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Partial interconnection dead-beat control of two-dimensional behaviors

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***Abstract*— In this paper the dead-beat control problem, by partial interconnection, of two-dimensional discrete behaviors, defined on the grid $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as (first) independent variable, is investigated. It turns out that the possibility of driving to zero (in a finite number of "steps") the "relevant variables" by means of a partial interconnection controller, making use of the "measurable variables", is equivalent to the reconstructibility of the relevant variables, that are not accessible for control. On the other hand, if we search for "admissible" dead-beat controllers, the only ones that provide meaningful results in practice, an additional condition has to be introduced: the zero-time-controllability of the relevant variables.**

I. Introduction

Most of the literature about two-dimensional (2D) and multi-dimensional (nD) systems has concentrated on mathematical models for which the two (n, in general) independent variables play the same role. In several engineering applications, however, one of the independent variables is time, and its role is distinguished from that of all the others, first of all because for that variable the concept of causality always makes sense. As a consequence, fundamental properties, like autonomy and controllability, need to be redefined at the light of this interpretation, in order to keep into account the privileged role of the time variable.

The interest in the behavioral approach to multidimensional systems for which one of the independent variables represents time, started with the works of Sasane and coauthors in [16], [17], [18], where the properties of time-autonomy and time-controllability for systems described by partial differential equations were thoroughly investigated. Recent years have witnessed a renewed interest in this class of systems, starting with the two journal papers [8], [13], where time-relevant autonomous 2D discrete behaviors and the related stability problems have been investigated. More recently, Oberst and Scheicher [11] have developed a general framework for the discrete nD case, and have provided characterizations of time-autonomy and time-controllability.

In [3], [4] time-controllability and zero-time-controllability of discrete 2D behaviors, defined on $\mathbb{Z}_+ \times \mathbb{Z}$ and having the time as an independent variable, have been characterized, and a complete solution of the full interconnection dead-beat control (DBC) problem has been provided. In these papers set-up, the first variable, defined on $\mathbb{Z}_+$, has been regarded as the time variable, while the second coordinate, defined on the whole integer set, is a space variable. According to this perspective, *vertical strips* in the grid $\mathbb{Z}_+ \times \mathbb{Z}$, namely the sets $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : 0 \le h \le N-1\}$, and the *half-planes* $\{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z} : h \ge N\}$, for $N \in \mathbb{Z}_+$, have been given special interpretation. The former are the sets where "initial conditions" on the system variables are given, while the latter are the supports of "long term evolutions", where both concepts clearly refer to the time coordinate. The definitions of time-controllability and of zero-time-controllability, as well as the DBC problem, have been accordingly introduced and investigated.

In this paper, we extend the results obtained in [3], [4], by investigating the dead-beat control problem under different assumptions on both the control action and the control target. Indeed, the system variables are split in two subsets: relevant variables, denoted by $\mathbf{w}_r$, and measured variables, $\mathbf{w}_m$. The control action is exerted only on the measured variables, and hence we deal with partial interconnection DBCs, and it targets the relevant variables alone. It turns out that the possibility of driving to zero in a finite number of "steps" the relevant variables by means of a partial interconnection controller is equivalent to the reconstructibility of such variables. On the other hand, if we search for "admissible" dead-beat controllers, the only ones that provide meaningful results in practice, we have to introduce the zero-time-controllability assumption on $\mathbf{w}_r$.

The paper is organized as follows. In section II, preliminary results about polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ are introduced. Basic properties of behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$ and the corresponding algebraic characterizations are discussed in section III. Section IV revisits the estimation problem of the relevant variables based on the knowledge of the measured ones. In section V, we first extend the zero-time-controllability property to the case when we target only the relevant variables (see [2] for the 1D case), and then provide necessary and sufficient conditions for the solvability of the DBC problem by partial interconnection.

II. TS-polynomial matrices

Let $\mathbb{R}[z_1, z_2, z_2^{-1}]$ be the ring of polynomials with real coefficients in the nonnegative powers of z_1 and in the integer powers of z_2. The first variable is associated with the time variable, while the second one with a space variable. Accordingly, we refer to such polynomials as *time-space polynomials* (for short, *TS-polynomials*) [4]. The ring of TS-polynomials is properly included in the ring of *Laurent polynomials* (*L-polynomials*) $\mathbb{R}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. So, for some purposes, it is convenient to regard TS-polynomials as L-polynomials.

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(Factor and zero) primeness properties for the class of TS-polynomial matrices naturally extend the analogous definitions for L-polynomial matrices [7], [19], [20]. A full column rank TS-polynomial matrix $H(z_1, z_2)$ is *right factor prime* if in every factorization $H(z_1, z_2) = \bar{H}(z_1, z_2)\Delta(z_1, z_2)$ over the ring $\mathbb{R}[z_1, z_2, z_2^{-1}]$, with $\Delta(z_1, z_2)$ nonsingular square, the matrix $\Delta(z_1, z_2)$ is *unimodular* in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ (by this meaning that $\det \Delta(z_1, z_2) = cz_2^k$, for some $c \neq 0$ and some $k \in \mathbb{Z}$). Also, $H(z_1, z_2)$ is *right zero prime* if it admits a left inverse in $\mathbb{R}[z_1, z_2, z_2^{-1}]$, namely there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ such that (s.t.) $L(z_1, z_2)H(z_1, z_2) = I$.

A full column rank TS-polynomial matrix $H(z_1, z_2)$ is said to be *right monomic* if it is right zero prime if regarded as an L-polynomial matrix, namely it admits an L-polynomial (but not necessarily TS-polynomial) left inverse. This amounts to saying that there exists $L(z_1, z_2)$ with entries in $\mathbb{R}[z_1, z_2, z_2^{-1}]$ s.t. $L(z_1, z_2)H(z_1, z_2) = z_1^h I$, for some $h \in \mathbb{Z}_+$. In particular, a square TS-polynomial matrix $\Delta(z_1, z_2)$ is called *square monomic* if it is unimodular if regarded as an L-polynomial matrix, and hence $\det \Delta(z_1, z_2) = cz_1^h z_2^k$, for suitable $c \neq 0, h \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$.

Of course, the concepts of *left* factor/zero prime or monomic TS-polynomial matrix can be introduced for full row rank matrices in a similar way, and enjoy analogous properties and characterizations.

Analogously to what happens with L-polynomial matrices, every TS-polynomial matrix $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times q}$ of rank r can always be factorized as

$$H(z_1, z_2) = L(z_1, z_2)\Delta(z_1, z_2)R(z_1, z_2), \tag{1}$$

for some suitable TS-polynomial matrices, with $L(z_1, z_2)$ $p \times r$ right factor prime, $\Delta(z_1, z_2)$ $r \times r$ nonsingular square, and $R(z_1, z_2)$ $r \times q$ left factor prime. This factorization is essentially unique, by this meaning that these three factors are uniquely determined up to (left and/or right) unimodular matrices.

The concepts of *left annihilator* and, in particular, of *minimal left annihilator* (MLA, for short) of a given TS-polynomial matrix $H(z_1, z_2)$ extend the concepts originally introduced in [14] for polynomial matrices in two indeterminates, and can be summarized as follows: if $H(z_1, z_2)$ is a $p \times q$ TS-polynomial matrix of rank r, a TS-polynomial matrix $L(z_1, z_2)$ is a left annihilator of $H(z_1, z_2)$ if $L(z_1, z_2)H(z_1, z_2) = 0$. A left annihilator $L_m(z_1, z_2)$ of $H(z_1, z_2)$ is an MLA if it is of full row rank and for any other left annihilator $L(z_1, z_2)$ of $H(z_1, z_2)$ we have $L(z_1, z_2) = P(z_1, z_2)L_m(z_1, z_2)$ for some TS-polynomial matrix $P(z_1, z_2)$. It can be easily proved that, when $r < p$, an MLA always exists, it is a $(p - r) \times p$ left factor prime matrix and is uniquely determined modulo a unimodular left factor. If the given $H(z_1, z_2)$ is of full row rank, then for consistency we define its MLA as the "void" matrix with 0 rows and p columns [12].

The two backward shift operators, σ_1 and σ_2, along the coordinate axes of the discrete grid $\mathbb{Z} \times \mathbb{Z}$ are defined as:

$$(\sigma_1 \mathbf{w})(h, k) := \mathbf{w}(h + 1, k), \ \ (\sigma_2 \mathbf{w})(h, k) := \mathbf{w}(h, k + 1),$$

respectively. The forward shift operators σ_1^{-1} and σ_2^{-1} are similarly defined. Notice that σ_i, $i = 1, 2$, and σ_2^{-1} map $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ into $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$, but this is not true for σ_1^{-1}.

If $H(z_1, z_2)$ is a $p \times q$ L-polynomial matrix, we associate with it the L-polynomial matrix operator $H(\sigma_1, \sigma_2)$, acting on any 2D sequence of size q. In the special case when $H(z_1, z_2)$ is a TS-polynomial matrix, the operator $H(\sigma_1, \sigma_2)$ maps $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ into $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$. If $H(z_1, z_2)$ is a TS-polynomial matrix, the associated map $H(\sigma_1, \sigma_2)$ is injective if and only if $H(z_1, z_2)$ is right zero prime, and it is surjective if and only if $H(z_1, z_2)$ is of full row rank.

III. 2D BEHAVIORS DEFINED ON $\mathbb{Z}_+ \times \mathbb{Z}$

A 2D *behavior* $\mathfrak{B}$ on $\mathbb{Z}_+ \times \mathbb{Z}$ is the set of solutions $\mathbf{w} = \{\mathbf{w}(h, k)\}_{(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}}$ of a family of linear 2D difference equations of the following type:

$$\sum_{(i,j) \in \Sigma_H} H_{ij} \mathbf{w}(h+i, k+j) = 0, \qquad \forall\, (h, k) \in \mathbb{Z}_+ \times \mathbb{Z}, \tag{2}$$

where the H_{ij}'s are real matrices with $\mathtt{w}$ columns (and say p rows), and the index set Σ_H is a finite subset of $\mathbb{Z}_+ \times \mathbb{Z}$. A 2D behavior $\mathfrak{B}$ described as in (2), is denoted by

$$\mathfrak{B} = \ker H(\sigma_1, \sigma_2), \tag{3}$$

where $H(z_1, z_2) = \sum_{(i,j) \in \Sigma_H} H_{ij} z_1^i z_2^j$ is a TS-polynomial matrix.

Given two TS-polynomial matrices $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$, with the same number of columns $\mathtt{w}$, condition $\ker H_1(\sigma_1, \sigma_2) \subseteq \ker H_2(\sigma_1, \sigma_2)$ holds if and only if $H_2(z_1, z_2) = P(z_1, z_2)H_1(z_1, z_2)$, for some TS-polynomial matrix $P(z_1, z_2)$ of suitable size. Also, $\ker H(\sigma_1, \sigma_2) = \{0\}$ if and only if $H(z_1, z_2)$ is right zero prime.

We now introduce autonomous behaviors.

Definition 1: [6], [15] Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times \mathtt{w}}$, a set of variables $\{\mathbf{w}_i : i \in \mathcal{I}\}$, $\mathcal{I}$ a proper subset of $\{1, 2, \ldots, \mathtt{w}\}$, is said to be a *set of free variables* for $\mathfrak{B}$ if the map $\pi_{\mathcal{I}} : \mathfrak{B} \to (\mathbb{R}^{|\mathcal{I}|})^{\mathbb{Z}_+ \times \mathbb{Z}}$, that projects every behavior trajectory onto the components indexed on $\mathcal{I}$, is surjective.
$\mathfrak{B}$ is said to be *autonomous* if it has no free variables.

Within the class of 2D autonomous behaviors we single out the nilpotent ones. Before introducing their definition, it is convenient to introduce some notation that will be used extensively in the rest of the paper (see [4]). For any pair of nonnegative integers t_0 and t_1, with $t_0 \leq t_1$, we define the *vertical strip* $\mathcal{S}_{t_0,t_1} := \{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : t_0 \leq h \leq t_1\}$. When $t_0 = t_1$ we use $\mathcal{S}_{t_0}$ to denote the *vertical line* $\{(t_0, k) : k \in \mathbb{Z}\}$, when $t_0 = 0 \leq t_1$ we use $\mathcal{S}_{\to t_1}$ and when $t_1 = +\infty$ we use $\mathcal{S}_{t_0 \to}$. Given any trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ and any set $\mathcal{S}_{t_0,t_1}$, we denote the trajectory restriction to the set $\mathcal{S}_{t_0,t_1}$ by $\mathbf{w}|_{\mathcal{S}_{t_0,t_1}}$. The *support* of a trajectory $\mathbf{w} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}_+ \times \mathbb{Z}}$ is the set of points where the trajectory takes nonzero values, i.e. $\{(h, k) \in \mathbb{Z}_+ \times \mathbb{Z} : \mathbf{w}(h, k) \neq 0\}$.

Definition 2: A 2D autonomous behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p \times \mathtt{w}}$, is said to be *nilpotent* (with respect to the half-plane $\mathbb{Z}_+ \times \mathbb{Z}$)

if there exists $N \in \mathbb{Z}_+$ s.t. all the trajectories $\mathbf{w} \in \mathfrak{B}$ satisfy $\mathbf{w}|_{\mathcal{S}_{N\rightarrow}} = 0$, or, equivalently $\mathbf{w}(h,k) = 0,\ \forall\ (h,k) \in \mathcal{S}_{N\rightarrow}$.

Nilpotency for a 2D behavior defined on $\mathbb{Z}_+ \times \mathbb{Z}$ does not mean that each trajectory has a finite support, as in the 1D case [5], but only that its support intersects finitely many (possibly zero) straight vertical lines $\mathcal{S}_t$ of $\mathbb{Z}_+ \times \mathbb{Z}$. Further insights into autonomous and nilpotent 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$ can be found in [4].

Proposition 1: [4] A 2D behavior $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$, with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2,\ z_2^{-1}]^{p\times \mathtt{w}}$, is

- autonomous iff $H(z_1, z_2)$ is of full column rank;
- nilpotent iff $H(z_1, z_2)$ is right monomic.

IV. Reconstructibility

Consider a 2D system whose behavior $\mathfrak{B}$ is described as in (3), for some matrix $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$. Assume that the system variables, grouped together in the vector $\mathbf{w}$, split into two groups: measured variables, denoted by $\mathbf{w}_m$, and variables that represent the target of our estimation problem (the "relevant" variables), denoted by $\mathbf{w}_r$. The TS-polynomial matrix $H(z_1, z_2)$ can be accordingly block-partitioned, thus leading to the following description of the 2D behavior trajectories:

$$[R_r(\sigma_1, \sigma_2)\ \ -R_m(\sigma_1, \sigma_2)] \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} = 0, \qquad (4)$$

or, equivalently

$$R_r(\sigma_1, \sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1, \sigma_2)\mathbf{w}_m(h,k), \qquad (5)$$

for all $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, where $R_r(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_r}$ and $R_m(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}_m}$. With respect to this partition of the system variables, the notions of observability and reconstructibility have been introduced in [1] for 2D behaviors defined on the half-plane $\mathcal{H}_0 = \{(h,k) \in \mathbb{Z}\times\mathbb{Z} : h+k \geq 0\}$, by also assuming that additional unmeasured and not relevant variables (for instance, disturbances) are involved in the system description. The adaption to the case of behaviors described as in (5) and defined over the half-plane $\mathbb{Z}_+ \times \mathbb{Z}$ is immediate.

Definition 3: Given a behavior $\mathfrak{B}$, described as in (5), we say that $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ if there exists $N \in \mathbb{Z}_+$ such that $(\mathbf{w}_r, \mathbf{w}_m), (\bar{\mathbf{w}}_r, \mathbf{w}_m) \in \mathfrak{B}$ implies $\mathbf{w}_r(h,k) - \bar{\mathbf{w}}_r(h,k) = 0,\ \forall\ (h,k) \in \mathcal{S}_{N\rightarrow}$.

A characterization of reconstructibility is provided in Proposition 2, below, and it is a simple adaption of the analogous result obtained in [1].

Proposition 2: Given a 2D behavior $\mathfrak{B}$, described as in (5), $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ if and only if $R_r(z_1, z_2)$ is right monomic.

In [1] it has been proved that reconstructibility is a necessary and sufficient condition for the existence of a dead-beat observer of $\mathbf{w}_r$ from $\mathbf{w}_m$, by this meaning a system (defined, again, on $\mathbb{Z}_+ \times \mathbb{Z}$) that, corresponding to every trajectory $(\mathbf{w}_r, \mathbf{w}_m)$ in $\mathfrak{B}$, produces an estimate $\hat{\mathbf{w}}_r$ of the trajectory $\mathbf{w}_r$ (based on the measured variable $\mathbf{w}_m$ alone), that coincides with the sequence $\mathbf{w}_r$ except, possibly, on an initial strip $\mathcal{S}_{\rightarrow N-1}$. The interested reader is referred to [1] for the details.

V. Zero-time-controllability and dead-beat controllers

For 2D behaviors defined on $\mathbb{Z}_+ \times \mathbb{Z}$, we have introduced the definition of zero-time-controllability [4]. For zero-time-controllable behaviors it is possible to patch any initial strip of a behavior trajectory (its portion in $\mathcal{S}_{\rightarrow N-1}$) with the zero trajectory. This means that any behavior trajectory can be driven to zero within a finite number of time instants, so that it is identically zero on some suitable half-plane $\mathcal{S}_{N+L\rightarrow}$. In this paper we extend the results obtained in [4] to the case when the system variables are partitioned into two families, and we are interested only in ensuring that one of these sets has trajectories that can be driven to zero.

Definition 4: Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_, \sigma_2)$ with $H(z_1, z_2) \in \mathbb{R}[z_1, z_2, z_2^{-1}]^{p\times \mathtt{w}}$, let I be a subset of $\{1, 2, \dots, \mathtt{w}\}$, and let $\mathbf{w}_I$ be the subset of the system variables, consisting of all the entries of $\mathbf{w}$ indexed on I. We say that $\mathbf{w}_I$ is *zero-time-controllable* if there exists a nonnegative integer $L \in \mathbb{Z}_+$ s.t. for every $N \in \mathbb{N}$ and every $\mathbf{w} \in \mathfrak{B}$, one can find $\bar{\mathbf{w}} \in \mathfrak{B}$ s.t.

$$\begin{aligned} \bar{\mathbf{w}}_I(h,k) &= \mathbf{w}_I(h,k), & \forall\ (h,k) \in \mathcal{S}_{\rightarrow N-1}, \quad (6) \\ \bar{\mathbf{w}}_I(h,k) &= 0, & \forall\ (h,k) \in \mathcal{S}_{N+L\rightarrow}, \quad (7) \end{aligned}$$

i.e. $\bar{\mathbf{w}}_I|_{\mathcal{S}_{\rightarrow N-1}} = \mathbf{w}_I|_{\mathcal{S}_{\rightarrow N-1}}$ and $\sigma_1^{N+L}\bar{\mathbf{w}}_I = 0$.
When $I = \{1, 2, \dots, \mathtt{w}\}$ we say, equivalently, that $\mathbf{w}$ or the behavior $\mathfrak{B}$ is zero-time-controllable.

Note that, when $I \subsetneq \{1, 2, \dots, \mathtt{w}\}$, no constraint is imposed on the evolution of the remaining variables $\mathbf{w}_{\bar{I}}$, where $\bar{I}$ is the complementary set of I with respect to $\{1, 2, \dots, \mathtt{w}\}$. In [4] we have derived algebraic characterizations of zero-time-controllable behaviors. The adaption to the case when we restrict our attention to a subset of the system variables is immediate, and it is based on the fact that the zero-time-controllability of $\mathbf{w}_I$ is equivalent to the zero-time-controllability (in the sense of [4]) of the projection[1] of the behavior $\mathfrak{B}$ on the variables $\mathbf{w}_I$, denoted by $\mathcal{P}_I\mathfrak{B}$.

Theorem 1: Given a 2D behavior, described by the following difference equation:

$$R_I(\sigma_1, \sigma_2)\mathbf{w}_I(h,k) = R_{\bar{I}}(\sigma_1, \sigma_2)\mathbf{w}_{\bar{I}}(h,k), \qquad (8)$$

for all $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, where $R_I(z_1, z_2)$ and $R_{\bar{I}}(z_1, z_2)$ are TS-polynomial matrices of sizes $q \times \mathtt{w}_I$ and $q \times (\mathtt{w} - \mathtt{w}_I)$, respectively, let $M_{\bar{I}}(z_1, z_2)$ be an MLA of $R_{\bar{I}}(z_1, z_2)$. The following facts are equivalent:

i) $\mathbf{w}_I$ is *zero-time-controllable*;

[1] It is well-known [4], [9], that if a behavior $\mathfrak{B}$ is described by $R_I(\sigma_1, \sigma_2)\mathbf{w}_I(h,k) = R_{\bar{I}}(\sigma_1, \sigma_2)\mathbf{w}_{\bar{I}}(h,k), (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, then $\mathcal{P}_I\mathfrak{B} = \ker(M_{\bar{I}}(\sigma_1, \sigma_2)R_I(\sigma_1, \sigma_2))$, where $M_{\bar{I}}(z_1, z_2)$ is an MLA of $R_{\bar{I}}(z_1, z_2)$. In the special case when $R_{\bar{I}}(z_1, z_2)$ is of full row rank, its MLA is a *void* matrix and $\mathcal{P}_I\mathfrak{B} = \ker 0 = (\mathbb{R}^{\mathtt{w}_I})^{\mathbb{Z}_+ \times \mathbb{Z}}$.

ii) either $R_{\bar{I}}(z_1,z_2)$ is of full row rank or $H_I(z_1,z_2) := M_{\bar{I}}(z_1,z_2)R_I(z_1,z_2)$ can be expressed as

$$H_I(z_1,z_2) = L(z_1,z_2)\Delta(z_1,z_2)R(z_1,z_2),$$

with $L(z_1,z_2)$ right monomic, $\Delta(z_1,z_2)$ nonsingular square with $\det\Delta(z_1,\ z_2) \in \mathbb{R}[z_2,z_2^{-1}]$, and $R(z_1,z_2)$ left factor prime.

By a *controller* $\mathcal{C}$ of a given 2D behavior $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$, we mean a system that constrains the system trajectories, and hence is described by a difference equation of the following type

$$C(\sigma_1,\sigma_2)\mathbf{w}(h,k) = 0, \qquad \forall\ (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}, \quad (9)$$

for a suitable TS-polynomial matrix $C(z_1,z_2)$. The overall *controlled behavior*, i.e. the behavior of the system obtained by *full interconnection* of the behavior $\mathfrak{B}$ and the controller (9), is described by

$$\begin{bmatrix} H(\sigma_1,\sigma_2) \\ C(\sigma_1,\sigma_2) \end{bmatrix} \mathbf{w}(h,k) = 0, \qquad \forall\ (h,k)\in\mathbb{Z}_+ \times \mathbb{Z}, \quad (10)$$

it is denoted by $\mathcal{K}$, and it is clearly the intersection of $\mathfrak{B}$ and $\mathcal{C} := \ker C(\sigma_1,\sigma_2)$. On the other hand, if we assume that $\mathfrak{B}$ is described as in

$$R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k) = R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k), (h,k)\in\mathbb{Z}_+ \times \mathbb{Z}$$

where $R_r(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}_r}$, $R_m(z_1,z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}_m}$, and $\mathbf{w}_r$ and $\mathbf{w}_m$ represent, as in the previous sections, the relevant and the measured variables, respectively, it makes sense to consider also control by *partial interconnection*, namely to apply the control action only to the measured variables. When so, the controlled behavior $\mathcal{K}$ is described as

$$\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -C_m(\sigma_1,\sigma_2) \end{bmatrix} \begin{bmatrix} \mathbf{w}_r(h,k) \\ \mathbf{w}_m(h,k) \end{bmatrix} = 0, \quad (11)$$

for all $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$. In either cases, whether we act by full interconnection or by partial interconnection, the target of the dead-beat control problem is to design, if possible, a controller s.t. the projection of the controlled behavior $\mathcal{K}$ on the subset $\mathbf{w}_I$ of its variables, $\mathcal{P}_I\mathcal{K}$, is nilpotent.

Definition 5: Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$, a controller $\mathcal{C}$, acting on $\mathfrak{B}$ by full or partial interconnection, is said to be a *dead-beat controller (DBC) for the system variables* $\mathbf{w}_I, I \subseteq \{1,2,\dots,\mathtt{w}\}$ (a DBC for $\mathfrak{B}$ if $I = \{1,2,\dots,\mathtt{w}\}$), if there exists $N \in \mathbb{Z}_+$ s.t. all the trajectories of the behavior $\mathcal{P}_I\mathcal{K} = \mathcal{P}_I(\mathfrak{B}\cap\mathcal{C})$ have supports included in the vertical strip $\mathcal{S}_{\to N-1}$, which amounts to saying that $\mathcal{P}_I\mathcal{K}$ is nilpotent.

A concept of *admissible DBC* has been introduced in [2] for 1D behaviors and in [4] for the case of control by full interconnection targeting all system variables (namely in the case $I = \{1,2\dots,\mathtt{w}\}$). Its adaption to the general set-up considered in this section is rather simple, but requires some preliminary steps. Admissibility represents the necessary requirement in order to have a DBC that can be practically implemented. Indeed, only admissible controllers can start acting any time on the trajectory without constraining a posteriori its past evolution. Also, it must be remarked that the common "regularity" requirement introduces much stronger conditions on the 2D behavior matrices, which are not required for ensuring admissibility, and hence it does not represent a priority when designing a DBC. The interested reader is referred to [2], [4] for a discussion about the significance of the admissibility property, and to [12] for necessary and sufficient conditions for the existence of regular controllers. Given a controller $\mathcal{C}$, acting either by full or by partial interconnection, we introduce the *delayed controllers* $\mathcal{C}_d, d \in \mathbb{Z}_+$, described by the difference equation

$$\sigma_1^d C(\sigma_1,\sigma_2)\mathbf{w}(h,k) = 0, \qquad (h,k) \in \mathbb{Z}_+ \times \mathbb{Z}.$$

If we denote by $\mathcal{K}_d$ the controlled behavior obtained corresponding to $\mathcal{C}_d$, then

$$\mathcal{K}_d = \ker \begin{bmatrix} H(\sigma_1,\sigma_2) \\ \sigma_1^d C(\sigma_1,\sigma_2) \end{bmatrix}. \quad (12)$$

Clearly, $\mathcal{C} = \mathcal{C}_0$ and $\mathcal{K} = \mathcal{K}_0$.

It is not difficult to see that if $\mathcal{C}$ is a DBC for the system variables $\mathbf{w}_I$, then every $\mathcal{C}_d$ is.

Lemma 1: Given a 2D behavior $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$, with $H(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{p\times \mathtt{w}}$, let $\mathcal{C} = \ker C(\sigma_1,\sigma_2)$, with $C(z_1,\ z_2) \in \mathbb{R}[z_1,z_2,z_2^{-1}]^{q\times \mathtt{w}}$, be a DBC for the system variables $\mathbf{w}_I$ (either acting by full interconnection or by partial interconnection). Then, for every $d \in \mathbb{Z}_+$, $\mathcal{C}_d$ is a DBC for $\mathbf{w}_I$ (of the same type in terms of interconnection).

Proof: It entails no loss of generality to rewrite the behavior equations in order to distinguish between the variables $\mathbf{w}_I$ and $\mathbf{w}_{\bar{I}}$:

$$R_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) = R_{\bar{I}}(\sigma_1,\sigma_2)\mathbf{w}_{\bar{I}}(h,k),\ (h,k) \in \mathbb{Z}_+\times\mathbb{Z}.$$

On the other hand, the controller $\mathcal{C}$ can accordingly be rewritten as

$$C_I(\sigma_1,\sigma_2)\mathbf{w}_I(h,k) = C_{\bar{I}}(\sigma_1,\sigma_2)\mathbf{w}_{\bar{I}}(h,k),\ (h,k) \in \mathbb{Z}_+\times\mathbb{Z},$$

independently of what are the variables it is acting on. If this is a DBC for the variables $\mathbf{w}_I$, then $\begin{bmatrix} R_{\bar{I}}(z_1,z_2) \\ C_{\bar{I}}(z_1,z_2) \end{bmatrix}$ cannot be of full row rank, otherwise $\mathcal{P}_I\mathcal{K} = (\mathbb{R}^{\mathtt{w}_I})^{\mathbb{Z}_+\times\mathbb{Z}}$. So, let $[\, M_R(z_1,z_2) \quad M_C(z_1,z_2)\,]$ be an MLA of $\begin{bmatrix} R_{\bar{I}}(z_1,z_2) \\ C_{\bar{I}}(z_1,z_2) \end{bmatrix}$. Then $\mathcal{P}_I\mathcal{K}$ is nilpotent or, equivalently,

$$H_I(z_1,z_2) := [\, M_R(z_1,z_2) \quad M_C(z_1,z_2)\,] \begin{bmatrix} R_I(z_1,z_2) \\ C_I(z_1,z_2) \end{bmatrix}$$

is right monomic. Now, consider $\mathcal{K}_d = \mathcal{B} \cap \mathcal{C}_d$:

$$\mathcal{K}_d = \ker \begin{bmatrix} R_I(\sigma_1,\sigma_2) & -R_{\bar{I}}(\sigma_1,\sigma_2) \\ \sigma_1^d C_I(\sigma_1,\sigma_2) & -\sigma_1^d C_{\bar{I}}(\sigma_1,\sigma_2) \end{bmatrix}.$$

It is clear that $[\, z_1^d M_R(z_1,z_2) \quad M_C(z_1,z_2)\,]$ is a left annihilator of $\begin{bmatrix} R_{\bar{I}}(z_1,z_2) \\ z_1^d C_{\bar{I}}(z_1,z_2) \end{bmatrix}$. So, if $[\, \hat{M}_R(z_1,z_2) \quad \hat{M}_C(z_1,z_2)\,]$ is an MLA of $\begin{bmatrix} R_{\bar{I}}(z_1,z_2) \\ z_1^d C_{\bar{I}}(z_1,z_2) \end{bmatrix}$, then $[\, z_1^d M_R(z_1,z_2) \quad M_C(z_1,z_2)\,] =$

$P(z_1,z_2)\,[\,\hat{M}_R(z_1,z_2)\quad \hat{M}_C(z_1,z_2)\,]$, for some TS-polynomial matrix $P(z_1,z_2)$. This ensures that

$$\begin{aligned}\mathcal{P}_I\mathcal{K}_d &= \ker\left([\,\hat{M}_R(\sigma_1,\sigma_2)\ \hat{M}_C(\sigma_1,\sigma_2)\,]\begin{bmatrix} R_I(\sigma_1,\sigma_2)\\ \sigma_1^d C_I(\sigma_1,\sigma_2)\end{bmatrix}\right)\\ &\subseteq \ker\left([\,\sigma_1^d M_R(\sigma_1,\sigma_2)\ M_C(\sigma_1,\sigma_2)\,]\begin{bmatrix} R_I(\sigma_1,\sigma_2)\\ \sigma_1^d C_I(\sigma_1,\sigma_2)\end{bmatrix}\right)\\ &= \ker\left(\sigma_1^d H_I(\sigma_1,\sigma_2)\right).\end{aligned}$$

Since $H_I(z_1,z_2)$ is right monomic, also $z_1^d H_I(z_1,z_2)$ is right monomic. This ensures that $\mathcal{P}_I\mathcal{K}_d$ is included in a nilpotent behavior, and hence it is nilpotent, too. This shows that $\mathcal{C}_d$ is a DBC for $\mathbf{w}_I$. ∎

Definition 6: Given a 2D behavior $\mathfrak{B}$, a dead-beat controller $\mathcal{C}$ for the system variables $\mathbf{w}_I, I\subseteq\{1,2,\dots,\mathtt{w}\}$, (a DBC for $\mathfrak{B}$) is said to be *admissible* if there exists $L\in\mathbb{Z}_+$ s.t. for every $\mathbf{w}\in\mathfrak{B}$ and every $N\in\mathbb{N}$, there exists $\bar{\mathbf{w}}\in\mathcal{K}_{L+N}$, the controlled behavior obtained corresponding to the controller $\mathcal{C}_{L+N}$, s.t. $\bar{\mathbf{w}}_I(h,k)|_{\mathcal{S}_{\to N-1}} = \mathbf{w}_I(h,k)|_{\mathcal{S}_{\to N-1}}$.

In the following we will characterize the existence of DBCs and of admissible DBCs by steadily assuming that the system variables are split into relevant and measured variables, and that we resort to a partial interconnection controller $\mathcal{C}=\mathcal{C}_{PI}$, acting on the measured variables only. This means that the overall controlled behavior $\mathcal{K}$ will always be described as in (11). Also, we will aim at driving to zero only the relevant variables ($\mathbf{w}_I=\mathbf{w}_r$).

We have the following characterization of the existence of a (not necessarily admissible) DBC, acting on the measured variables alone.

Proposition 3: Consider a 2D behavior $\mathfrak{B}$, described by

$$R_r(\sigma_1,\sigma_2)\mathbf{w}_r(h,k)=R_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k),\ (h,k)\in\mathbb{Z}_+\times\mathbb{Z}, \tag{13}$$

where $R_r(z_1,z_2)$ and $R_m(z_1,z_2)$ are TS-polynomial matrices of size $p\times\mathtt{w}_r$ and $p\times\mathtt{w}_m$, respectively. The following conditions are equivalent:

i) there exists a DBC $\mathcal{C}$ for the variable $\mathbf{w}_r$, described as

$$C_m(\sigma_1,\sigma_2)\mathbf{w}_m(h,k)=0,\qquad (h,k)\in\mathbb{Z}_+\times\mathbb{Z}, \tag{14}$$

for some TS-polynomial matrix (of size say $q\times\mathtt{w}_m$) $C_m(z_1,z_2)$;

ii) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$ (equivalently, $R_r(z_1,z_2)$ is right monomic).

Proof: i) ⇒ ii) If i) holds, then the behavior

$$\mathcal{K}=\ker\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2)\\ 0 & -C_m(\sigma_1,\sigma_2)\end{bmatrix}, \tag{15}$$

has projection $\mathcal{P}_{\mathbf{w}_r}\mathfrak{B}$ that is nilpotent. This means that if $[\,M_R(z_1,z_2)\quad M_C(z_1,z_2)\,]$ is an MLA of $\begin{bmatrix} R_m(z_1,z_2)\\ C_m(z_1,z_2)\end{bmatrix}$, then

$$[\,M_R(z_1,z_2)\ M_C(z_1,z_2)\,]\begin{bmatrix} R_r(z_1,z_2)\\ 0\end{bmatrix}=M_R(z_1,z_2)R_r(z_1,z_2)$$

is right monomic. But this implies that there exists a TS-polynomial matrix $L(z_1,z_2)$ and $\delta\in\mathbb{Z}_+$ such that

$$\begin{aligned} z_1^\delta I_{\mathtt{w}_r} &= L(z_1,z_2)[M_R(z_1,z_2)R_r(z_1,z_2)]\\ &= [L(z_1,z_2)M_R(z_1,z_2)]R_r(z_1,z_2),\end{aligned}$$

thus ensuring that $R_r(z_1,z_2)$ is right monomic, too.
ii) ⇒ i) It is easily seen that, under the assumption that $R_r(z_1,z_2)$ is right monomic, the trivial controller $\mathcal{C}=\ker I_{\mathtt{w}_m}$ is a DBC for the whole behavior $\mathfrak{B}$, and hence, in particular, for $\mathbf{w}_r$. ∎

We now address the less trivial case of admissible DBCs. The following theorem represents the main contribution of the paper, and it extends the analogous result about full interconnection DBCs, targeting all the system variables, presented in [4].

Theorem 2: Consider a 2D behavior $\mathfrak{B}$, described as in (13), for suitable TS-polynomial matrices $R_r(z_1,z_2)$ and $R_m(z_1,z_2)$, of size $p\times\mathtt{w}_r$ and $p\times\mathtt{w}_m$, respectively. There exists an admissible DBC $\mathcal{C}$ for the variable $\mathbf{w}_r$, described as in (14), for some TS-polynomial matrix (of size say $q\times\mathtt{w}_m$) $C_m(z_1,z_2)$ if and only if the following two conditions hold:

a) $\mathbf{w}_r$ is reconstructible from $\mathbf{w}_m$;
b) the variable $\mathbf{w}_r$ is zero-time-controllable.

If these two conditions are satisfied, then every DBC for $\mathbf{w}_r$ is admissible.

Proof: [Necessity] Assume, first, that there exists an admissible DBC $\mathcal{C}$ for $\mathbf{w}_r$, described as in (14). This means that the overall controlled behavior $\mathcal{K}$, described as in (15), has projection $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ that is nilpotent and hence there exists $M\in\mathbb{Z}_+$ such that all trajectories in $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}$ are zero in $\mathcal{S}_{M\to}$. If we let $[\,M_R(z_1,z_2)\quad M_C(z_1,z_2)\,]$ be an MLA of $\begin{bmatrix} R_m(z_1,z_2)\\ C_m(z_1,z_2)\end{bmatrix}$, then

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}=\ker(M_R(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2)).$$

Now, consider

$$\mathcal{K}_d=\mathcal{B}\cap\mathcal{C}_d=\ker\begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2)\\ 0 & -\sigma_1^d C_m(\sigma_1,\sigma_2)\end{bmatrix}.$$

We have already shown in the proof of Lemma 1 that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_d\subseteq\ker\left(\sigma_1^d M_R(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2)\right).$$

So, the trajectories of $\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_d$ have support included in $\mathcal{S}_{\to d+M-1}$, and this is true for every $d\in\mathbb{Z}_+$.

Since $\mathcal{C}$ is an admissible DBC for $\mathbf{w}_r$, there exists $L\in\mathbb{Z}_+$ such that for every $N\in\mathbb{N}$ and every $(\mathbf{w}_r,\mathbf{w}_m)\in\mathfrak{B}$ a trajectory $(\bar{\mathbf{w}}_r,\bar{\mathbf{w}}_m)\in\mathcal{K}_{L+N}\subseteq\mathfrak{B}$ can be found, with $\bar{\mathbf{w}}_r=\mathbf{w}_r$ in $\mathcal{S}_{\to N-1}$. Such a trajectory $\bar{\mathbf{w}}_r$ is surely zero in $\mathcal{S}_{L+N+M\to}$. So, we have proved that there exists $L^*\in\mathbb{N}$, specifically $L^*:=M+L$, such that for every $(\mathbf{w}_r,\mathbf{w}_m)\in\mathfrak{B}$ there exists a trajectory $(\bar{\mathbf{w}}_r,\bar{\mathbf{w}}_m)\in\mathcal{K}_{L+N}\subseteq\mathfrak{B}$ such that $\bar{\mathbf{w}}_r=\mathbf{w}_r$ in $\mathcal{S}_{\to N-1}$ and $\bar{\mathbf{w}}_r=0$ in $\mathcal{S}_{N+L^*\to}$. This proves that $\mathbf{w}_r$ is zero-time-controllable.
[Sufficiency] Assume now that conditions a) and b) hold. We want to prove that an admissible DBC for $\mathbf{w}_r$ always exists.

Set $C_m(z_1,z_2) = R_m(z_1,z_2)$. Correspondingly

$$\mathcal{K} = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -R_m(\sigma_1,\sigma_2) \end{bmatrix},$$

and it is easy to verify that if $M_m(z_1,z_2)$ is an MLA of $R_m(z_1,z_2)$, then $\begin{bmatrix} M_m(z_1,z_2) & 0 \\ I_p & -I_p \end{bmatrix}$ is an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ R_m(z_1,z_2) \end{bmatrix}$. Therefore

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K} = \ker \left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ R_r(\sigma_1,\sigma_2) \end{bmatrix} \right) = \ker R_r(\sigma_1,\sigma_2).$$

By assumption a), $R_r(z_1,z_2)$ is right monomic and hence $C_m(z_1,z_2) = R_m(z_1,z_2)$ defines a DBC for $\mathbf{w}_r$.
We now need to prove that this DBC is admissible. By the zero-time-controllability property, there exists a nonnegative integer L such that for every $N \in \mathbb{N}$ and every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$, one can find $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathfrak{B}$ such that

$$\begin{aligned} \bar{\mathbf{w}}_r(h,k) &= \mathbf{w}_r(h,k), \quad \forall\, (h,k) \in \mathcal{S}_{\to N-1} \\ \bar{\mathbf{w}}_r(h,k) &= 0, \quad \forall\, (h,k) \in \mathcal{S}_{N+L\to}. \end{aligned} \tag{16}$$

We want to show that this same nonnegative integer L makes the definition of admissible DBC for $\mathbf{w}_r$ satisfied. To this end we have to show that for every $N \in \mathbb{N}$ and every $(\mathbf{w}_r, \mathbf{w}_m) \in \mathfrak{B}$ a trajectory $(\bar{\mathbf{w}}_r, \bar{\mathbf{w}}_m) \in \mathcal{K}_{L+N} \subseteq \mathfrak{B}$ can be found, with $\bar{\mathbf{w}}_r = \mathbf{w}_r$ in $\mathcal{S}_{\to N-1}$. Now, consider

$$\mathcal{K}_{L+N} = \mathcal{B} \cap \mathcal{C}_{L+N} = \ker \begin{bmatrix} R_r(\sigma_1,\sigma_2) & -R_m(\sigma_1,\sigma_2) \\ 0 & -\sigma_1^{L+N} R_m(\sigma_1,\sigma_2) \end{bmatrix}.$$

It is easily seen that $\begin{bmatrix} M_m(z_1,z_2) & 0 \\ z_1^{L+N} I & -I \end{bmatrix}$ is an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ z_1^{L+N} R_m(z_1,z_2) \end{bmatrix}$. This implies that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N} = \ker \left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ \sigma_1^{L+N} R_r(\sigma_1,\sigma_2) \end{bmatrix} \right).$$

So, it is easy to see that the same trajectory $\bar{\mathbf{w}}_r$ that satisfies (16), and whose existence is ensured by the zero-controllability property, is necessarily a trajectory of both $\ker(M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2))$ and $\ker(\sigma_1^{L+N} R_r(\sigma_1,\sigma_2))$. Therefore $\bar{\mathbf{w}}_r \in \mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N}$ and this makes the definition of admissible DBC satisfied. This completes the proof of sufficiency.

By repeating the same reasoning we just used, we can show that if a) and b) hold, and $C_m(z_1,z_2)$ defines a DBC, then an MLA of $\begin{bmatrix} R_m(z_1,z_2) \\ C_m(z_1,z_2) \end{bmatrix}$ can always be expressed as

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ M_{R_m}(z_1,z_2) & M_{C_m}(z_1,z_2) \end{bmatrix},$$

while a left annihilator of $\begin{bmatrix} R_m(z_1,z_2) \\ z_1^{N+L} C_m(z_1,z_2) \end{bmatrix}$ is

$$\begin{bmatrix} M_m(z_1,z_2) & 0 \\ z_1^{N+L} M_{R_m}(z_1,z_2) & M_{C_m}(z_1,z_2) \end{bmatrix}$$

and this ensures that

$$\mathcal{P}_{\mathbf{w}_r}\mathcal{K}_{L+N} \subseteq \ker \left(\begin{bmatrix} M_m(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \\ \sigma_1^{L+N} M_{R_m}(\sigma_1,\sigma_2)R_r(\sigma_1,\sigma_2) \end{bmatrix} \right).$$

So, zero-time-controllability of $\mathbf{w}_r$ ensures the admissibility of any DBC for $\mathbf{w}_r$. ∎

The following example enlightens both the meaning of the previous results and how they improve on the results obtained in [3], [4].

Example 1: Consider the behavior $\mathfrak{B}$ described, for every $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, by the following difference equations:

$$\begin{bmatrix} \sigma_2 & \sigma_1 - \sigma_1^2 \\ 0 & \sigma_1 - 1 \end{bmatrix} \begin{bmatrix} w_r(h,k) \\ w_m(h,k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\mathcal{B}$ is not zero-time-controllable [4], so an admissible DBC for both variables does not exist. However, it is easy to see that w_r is both reconstructible from w_m and zero-time-controllable. Consequently, an admissible DBC that drives to zero only w_r, by acting only on w_m (and hence obtained by partial interconnection) exists: for instance, the trivial controller described by $C_m(z_1,z_2) = 1$.

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