is to further extend these results to the stability and

persistence analysis of large-scale interconnected systems constructed from positive and stable subsystems and a nonnegative interconnection matrix. To this end, we first show that the interconnected system is admissible [2] and stable if and only if a Metzler matrix built from the coefficient matrices of the positive subsystems and the interconnection matrix is Hurwitz stable. By means of this key lemma, we clarify that the interconnected system is admissible and stable if and only if there exists a set of weighting vectors that renders the weighted L_1 -induced norm of each positive subsystem less than unity. This is a crucial extension of [2] to the case where each subsystem has nonzero direct feed-through term. Moreover, in the case where every subsystem is SISO, the condition is drastically simplified and given in terms of the unweighted L_1 -induced norm (this is nothing but the steady-state gain) of each positive subsystem and the Frobenius eigenvalue [9] of the interconnection matrix.

On the other hand, in the case where every subsystem is positive, stable, and SISO, we provide explicit conditions under which the interconnected system has the property of persistence, i.e., the state of the interconnected system remains to be nonnegative and converges to a unique strictly positive vector (up to a strictly positive constant multiplicative factor) irrespective of nonnegative and nonzero initial states. We prove that the persistence is achieved if every subsystem shares identical unweighted L_1 -induced norm γ and the interconnection matrix has the Frobenius eigenvalue equal to the reciprocal of γ . We illustrate the usefulness of this result via a sort of formation control problem of multi-agent systems [7], [12], [17], [18]. The goal is to design a communication scheme over the agents with respect to each agent's position so that prescribed formation can be achieved. We show that such communication scheme synthesis is possible even if the agents have different dynamics as long as they are positive, stable and share identical unweighted L_1 -induced norm.

We use the following notations. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i, j) , where A_{ij} (B_{ij}) stands for the (i, j) -entry of A (B). In relation to this notation, we also define $\mathbb{R}_{++}^n := \{x \in \mathbb{R}^n : x > 0\}$ and $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq 0\}$. We also define $\mathbb{R}_{++}^{n \times m}$ and $\mathbb{R}_+^{n \times m}$ with obvious modifications. In addition, we denote by \mathbb{D}_{++}^n the set of diagonal and strictly positive matrices of the size n . For $A \in \mathbb{R}^{n \times n}$, we denote by $\sigma(A)$ and $\rho(A)$ the set of the eigenvalues of A and the spectral radius of A , respectively. For $A \in \mathbb{R}_+^{n \times n}$, Theorem 8.3.1 in [9] states

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that there is an eigenvalue equal to $\rho(A)$. This eigenvalue is related to the Frobenius Theorem and denoted by $\lambda_F(A)$. Finally, for given $v \in \mathbb{R}^N$ and $x_i \in \mathbb{R}^{n_{x_i}}$ ($i = 1, \dots, N$), we define

$$v \otimes_{i=1}^N x_i = [v_1 x_1^T \cdots v_N x_N^T]^T \in \mathbb{R}^{n_x}, \quad n_x := \sum_{i=1}^N n_{x_i}.$$

We note that this operation can be seen as a special case of the Khatri-Rao product defined in [11].

II. PRELIMINARIES

In this section, we gather basic definitions and fundamental results for positive system analysis.

Definition 1 (Metzler Martrix): [5] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ ($i \neq j$).

In the sequel, we denote by \mathbb{M}^n (\mathbb{H}^n) the set of the Metzler (Hurwitz stable) matrices of the size n . Under these notations, the next lemmas hold.

Lemma 1: [5], [10] For given $A \in \mathbb{M}^n$, the following conditions are equivalent.

- (i) The matrix A is Hurwitz stable, i.e., $A \in \mathbb{H}^n$.
- (ii) The matrix A is nonsingular and $A^{-1} \leq 0$.
- (iii) There exists $h \in \mathbb{R}_{++}^n$ such that $h^T A < 0$.

Lemma 2: [2] For given $P \in \mathbb{M}^{n_1}$, $Q \in \mathbb{R}_{++}^{n_1 \times n_2}$, $R \in \mathbb{R}_{++}^{n_2 \times n_1}$, and $S \in \mathbb{M}^{n_2}$, the following conditions are equivalent.

- (i) $\Pi := \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathbb{H}^{n_1+n_2}$.
- (ii) $P \in \mathbb{H}^{n_1}$, $S - RP^{-1}Q \in \mathbb{H}^{n_2}$.
- (iii) $S \in \mathbb{H}^{n_2}$, $P - QS^{-1}R \in \mathbb{H}^{n_1}$.

To move on to the definition of positive systems, let us consider the linear system described by

$$G: \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases} \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$, and $D \in \mathbb{R}^{n_z \times n_w}$. The definition and a basic result of positive systems are given in the following.

Definition 2 (Positive Linear System): [5] The linear system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Theorem 1: [5] The system (1) is positive if and only if $A \in \mathbb{M}^n$, $B \in \mathbb{R}_{++}^{n \times n_w}$, $C \in \mathbb{R}_{++}^{n_z \times n}$, and $D \in \mathbb{R}_{++}^{n_z \times n_w}$.

For a given linear positive system, its L_1 -induced norm with weighting vectors are defined as follows.

Definition 3: [2] Suppose G given by (1) is positive. Then, its L_1 -induced norm with given weighting vectors $q_z \in \mathbb{R}_{++}^{n_z}$ and $q_w \in \mathbb{R}_{++}^{n_w}$ is defined by

$$\|G_{q_z, q_w}\|_{1+} := \sup_{\|q_w^T w\|_1=1, w \in L_1} \|q_z^T z\|_1 \quad (2)$$

where

$$\|s\|_1 := \int_0^\infty |s(t)| dt,$$

$$L_{1+} := \{s(t) : \|s_i\|_1 < \infty, s_i(t) \geq 0 \forall t \in [0, \infty)\}.$$

It has been shown in [2] that the L_1 -induced norm with weighting vectors is useful to characterize the stability of interconnected positive systems. The goal of this paper is to further extend this result. In particular, we derive several interesting results in the case where every subsystem under interconnection is SISO. If G given by (1) is stable and SISO, it is shown in [2] that $\|G_{1,1}\|_{1+} = G(0) = -CA^{-1}B + D$. Namely, the unweighted L_1 -induced norm coincides with the steady-state gain.

III. STABILITY ANALYSIS OF INTERCONNECTED POSITIVE SYSTEMS

Let us consider the positive subsystem G_i ($i = 1, \dots, N$) represented by

$$G_i: \begin{cases} \dot{x}_i = A_i x_i + B_i w_i, \\ z_i = C_i x_i + D_i w_i, \end{cases} \quad (3)$$

$$A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\}, B_i \in \mathbb{R}_{++}^{n_i \times n_{w_i}},$$

$$C_i \in \mathbb{R}_{++}^{n_{z_i} \times n_i}, D_i \in \mathbb{R}_{++}^{n_{z_i} \times n_{w_i}}.$$

As clearly shown in (3), we have assumed that G_i ($i = 1, \dots, N$) are all stable.

With these positive subsystems, let us define a positive and stable system \mathcal{G} by

$$\mathcal{G} := \text{diag}(G_1, \dots, G_N). \quad (4)$$

The state space realization of \mathcal{G} is given by

$$\mathcal{G}: \begin{cases} \dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}\hat{w}, \\ \hat{z} = \mathcal{C}\hat{x} + \mathcal{D}\hat{w} \end{cases} \quad (5)$$

where

$$\mathcal{A} := \text{diag}(A_1, \dots, A_N), \mathcal{B} := \text{diag}(B_1, \dots, B_N),$$

$$\mathcal{C} := \text{diag}(C_1, \dots, C_N), \mathcal{D} := \text{diag}(D_1, \dots, D_N), \quad (6)$$

$$\hat{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{x}}}, \quad n_{\hat{x}} := \sum_{i=1}^N n_i,$$

$$\hat{w} := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{w}}}, \quad n_{\hat{w}} := \sum_{i=1}^N n_{w_i}, \quad (7)$$

$$\hat{z} := \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{z}}}, \quad n_{\hat{z}} := \sum_{i=1}^N n_{z_i}.$$

For a given interconnection matrix $\Omega \in \mathbb{R}_{++}^{n_{\hat{w}} \times n_{\hat{z}}}$, we are interested in the stability of the interconnected system $\mathcal{G} \star \Omega$ defined by (5) and $\hat{w} = \Omega \hat{z}$. In relation to the well-posedness of this interconnection, we make the next definition.

Definition 4: The interconnected system $\mathcal{G} \star \Omega$ is said to be *admissible* if the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz stable.

In the sequel, we require the admissibility of the interconnected system $\mathcal{G} \star \Omega$ whenever we analyze its stability. The meaning of this presupposition, and its rationality as well, can be explained as follows. If $\det(\mathcal{D}\Omega - I) \neq 0$, then the interconnection is well-posed, and the state-space description of the interconnected system is represented by

$$\begin{aligned} \hat{x} &= \mathcal{A}_{\text{cl}} \hat{x}, \\ \mathcal{A}_{\text{cl}} &:= \mathcal{A} + \mathcal{B}\Omega(I - \mathcal{D}\Omega)^{-1}\mathcal{C}. \end{aligned} \quad (8)$$

Thus, if the admissibility is ensured, we see that

- (i) the interconnection $\mathcal{G} \star \Omega$ is well-posed;
- (ii) the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz and hence $(I - \mathcal{D}\Omega)^{-1} \geq 0$ holds from Lemma 1. Therefore the matrix \mathcal{A}_{cl} is Metzler. It follows that the positive nature of the subsystems G_i ($i = 1, \dots, N$) is inherited to the interconnected system, i.e., the nonnegativity of the states x_i ($i = 1, \dots, N$) for any nonnegative initial states is still preserved under the interconnection.

Note that in the case $\mathcal{D} = 0$ the admissibility is trivially satisfied and hence this is out of the issue. In the case $\mathcal{D} \neq 0$ the admissibility is in general a sufficient condition for the positivity of $\mathcal{G} \star \Omega$.

For the admissibility and stability of the interconnected system $\mathcal{G} \star \Omega$, we can obtain the next lemma that plays an important role in this paper.

Lemma 3: The interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix

$$\Pi := \begin{bmatrix} \mathcal{A} & \mathcal{B}\Omega \\ \mathcal{C} & \mathcal{D}\Omega - I \end{bmatrix} \quad (9)$$

is Hurwitz stable.

Proof of Lemma 3: From Definition 4, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrices $\mathcal{D}\Omega - I$ and $\mathcal{A}_{\text{cl}} := \mathcal{A} + \mathcal{B}\Omega(I - \mathcal{D}\Omega)^{-1}\mathcal{C}$ are both Hurwitz stable. Thus the assertion readily follows from Lemma 2. ■

From this key lemma, we can obtain various conditions for the admissibility and stability of the interconnected system according to the properties of the subsystems G_i ($i = 1, \dots, N$) and the interconnection matrix Ω . Typical examples are given in the following Theorems.

Theorem 2: Let us consider the case where the i -th subsystem G_i is represented by (3) with the following specific structure:

$$G_i : \begin{cases} \dot{x}_i = A_i x_i + \sum_{k=1, k \neq i}^N B_{ik} w_{ik}, \\ z_{ji} = C_{ji} x_i + \sum_{k=1, k \neq i}^N D_{jik} w_{ik} \quad (j \neq i) \end{cases} \quad (10)$$

$$A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\}, B_{ik} \in \mathbb{R}_+^{n_i \times n_{w_{ik}}}, \\ C_{ji} \in \mathbb{R}_+^{n_{z_{ji}} \times n_i}, D_{jik} \in \mathbb{R}_+^{n_{z_{ji}} \times n_{w_{ik}}}.$$

We assume that the size of w_{ij} and z_{ij} are identical, and the N subsystems are interconnected by

$$w_{ij} = z_{ij} \quad (i, j = 1, \dots, N, i \neq j). \quad (11)$$

Then, the interconnected system is admissible and stable if and only if there exist weighting vectors $q_{ij} \in \mathbb{R}_+^{n_{w_{ij}}}$ ($i, j = 1, \dots, N, i \neq j$) such that

$$\|G_{i, q_{z, i}, q_{w, i}}\|_{1+} < 1, \\ q_{z, i} = \begin{bmatrix} q_{1, i} \\ \vdots \\ q_{i-1, i} \\ q_{i+1, i} \\ \vdots \\ q_{N, i} \end{bmatrix}, q_{w, i} = \begin{bmatrix} q_{i, 1} \\ \vdots \\ q_{i, i-1} \\ q_{i, i+1} \\ \vdots \\ q_{i, N} \end{bmatrix} \quad (i = 1, \dots, N). \quad (12)$$

The interconnection structure assumed in Theorem 2 is illustrated in Fig. 1 for the case $N = 3$. The implication of the theorem is that the interconnected system is admissible and stable if and only if there exists a set of weighting vectors that renders the L_1 -induced norm of each positive subsystem less than unity. Namely, the condition for the admissibility and stability is separated into the L_1 -induced norm conditions of each subsystem, where the L_1 -induced norm conditions are correlated with each other through the weighting vectors. This theorem is a crucial extension of the one in [2] where the validity is ensured only for the case $D_{jik} = 0$ ($i, j = 1, \dots, N, j \neq i, k \neq i$). The proof of this theorem is omitted due to limited space.

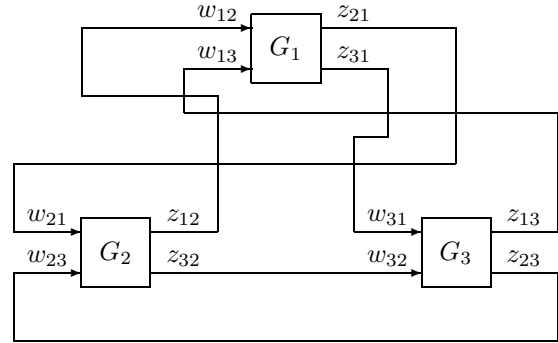


Fig. 1. Interconnected positive system ($N = 3$).

Theorem 3: Let us consider the case where the i -th subsystem G_i is represented by (3). We assume that every subsystem G_i ($i = 1, \dots, N$) is SISO. Then, for given $\Omega \in \mathbb{R}_+^{N \times N}$, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix $\Psi\Omega - I$ is Hurwitz stable where $\Psi \in \mathbb{R}_+^{N \times N}$ is constructed from the unweighted L_1 -induced norm of each subsystem as in $\Psi := \text{diag}(\|G_{1,1,1}\|_{1+}, \dots, \|G_{N,1,1}\|_{1+})$.

Proof of Theorem 3: From Lemma 3, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if the Metzler matrix Π defined by (9) is Hurwitz stable. From Lemma 2 and the fact that $\|G_{i,1,1}\|_{1+} = -C_i A_i^{-1} B_i + D_i$ ($i = 1, \dots, N$), this condition holds if and only if both of the Metzler matrices \mathcal{A} and $\mathcal{D}\Omega - I - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\Omega = \Psi\Omega - I$ are Hurwitz stable. Thus the assertion readily follows since \mathcal{A} is Hurwitz stable from the assumption $A_i \in \{\mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}\}$ ($i = 1, \dots, N$). ■

Theorem 4: Let us consider the case where the i -th subsystem G_i is represented by (3). We assume that every subsystem G_i ($i = 1, \dots, N$) is SISO and has identical unweighted L_1 -induced norm $\|G_{1,1,1}\|_{1+} = \dots = \|G_{N,1,1}\|_{1+} =: \gamma$. Then, for given $\Omega \in \mathbb{R}_+^{N \times N}$, the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if $\gamma \lambda_F(\Omega) < 1$.

Proof of Theorem 4: From Theorem 3, we see that the interconnected system $\mathcal{G} \star \Omega$ is admissible and stable if and only if $\gamma\Omega - I \in \mathbb{M}^N$ is Hurwitz stable. This condition can be restated equivalently as in $\gamma \lambda_F(\Omega) < 1$. This completes the proof. ■

These three theorems clearly show that the admissibility and stability of interconnected positive systems can be fully characterized in terms of the L_1 -induced norm of each subsystem evaluated with weighting vectors.

IV. PERSISTENCE ANALYSIS OF INTERCONNECTED POSITIVE SYSTEMS

In this section, we are interested in the *persistence* of the interconnected system $\mathcal{G} \star \Omega$. We first give the precise definition of what we call persistence.

Definition 5: For given positive subsystems G_i ($i = 1, \dots, N$) represented by (3) and a given interconnection matrix $\Omega \in \mathbb{R}_+^{n_{\hat{x}} \times n_{\hat{z}}}$, consider the interconnected system $\mathcal{G} \star \Omega$. Then, the interconnected system $\mathcal{G} \star \Omega$ is said to have the property of *persistence* if it is admissible and if there exist strictly positive vectors $\xi_0, \xi_\infty \in \mathbb{R}_{++}^{\hat{x}}$ such that

$$\lim_{t \rightarrow \infty} \hat{x}(t) = \xi_0^T \hat{x}(0) \xi_\infty.$$

This definition implies that, if $\mathcal{G} \star \Omega$ has the property of persistence, then for any $\hat{x}(0) \in \mathbb{R}_+^{\hat{x}} \setminus \{0\}$ the state \hat{x} converges to strictly positive vector $\xi_\infty \in \mathbb{R}_{++}^{\hat{x}}$ up to strictly positive multiplicative factor $\xi_0^T \hat{x}(0) \in \mathbb{R}_{++}$.

To state our main result on the persistence of $\mathcal{G} \star \Omega$, we first need to review the definition and related results on *irreducible* matrices.

Definition 6: [Reducible Matrix [9](p. 360)] A matrix $M \in \mathbb{R}^{n \times n}$ is said to be *reducible* if either

- (a) $n = 1$ and $M = 0$ or
- (b) $n \geq 2$ and there exist a permutation matrix $P \in \mathbb{R}^{n \times n}$ and r with $1 \leq r \leq n - 1$ such that
$$P^T M P = \begin{bmatrix} Q & R \\ 0_{n-r, r} & S \end{bmatrix}, \quad Q \in \mathbb{R}^{r \times r}, \quad S \in \mathbb{R}^{(n-r) \times (n-r)}.$$

Definition 7: [Irreducible Matrix [9](p. 361)] A matrix $M \in \mathbb{R}^{n \times n}$ is said to be *irreducible* if it is not reducible.

Definition 8: [Directed Graph of Matrices [9](p. 357)] The directed graph of $M \in \mathbb{R}^{n \times n}$, denoted by $\Gamma(M)$, is the directed graph on n nodes P_1, P_2, \dots, P_n such that there is a directed arc in $\Gamma(M)$ from P_i to P_j if and only if $M_{ij} \neq 0$ or equivalently, $\text{In}(M)_{ij} \neq 0$. Here, $\text{In}(M)$ stands for the indicator matrix of M defined by $\text{In}(M)_{i,j} = 1$ if $M_{i,j} \neq 0$ and $\text{In}(M)_{i,j} = 0$ if $M_{i,j} = 0^\dagger$.

Definition 9: [Strongly Connected Graph [9](p. 358)] A directed graph Γ is said to be *strongly connected* if between every pair of distinct nodes P_i, P_j in Γ there is a directed path of finite length that begins at P_i and ends at P_j .

Under these definitions, it is known that the next results hold.

Theorem 5: [9](p. 362) For given $M \in \mathbb{R}^{n \times n}$, the following conditions are equivalent.

- (a) M is irreducible.
- (b) $(I_n + \text{In}(M))^{n-1} > 0$.
- (c) $\Gamma(M)$ is strongly connected.

Theorem 6: [9](p. 508) Suppose $M \in \mathbb{R}_+^{n \times n}$ is irreducible. Then the following conditions hold.

- (i) $\rho(M) > 0$ and $\rho(M)$ is an eigenvalue of M .

[†]The indicator matrix $\text{In}(M)$ coincides with the adjacency matrix of the graph $\Gamma(M)$

- (ii) There is a vector $v \in \mathbb{R}_{++}^n$ such that $Mv = \rho(M)v$.
- (iii) $\rho(M)$ is an algebraically (and hence geometrically) simple eigenvalue of M .

Corollary 1: Suppose $M \in \mathbb{M}^n$ is irreducible. Then the following conditions hold.

- (i) M has an eigenvalue $\alpha \in \mathbb{R}$ such that $\alpha = \max_{\lambda \in \sigma(M)} \text{Re}(\lambda)$.
- (ii) There is a vector $v \in \mathbb{R}_{++}^n$ such that $Mv = \alpha v$.
- (iii) α is an algebraically (and hence geometrically) simple eigenvalue of M .

We are now ready to state our main result on the persistence of $\mathcal{G} \star \Omega$.

Theorem 7: Let us consider the case where every subsystem G_i represented by (3) is SISO. Suppose G_i ($i = 1, \dots, N$) and a given interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$ satisfy the following conditions.

- (i) The pair (A_i, B_i) is controllable ($i = 1, \dots, N$).
- (ii) The pair (A_i, C_i) is observable ($i = 1, \dots, N$).
- (iii) For $G_i(s) := C_i(sI - A_i)^{-1}B_i + D_i$ ($i = 1, \dots, N$), the condition $\|G_{1,1,1}\|_{1+} = \dots = \|G_{N,1,1}\|_{1+} =: \gamma(> 0)$ holds.
- (iv) The matrix $\Omega \in \mathbb{R}_+^{N \times N}$ is irreducible (i.e., the directed graph $\Gamma(\Omega)$ is strongly connected).
- (v) The condition $\gamma \lambda_F(\Omega) = 1$ holds.

Then, for the interconnected system $\mathcal{G} \star \Omega$, the next results hold.

- (I) The interconnected system $\mathcal{G} \star \Omega$ is admissible, i.e., the Metzler matrix $\mathcal{D}\Omega - I_N$ is Hurwitz stable.
- (II) The matrix \mathcal{A}_{cl} given by (8) satisfies $\sigma(\mathcal{A}_{\text{cl}}) \subset \overline{\mathbb{C}}_-$, i.e., $\text{Re}(\lambda) \leq 0$ ($\forall \lambda \in \sigma(\mathcal{A}_{\text{cl}})$).
- (III) If we denote the right and left eigenvector of Ω associated with the eigenvalue $\lambda_F(\Omega)$ by v_R and v_L , respectively, we have $\mathcal{A}_{\text{cl}} \xi_R = 0$ and $\xi_L^T \mathcal{A}_{\text{cl}} = 0$ where
$$\xi_R = -v_R \bigotimes_{i=1}^N A_i^{-1} B_i \in \mathbb{R}_{++}^{n_{\hat{x}}},$$

$$\xi_L = -v_L \bigotimes_{i=1}^N A_i^{-T} C_i^T \in \mathbb{R}_{++}^{n_{\hat{x}}}.$$
- (IV) The matrix \mathcal{A}_{cl} has eigenvalue 0 that is algebraically (and hence geometrically) simple.
- (V) For any initial state $\hat{x}(0) \in \mathbb{R}_+^{n_{\hat{x}}} \setminus \{0\}$, the state \hat{x} of the interconnected system $\mathcal{G} \star \Omega$ remains to be nonnegative and converges to the steady-state \hat{x}_∞ given by $\hat{x}_\infty = (\xi_L^T \hat{x}(0) / \xi_L^T \xi_R) \xi_R \in \mathbb{R}_{++}^{n_{\hat{x}}}$.

We need the following lemma for the proof of Theorem 7. The proof of this lemma is omitted due to limited space.

Lemma 4: For given $A \in \{\mathbb{M}^n \cap \mathbb{H}^n\}$, $B \in \mathbb{R}_+^{n \times 1}$, and $C \in \mathbb{R}_+^{1 \times n}$, we have $A^{-1}B < 0$ if (A, B) is controllable. Similarly, we have $CA^{-1} < 0$ if (A, C) is observable.

Proof of Theorem 7:

Proof of (I): From (i), (ii) and Lemma 4, it is clear that $-C_i A_i^{-1} B_i > 0$ ($i = 1, \dots, N$). If we define $\mathcal{S} := \text{diag}(-C_1 A_1^{-1} B_1, \dots, -C_N A_N^{-1} B_N) > 0$, we have $\mathcal{D} = \gamma I_N - \mathcal{S}$ from (iii). On the other hand, from (iv) and Theorem 6, we see that there exists $v_R \in \mathbb{R}_{++}^N$ such that $\Omega v_R = \lambda_F(\Omega) v_R$. This implies $(\gamma \Omega - I_N) v_R = 0$ since (v) holds. Therefore we have $(\mathcal{D}\Omega - I_N) v_R = ((\gamma I_N - \mathcal{S})\Omega - I_N) v_R$

$= -\mathcal{S}\Omega v_R = -\lambda_F(\Omega)\mathcal{S}v_R < 0$. It follows from (the dual version of) (iii) of Lemma 1 that $\mathcal{D}\Omega - I_N$ is Hurwitz stable. Proof of (II): From Theorem 4, we see that $\sigma(\mathcal{A}_{cl}) \subset \mathbb{C}_-$ if $\gamma\lambda_F(\Omega) < 1$. Since at present $\gamma\lambda_F(\Omega) = 1$ holds from (v), we see that $\sigma(\mathcal{A}_{cl}) \subset \mathbb{C}_-$ holds from the continuity of the eigenvalues of \mathcal{A}_{cl} with respect to the variation of Ω .

Proof of (III): By defining $\Omega_{\mathcal{D}} := \Omega(I - \mathcal{D}\Omega)^{-1}$, we readily see

$$\begin{aligned} \mathcal{A}_{cl}\xi_R &= -(\mathcal{A} + \mathcal{B}\Omega_{\mathcal{D}}\mathcal{C}) \left(v_R \otimes_{i=1}^N A_i^{-1} B_i \right) \\ &= - \left(v_R \otimes_{i=1}^N B_i \right) - \mathcal{B}\Omega_{\mathcal{D}} \left(v_R \otimes_{i=1}^N C_i A_i^{-1} B_i \right) \\ &= -\mathcal{B}v_R - \mathcal{B}\Omega_{\mathcal{D}}(\mathcal{D} - \gamma I)v_R \\ &= -\mathcal{B} \left(I + \Omega(I - \mathcal{D}\Omega)^{-1}(\mathcal{D} - \gamma I) \right) v_R \\ &= -\mathcal{B} \left(I + (I - \Omega\mathcal{D})^{-1}\Omega(\mathcal{D} - \gamma I) \right) v_R \\ &= -\mathcal{B}(I - \Omega\mathcal{D})^{-1} (I - \gamma\Omega) v_R \\ &= -\mathcal{B}(I - \Omega\mathcal{D})^{-1} (1 - \gamma\lambda_F(\Omega)) v_R \\ &= 0. \end{aligned}$$

The equality $\xi_L^T \mathcal{A}_{cl} = 0$ follows similarly. Since (A_i, B_i) is controllable and (A_i, C_i) is observable, we see $-A_i^{-1}B_i > 0$ and $-C_i A_i^{-1} > 0$ ($i = 1, \dots, N$) from Lemma 4. Moreover, since Ω is irreducible, we have $v_R > 0$ and $v_L > 0$ from Theorem 6. Therefore we have

$$\begin{aligned} \xi_R &= -v_R \otimes_{i=1}^N A_i^{-1} B_i \in \mathbb{R}_{++}^{n\hat{x}}, \\ \xi_L &= -v_L \otimes_{i=1}^N A_i^{-T} C_i^T \in \mathbb{R}_{++}^{n\hat{x}}. \end{aligned}$$

Proof of (IV): We can prove that \mathcal{A}_{cl} is irreducible and hence the assertion readily follows from (II), (III) and Corollary 1. The proof for the irreducibility of \mathcal{A}_{cl} , which is indeed the core of the proof of Theorem 7, is unfortunately omitted due to limited space.

Proof of (V): Since $\sigma(\mathcal{A}_{cl}) \subset \overline{\mathbb{C}_-}$ from (II), since \mathcal{A}_{cl} has eigenvalue 0 that is algebraically (and hence geometrically) simple from (IV), and since $\xi_R \in \mathbb{R}_{++}^{Nn}$ is the right eigenvector of \mathcal{A}_{cl} corresponding to the eigenvalue 0, it is an elementary fact that the state \hat{x} of the interconnected system $\mathcal{G} \star \Omega$ converges to $\alpha\xi_R$ for some $\alpha \in \mathbb{R}$. Furthermore, for the dynamics of the interconnected system represented by $\hat{\dot{x}} = \mathcal{A}_{cl}\hat{x}$, we can readily see that $\xi_L^T \hat{\dot{x}} = 0$. Therefore we have $\xi_L^T \hat{x}(0) = \alpha \xi_L^T \xi_R$. Since $\xi_R, \xi_L \in \mathbb{R}_{++}^{n\hat{x}}$, we have $\xi_L^T \xi_R > 0$ and hence $\alpha = (\xi_L^T \hat{x}(0) / \xi_L^T \xi_R)$. Moreover, since $\xi_L \in \mathbb{R}_{++}^{n\hat{x}}$ and $\hat{x}(0) \in \mathbb{R}_{++}^{n\hat{x}} \setminus \{0\}$, it is obvious that $\alpha > 0$. This completes the proof. ■

The next result is a direct consequence of Theorem 7 and illustrates its usefulness in the application to the formation control of multi-agent systems [17], [18], [7].

Corollary 2: Let us consider the case where every subsystem G_i represented by (3) is SISO and satisfies conditions (i), (ii), and (iii) in Theorem 7. Then, for given $v_{obj} \in \mathbb{R}_{++}^N$, the interconnected system $\mathcal{G} \star \Omega$ satisfies

$$\hat{z}_\infty := \lim_{t \rightarrow \infty} \hat{z}(t) = \gamma\alpha v_{obj}, \quad \alpha = \xi_L^T \hat{x}(0) / \xi_L^T \xi_R \quad (14)$$

if we let

$$\Omega = \frac{1}{2\gamma} \Omega_{v_{obj}} \in \mathbb{R}_{++}^{N \times N} \quad (15)$$

where

$$\Omega_{v_{obj},i,j} = \begin{cases} v_{obj,1}/v_{obj,N} & (i=1, j=N), \\ v_{obj,i}/v_{obj,j} & (1 \leq i \leq N, j=i-1, i+1), \\ v_{obj,N}/v_{obj,1} & (i=N, j=1), \\ 0 & \text{otherwise.} \end{cases}$$

Namely, we can achieve the convergence of the output $\hat{z} = [z_1 \ \dots \ z_N]^T$ to $\gamma\alpha v_{obj} \in \mathbb{R}_{++}^N$ by the interconnection with $\Omega \in \mathbb{R}_{++}^{N \times N}$ given by (15).

In comparison with the consensus algorithm conceived in [13], the results in Corollary 2 can be regarded as an extension to the case where each agent allows to have distinct dynamics, the interconnection matrix Ω is not restricted to the graph Laplacian, and the vector v_{obj} that defines the asymptotic behaviour of the output is not restricted to the all-ones-vector.

V. NUMERICAL EXAMPLE

Let us consider the formation control problem of a multi-agent system constructed from N agents. The i -th agent ($i = 1, \dots, N$) can move on the (x, y) -plane with the dynamics $Z_{i,x}(s)$ and $Z_{i,y}(s)$. Assume

$$Z_{i,j}(s) = \frac{k_{i,j}}{s(s+a_{i,j})} U_{i,j}(s) \quad (i=1, \dots, N, j=x, y)$$

where $k_{i,j}, a_{i,j} > 0$. Applying the local feedback

$$U_{i,j}(s) = -f_{i,j}(Z_{i,j}(s) - W_{i,j}(s))$$

with $0 < f_{i,j} < a_{i,j}^2/4k_{i,j}$, we have

$$\begin{aligned} Z_{i,j}(s) &= G_{i,j}(s)W_{i,j}(s), \\ G_{i,j}(s) &= \left[\begin{array}{cc|c} -p_{i,j} & 1 & 0 \\ 0 & -q_{i,j} & p_{i,j}q_{i,j} \\ \hline 1 & 0 & 0 \end{array} \right], \\ p_{i,j} + q_{i,j} &= a_{i,j}, \quad p_{i,j}q_{i,j} = f_{i,j}k_{i,j}. \end{aligned}$$

It follows that each subsystem $G_{i,j}$ ($i = 1, \dots, N, j = x, y$) is SISO, stable, positive and satisfies $\|G_{i,j,1,1}\| = 1$.

Assuming that the agent i can communicate with agent $i-1$ and $i+1$ (agent 0 and $N+1$ should be regarded as agent N and 1, respectively), our goal here is to design a communication scheme over the agents with respect to each agent's position so that prescribed formation can be achieved. To this end, for given two vectors $v_{obj,x}, v_{obj,y} \in \mathbb{R}_{++}^N$ that array (x, y) coordinates of N agents corresponding to the desired formation, we constructed two interconnection matrices $\Omega_{v_{obj,x}}, \Omega_{v_{obj,y}}$ by following Corollary 2.

We generated $a_{i,j}$ and $k_{i,j}$ randomly (and uniformly) over the closed interval [1 2] and then chose $f_{i,j}$ randomly from $(0, a_{i,j}^2/4k_{i,j})$. We thus constructed $G_{i,j}$ ($i = 1, \dots, N, j = x, y$). With $\Omega_{v_{obj,x}}$ and $\Omega_{v_{obj,y}}$ designed above, we next computed the position of i -th agent $(z_{i,x}(t), z_{i,y}(t))$, which is nothing but the i -th output of $\mathcal{G}_x \star \Omega_{v_{obj,x}}$ and $\mathcal{G}_y \star \Omega_{v_{obj,y}}$, respectively, where $\mathcal{G}_j := \text{diag}(G_{1,j}, \dots, G_{N,j})$ ($j = x, y$). In Fig. 2, we show the initial position of the agents for $N = 30$. If we let $[v_{obj,x} \ v_{obj,y}]_i = [i/N \ i/N]$, the agents gradually form a line and converge to the position shown in Fig. 3. On the other hand, if we let $[v_{obj,x} \ v_{obj,y}]_i = [2 + \cos(2\pi i/N) \ 2 + \sin(2\pi i/N)]$, the agents gradually form an ellipsoid and converge to the position shown in Fig. 4. The convergence is rather slow, mainly because the second

largest real part of the eigenvalues of \mathcal{A}_{c1} is not small enough in both cases.

VI. CONCLUSION

In this paper, we studied stability and persistence of large-scale interconnected positive systems. We showed that the stability and the persistence conditions can be characterized completely in terms of the (weighted) L_1 -induced norm of each positive subsystem and the Frobenius eigenvalue of the positive interconnection matrix. As an application of our results on the persistence analysis, we showed that a sort of formation control of multi-agent systems can be achieved even if the dynamics of the agents are distinct, provided that they are all positive, stable, SISO and and share an identical unweighted L_1 -induced norm.

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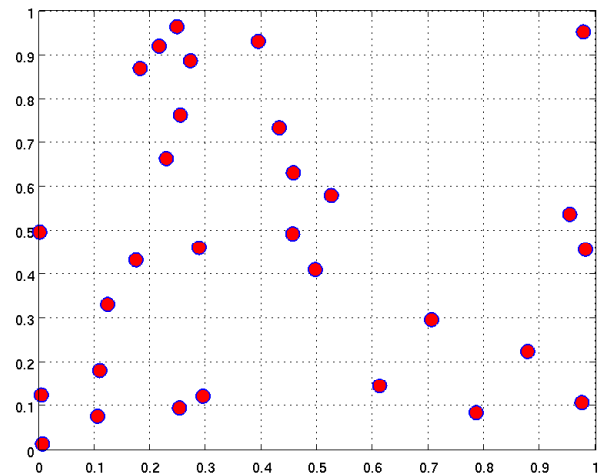


Fig. 2. Plot of the initial positions of agents.

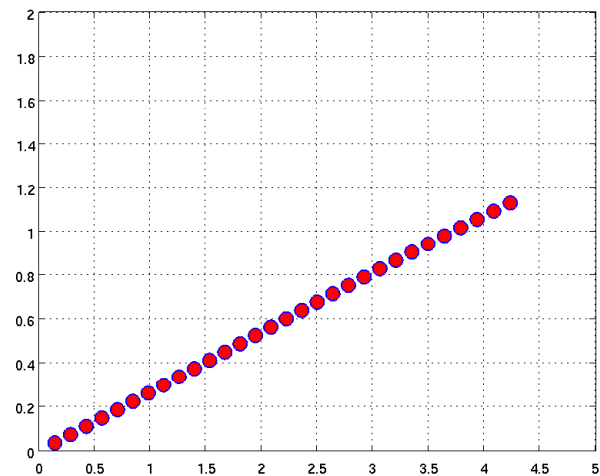


Fig. 3. Plot of the positions of agents at $t = 3000[\text{sec}]$

$$([v_{\text{obj},x} \ v_{\text{obj},y}]_i = [i/N \ i/N]).$$

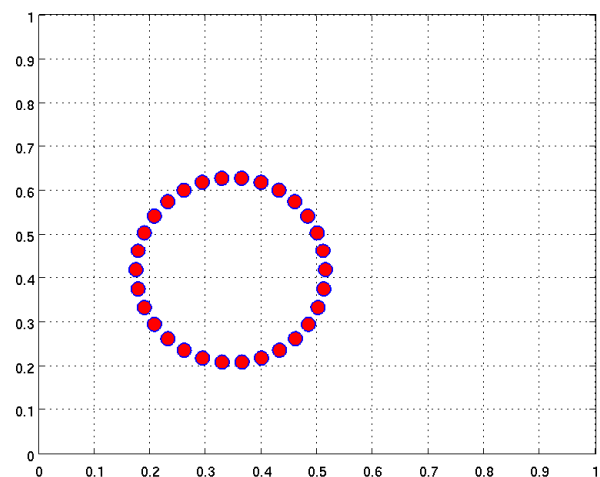


Fig. 4. Plot of the positions of agents at $t = 3000[\text{sec}]$

$$([v_{\text{obj},x} \ v_{\text{obj},y}]_i = [2 + \cos(2\pi i/N) \ 2 + \sin(2\pi i/N)].$$