

Internal and External Force-Based Impedance Control for Cooperative Manipulation

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Abstract—An asymptotically stable cascaded control algorithm is proposed for cooperative manipulation of a common object. This algorithm controls motion and internal forces of the object, as well as the contact forces between the object and environment. The motion of each manipulator is controlled using an inverse dynamics type of controller. Only knowledge of the kinematics of the manipulated object is required, since the interaction forces and moments between the object and manipulators are measured. The internal stresses in the object are controlled based on enforced impedance relationships between the object and each manipulator. The internal forces and moments are computed using the object kinematics. Contact with the environment is controlled with an enforced impedance relationship between the object and the environment. For both internal and external forces, reference trajectories can be specified. Asymptotic stability of each controller is proven using Lyapunov stability theory and LaSalle's invariance principle. Guidelines are suggested to compute control parameters of the internal impedance parameters. Merits of the control algorithm are demonstrated in simulations.

I. INTRODUCTION

Multiple robotic manipulators working together on a common task is referred to as cooperative manipulation. In many cooperative tasks, the manipulators grasp a common object. Eventually, this object makes contact with the environment, so it is important to simultaneously control the motion of the system, interaction forces between the object and the manipulators (internal forces), and the contact forces between the object and the environment (external forces).

Over the past decades, several control structures have been proposed and tested on various systems with cooperative manipulators (see [1], or [2] for a more recent overview). The proposed algorithms that can control both the forces and motion can be classified as hybrid position/force control [3], [4] and impedance/admittance control [5], [6], [7], [8]. In hybrid control, the coordination space is decoupled into motion and force controlled directions, using a predefined and fixed selection matrix. Contact in a motion controlled direction can lead to damage of the object and manipulators, since the forces in this direction are not controlled [9].

The main advantage of impedance control is that it takes the relation between the forces and motion of the system dynamically into account. In this way, no prior knowledge of the contact directions is required. Furthermore, it can be used as an outer loop in a cascade control structure, such that traditional motion controllers can still be applied.

In this article we adapt the control architecture of [5] to control the motion and interaction forces, and combine it with the cascade architecture of [8] and an extension of the impedance controller of [7] to control the environment

contact forces. In the controller, the angle/axis representation is used as angular parametrization [10].

As one of the main contributions, this work complements the asymptotic stability analysis of the interaction force-based impedance controller of [5]. We propose guidelines on how to tune the parameters of the internal force-based impedance relationship. Furthermore, using our object impedance controller, any required contact force can be achieved.

We proceed as follows. First, the kinematics, dynamics and internal force computation are introduced in Section II. Then, we introduce the control architecture in 3 steps, each containing an asymptotic stability analysis. In Section III the design of the motion controller is discussed. The control architecture is extended in Section IV with an internal force-based impedance controller. In Section V we introduce an additional extension to control the external contact forces with the environment. Finally, in Section VI the simulation results of a spatial two manipulator setup are presented.

II. SYSTEM DESCRIPTION

Consider a system of n non-redundant manipulators, manipulating a rigid object. The links of the manipulators are rigid and the joints feature no flexibility. The grasp on the object is tight, such that no degrees of freedom (DOF) exist in the grasp and the manipulators can exert both forces and moments on the object. An example of a two manipulator system is shown in Fig. 1. The kinematics, dynamics and internal force computation are discussed below.

A. Kinematics

The joint angles $\mathbf{q}_i \in \mathbb{R}^6$ of manipulator $i \in \{1, 2, \dots, n\}$ are kinematically related to the end-effector position $\mathbf{x}_{it} \in \mathbb{R}^3$ and orientation, represented by the rotation matrix $\mathbf{R}_i \in SO(3)$. Both $\mathbf{x}_{it}$ and $\mathbf{R}_i$ are expressed relative to the world frame $\mathcal{T}_w$. In these and the following notations, no superscript is used when a quantity is expressed with respect to the world frame. A proper superscript is used only when a matrix or vector is referred to a frame other than the world one.

The translational $\dot{\mathbf{x}}_{it} \in \mathbb{R}^3$ and angular $\boldsymbol{\omega}_i \in \mathbb{R}^3$ velocities are related to the joint velocities $\dot{\mathbf{q}}_i$ by means of the *geometric* manipulator Jacobian $\mathbf{J}_i(\mathbf{q}_i) \in \mathbb{R}^{6 \times 6}$

$$\begin{bmatrix} \dot{\mathbf{x}}_{it} \\ \boldsymbol{\omega}_i \end{bmatrix} = \mathbf{J}_i(\mathbf{q}_i) \dot{\mathbf{q}}_i. \quad (1)$$

The position $\mathbf{x}_{ot} \in \mathbb{R}^3$ and rotation matrix $\mathbf{R}_o \in SO(3)$ of the object fixed frame $\mathcal{T}_o$, expressed in $\mathcal{T}_w$, are related to $\mathbf{x}_{it}$ and $\mathbf{R}_i$ via the object kinematics. The velocities $\dot{\mathbf{x}}_{it}$ and $\boldsymbol{\omega}_i$ are related to the translational $\dot{\mathbf{x}}_{ot} \in \mathbb{R}^3$ and angular

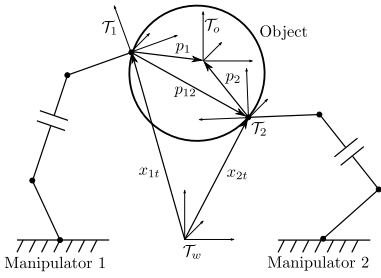


Fig. 1. Two cooperative manipulators handling a spherical object.

$\omega_o \in \mathbb{R}^3$ object velocities by means of the positive definite object Jacobian $J_{oi}(p_i) \in \mathbb{R}^{6 \times 6}$ of manipulator i [5]

$$\begin{bmatrix} \dot{x}_{it} \\ \omega_i \end{bmatrix} = J_{oi}(p_i) \begin{bmatrix} \dot{x}_{ot} \\ \omega_o \end{bmatrix} = \begin{bmatrix} I_3 & \Lambda(p_i) \\ O_3 & I_3 \end{bmatrix} \begin{bmatrix} \dot{x}_{ot} \\ \omega_o \end{bmatrix}. \quad (2)$$

Here, $\Lambda(p_i)$ is a skew symmetric matrix of the virtual stick $p_i = R_i p_i^1 \in \mathbb{R}^3$ (see Fig. 1) [4]. Note that $p_i^1 \in \mathbb{R}^3$ is a constant vector from frame T_i to T_o , expressed in frame T_1 .

B. Dynamics

Consider the equations of motion of each manipulator i

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i - J_i^T(q_i)h_i \quad (3)$$

where $M_i(q_i) \in \mathbb{R}^{6 \times 6}$ denotes the symmetric positive definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^6$ the Coriolis and centrifugal force vector, $g_i(q_i) \in \mathbb{R}^6$ the gravitational force vector, $\tau_i \in \mathbb{R}^6$ the vector of applied joint torques and $h_i = [f_i^T, \mu_i^T]^T \in \mathbb{R}^6$ the vector of *measured* interaction forces $f_i \in \mathbb{R}^3$ and moments $\mu_i \in \mathbb{R}^3$ at the end-effector of manipulator i , both expressed in the world frame.

The models (3) of all manipulators can be combined to

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau - J^T(q)h \quad (4)$$

where $M = \text{diag}(M_1, \dots, M_n)$, $h = [h_1^T, \dots, h_n^T]^T$, $q = [q_1^T, \dots, q_n^T]^T$, etc.

The object dynamics are modeled in the task space as

$$M_o(x_o)\ddot{x}_o + C_o(x_o, \dot{x}_o)\dot{x}_o + g_o(x_o) = h_o \quad (5)$$

with $M_o(x_o) \in \mathbb{R}^{6 \times 6}$ the inertia matrix, $C_o(x_o, \dot{x}_o)\dot{x}_o \in \mathbb{R}^6$ the Coriolis and centrifugal forces, $g_o(x_o) \in \mathbb{R}^6$ the gravity forces, and $h_o = [f_o^T, \mu_o^T]^T \in \mathbb{R}^6$ is the net forces $f_o \in \mathbb{R}^3$ and moments $\mu_o \in \mathbb{R}^3$ acting on the object (frame T_o). The vector h_o is related to the interaction forces h via $h_o = J_o^T(p)h$, where $J_o(p) = [J_{o1}^T(p_1), \dots, J_{on}^T(p_n)]^T \in \mathbb{R}^{6n \times 6}$ and $p = \{p_1, \dots, p_n\}$.

C. Internal force

The interaction forces and moments h can be decomposed into motion inducing forces $h_M = [h_{M1}^T, \dots, h_{Mn}^T]^T \in \mathbb{R}^{6n}$ and internal forces $h_I = [h_{I1}^T, \dots, h_{In}^T]^T \in \mathbb{R}^{6n}$ that do not contribute to the motion of the object [5]

$$\begin{aligned} h &= h_M + h_I, \\ h_M &= (J_o^T(p))^\# J_o^T(p)h, \\ h_I &= (I - (J_o^T(p))^\# J_o^T(p))h. \end{aligned}$$

Here, $(J_o^T(p))^\#$ denotes the generalized inverse of $J_o^T(p)$ and $(J_o^T(p))^\# J_o^T(p) \in \mathbb{R}^{6n \times 6n}$ is not of full rank.

III. MOTION CONTROLLER

The following controller, based on the inverse dynamics approach and the assumed knowledge of the manipulator dynamics, can be applied on each manipulator to achieve asymptotic tracking of the reference trajectories $x_{ir} = \{x_{ir,t} \in \mathbb{R}^3, R_{ir} \in SO(3)\}$ in the Cartesian space, away from kinematic singularities

$$\tau_i = C_i \dot{q}_i + g_i + J_i^T h_i + M_i J_i^{-1} (u_i - \dot{J}_i \dot{q}_i) \quad (6)$$

where J_i^{-1} is the inverse of J_i and u_i is the new control input, computed in the task space (see [10])

$$u_i = \begin{bmatrix} \ddot{x}_{ir,t} + K_{vi,t} \Delta \dot{x}_{it} + K_{pi,t} \Delta x_{it} \\ L_i^{-1} (\dot{L}_i^T \omega_{ir} + L_i^T \dot{\omega}_{ir} - \dot{L}_i \omega_i + K_{vi,\alpha} \dot{\xi}_i + K_{pi,\alpha} \xi_i) \end{bmatrix} \quad (7)$$

where $K_{pi,t}, K_{vi,t}, K_{pi,\alpha}, K_{vi,\alpha} \in \mathbb{R}^{3 \times 3}$ are positive definite matrices. With $\Delta x_{it}, \xi_i \in \mathbb{R}^3$ the translational and rotational error, the error $\Delta x_i \in \mathbb{R}^6$ is expressed by [10]

$$\Delta x_i = \begin{bmatrix} \Delta x_{it} \\ \xi_i \end{bmatrix} = \begin{bmatrix} x_{ir,t} - x_{it} \\ \frac{1}{2} (n_i \times n_{ir} + s_i \times s_{ir} + a_i \times a_{ir}) \end{bmatrix} \quad (8)$$

The rotation matrices can be written as $R_i = [n_i, s_i, a_i]$ and $R_{ir} = [n_{ir}, s_{ir}, a_{ir}]$, with $n_i, s_i, a_i, n_{ir}, s_{ir}$ and a_{ir} denoting the columns of the respective rotation matrices. The matrix $L_i \in \mathbb{R}^{3 \times 3}$ of the angle/axis representation

$$L_i = -\frac{1}{2} (\Lambda(n_{ir})\Lambda(n_i) + \Lambda(s_{ir})\Lambda(s_i) + \Lambda(a_{ir})\Lambda(a_i)) \quad (9)$$

depends on the columns of R_i and R_{ir} , with $\Lambda(z) \in \mathbb{R}^{3 \times 3}$ the skew symmetric matrix of the vector $z \in \mathbb{R}^3$. Note that L_i is nonsingular for $n_i^T n_{ir}, s_i^T s_{ir}, a_i^T a_{ir} > 0$ [10].

Substitution of (6) into (3), using (1) and assuming that J_i is away from kinematic singularities, results after rewriting in the closed-loop error dynamics

$$\Delta \ddot{x}_i + K_{vi} \Delta \dot{x}_i + K_{pi} \Delta x_i = 0 \quad (10)$$

with Δx_i as in (8), $\Delta \dot{x}_i$ and $\Delta \ddot{x}_i$ time derivatives of Δx_i , $K_{pi} = \text{diag}\{K_{pi,t}, K_{pi,\alpha}\}$ and $K_{vi} = \text{diag}\{K_{vi,t}, K_{vi,\alpha}\}$. Defining $\Delta x = [\Delta x_1^T, \dots, \Delta x_n^T]^T \in \mathbb{R}^{6n}$, $K_p = \text{diag}\{K_{p1}, \dots, K_{pn}\} \in \mathbb{R}^{6n \times 6n}$, etc., asymptotic stability of (10) can be proven straightforward with the candidate Lyapunov function

$$V_1 = \frac{1}{2} \Delta \dot{x}^T \Delta \dot{x} + \frac{1}{2} \Delta x^T K_p \Delta x \quad (11)$$

and LaSalle's invariance principle [11].

IV. INTERNAL FORCE-BASED IMPEDANCE CONTROLLER

To prevent damage of the object due to large internal stresses, the internal forces f_I and moments μ_I are controlled by enforcing impedance relationships between the object and each end-effector of the manipulator. The internal force error $\Delta h_{Ii} = h_{Iid} - h_{Ii}$, with $h_{Iid} = [f_{Iid}^T, \mu_{Iid}^T]^T \in \mathbb{R}^6$ the desired internal force, is reduced by computing the reference trajectory x_{ir} for the motion controller (6) from the desired end-effector trajectory $x_{id} =$

$\{x_{id,t} \in \mathbb{R}^3, R_{id} \in SO(3)\}$. The enforced impedance relationship for each manipulator, with $D_i, B_i, S_i \in \mathbb{R}^{6 \times 6}$ positive definite tunable matrices, reads

$$D_i \Delta \ddot{x}_i + B_i \Delta \dot{x}_i + S_i \Delta \tilde{x}_i = \Delta h_{I_i} \quad (12)$$

The error $\Delta \tilde{x}_i$ between the desired x_{id} and reference x_{ir} trajectories of end-effector i on position and velocity level read respectively ($R_{id} = [n_{id}, s_{id}, a_{id}]$)

$$\Delta \tilde{x}_i = \begin{bmatrix} \Delta \tilde{x}_{it} \\ \tilde{\xi}_i \end{bmatrix} = \begin{bmatrix} x_{id,t} - x_{ir,t} \\ \frac{1}{2} (n_{ir} \times n_{id} + s_{ir} \times s_{id} + a_{ir} \times a_{id}) \end{bmatrix} \quad (13)$$

$$\Delta \dot{x}_i = \begin{bmatrix} \Delta \dot{x}_{it} \\ \dot{\xi}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_{id,t} - \dot{x}_{ir,t} \\ \tilde{L}_i^T \omega_{id} - \tilde{L}_i \omega_{ir} \end{bmatrix} \quad (14)$$

where $\dot{x}_{id,t}$ is the translational and ω_{id} the angular velocity of the desired trajectory of the end-effector of manipulator i . The matrix $\tilde{L}_i$ of the angle/axis representation is defined as

$$\tilde{L}_i = -\frac{1}{2} (\Lambda(n_{id})\Lambda(n_{ir}) + \Lambda(s_{id})\Lambda(s_{ir}) + \Lambda(a_{id})\Lambda(a_{ir})). \quad (15)$$

Asymptotic stability of the n -manipulator system (4) with controllers (6) and (12) has been addressed by other authors. For instance, in [5] "the system has been shown to be stable, but not asymptotically stable". In [6] asymptotic stability is also studied, but as is shown in the proof of Lemma 3, the authors did not find the unique solution. Therefore, conditions are derived to guarantee asymptotic stability of the internal force-based impedance controller (12).

Proposition 1. *If the matrices K_{pi}, K_{di}, D_i, B_i and S_i are selected positive definite, and the orientation errors $\tilde{\xi}_i$ are sufficiently small (see proof), then the system described by (10) and (12) is locally asymptotically stable, i.e. $\Delta x, \Delta \tilde{x} \rightarrow 0$ for $t \rightarrow \infty$.*

Proof. See Appendix. $\square$

V. EXTERNAL FORCE-BASED IMPEDANCE CONTROLLER

To prevent damage of the object and the environment due to contact, the control architecture is extended with an impedance relationship between the object and the environment

$$D_o \Delta \ddot{x}_{ocd} + B_o \Delta \dot{x}_{ocd} + S_o \Delta x_{ocd} = \Delta h_{ext,cd}. \quad (16)$$

The contact forces and moments $h_{ext} = [f_{ext}^T, \mu_{ext}^T]^T$ are controlled by computing the *desired* object trajectory $x_{od} = \{x_{od,t} \in \mathbb{R}^3, R_{od} \in SO(3)\}$ from a *commanded* object trajectory $x_{oc} = \{x_{oc,t} \in \mathbb{R}^3, R_{oc} \in SO(3)\}$ and the tunable matrices $D_o, B_o, S_o \in \mathbb{R}^{6 \times 6}$ can be designed to shape the transient dynamics. Using the object kinematics, x_d is then computed from x_{od} . The tracking error Δx_{ocd} and contact force error $\Delta h_{ext,cd}$ are defined as

$$\Delta x_{ocd} = x_{oc} - x_{od} = \begin{bmatrix} x_{oc,t} \\ x_{oc,\alpha} \end{bmatrix} - \begin{bmatrix} x_{od,t} \\ x_{od,\alpha} \end{bmatrix} \quad (17)$$

$$\Delta h_{ext,cd} = h_{ext,c} - h_{ext,d} = \begin{bmatrix} f_{ext,c} \\ \mu_{ext,c} \end{bmatrix} - \begin{bmatrix} f_{ext,d} \\ \mu_{ext,d} \end{bmatrix} \quad (18)$$

We assume that the manipulators operate in a known environment, that is, the direction of contact (modeled by a

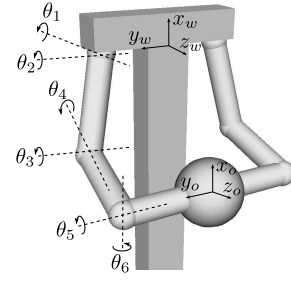


Fig. 2. Cooperative manipulator system.

diagonal selection matrix Σ), damping B_{env} and stiffness S_{env} properties of the environment are known and the contact forces can be modeled as

$$h_{ext,j} = \Sigma S_{env} (x_{oj} - x_{env}) + \Sigma B_{env} (\dot{x}_{oj} - \dot{x}_{env}), \quad (19)$$

with $j \in \{c, d\}$. Computing $h_{ext,cd}$ from (19) yields

$$\Delta h_{ext} = \Sigma S_{env} \Delta x_{ocd} + \Sigma B_{env} \Delta \dot{x}_{ocd}. \quad (20)$$

Substitution into (16) and rewriting, gives

$$D_o \Delta \ddot{x}_{ocd} + (B_o - \Sigma B_{env}) \Delta \dot{x}_{ocd} + (S_o - \Sigma S_{env}) \Delta x_{ocd} = 0. \quad (21)$$

Proposition 2. *If Proposition 1 is satisfied and the matrices $D_o, B_o - \Sigma B_{env}$ and $S_o - \Sigma S_{env}$ are designed positive definite, then the system, described by (4), (6), (12), (16), is asymptotically stable, i.e. $\Delta x, \Delta \tilde{x}, \Delta x_{ocd} \rightarrow 0$ for $t \rightarrow \infty$.*

Proof. If the matrices $D_o, B_o - \Sigma B_{env}$ and $S_o - \Sigma S_{env}$ are selected positive definite, asymptotic stability can be proven straightforward, using (21), (25) and Proposition 1, with the Lyapunov function

$$V_3 = V_2 + \frac{1}{2} \Delta \dot{x}_{ocd}^T D_o \Delta \dot{x}_{ocd} + \frac{1}{2} \Delta x_{ocd}^T (S_o - \Sigma S_{env}) \Delta x_{ocd}$$

Remark 1: An advantage over the object impedance controller of [7], [8] is that the actual contact force is not required for the controller. The idea is that when x_d is tracked by the internal force-based impedance controller (12) and the motion controller (6), h_{ext} converges to $h_{ext,c}$. Furthermore, we can specify a commanded contact force and aim to apply any force on the environment, rather than controlling the contact force to zero as in [7], [8].

Remark 2: In case S_{env} and B_{env} are not known exactly, an over approximation can be used to guarantee asymptotic stability. As long as $B_o - \Sigma B_{env}$ and $S_o - \Sigma S_{env}$ are positive definite, a desired trajectory x_{od} will be computed that reduces $\Delta h_{ext,c}$.

VI. SIMULATIONS

The simulation results are presented in this section, but first a way to tune the impedance parameters of (12) is discussed. Tuning these parameters by hand can be difficult and time consuming. Instead, the procedure of [12] is followed. Here, the desired inertia matrix D_i is chosen to represent the mass and mass moments of inertia felt at the end-effector

$$D_i(q_i) = (J_i(q_i) M_i(q_i)^{-1} J_i(q_i)^T)^{-1} \quad (22)$$

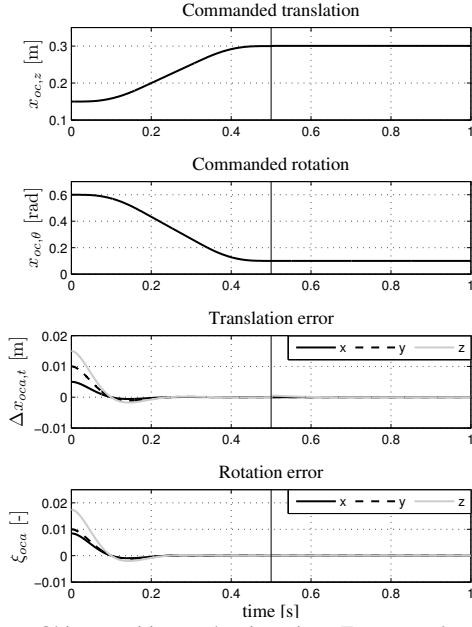


Fig. 3. Object position and orientation. Top two plots: commanded trajectories. Bottom two plots: motion errors. The solid vertical lines at $t = 0.5$ s indicate the time of contact with the environment.

such that convergence to zero is assured with equal rate for all 6 directions of $\Delta \tilde{x}_i$ and $\Delta \tilde{h}_{I_i}$. The desired stiffness S_i and damping B_i matrices are based on D_i via

$$S_i = Q_i S_{i0} Q_i^T \quad (23)$$

$$B_i = 2Q_i B_{i0} S_{i0}^{1/2} Q_i^T \quad (24)$$

where S_{i0} is the diagonal stiffness matrix and B_{i0} is a diagonal matrix containing the damping coefficients ($B_{i0} = I_6$ represents critical damping). These matrices can be tuned such as to achieve the required response. The matrix Q_i follows from $D_i = Q_i Q_i^T$. Note that for the stability analysis of Proposition 1, the matrices D_i , B_i and S_i should be computed once and kept constant during the task.

In our simulation case-study, the cooperative manipulator system consists of two identical 6 DOF manipulators, handling a spherical, rigid object, as is shown in Fig. 2. The manipulators have dimensions similar to an arm of an average adult human. The object has radius $r_o = 0.1$ m and mass $m_o = 0.1$ kg. In order to simulate the cooperative manipulator system, the dynamics of the manipulators (4) and the object (5) are combined with the velocity constraints by following a procedure presented in [5] (Section VII).

The commanded task consists of free motion and constrained motion. During the free motion phase (0-0.5 s), the object is simultaneously rotated and translated to the environment (see top plots of Fig. 3; $x_{oc,\theta}$ is a rotation about y_w of Fig. 2) and compressed with a total internal force of 10 N (see the top plot in Fig. 4 for the contribution of manipulator 1). At $t = 0.5$ s the object makes a contact with the environment, which is located at position $z_{env} = 0.4$ m, and characterized by the stiffness $s_{env} = 1 \cdot 10^4$ N/m and damping constant $b_{env} = 1$ Ns/m. During the constrained motion (0.5-1 s), an external force of 5 N is commanded (second plot of Fig. 4). The motion control gains are selected as $K_{pi} = 700I_6$ and $K_{vi} = 30I_6$. The gains of the internal

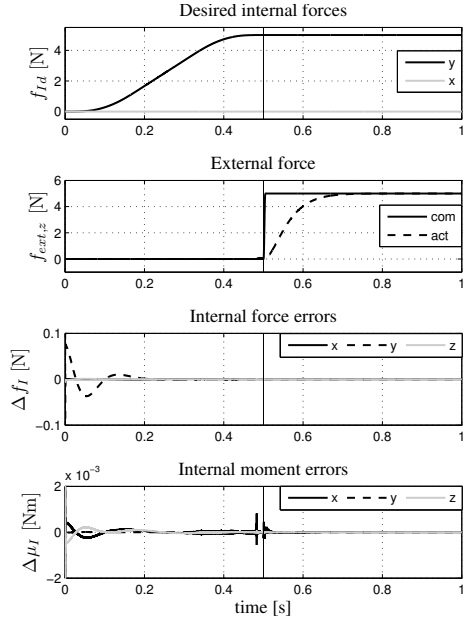


Fig. 4. Object internal and external forces. Top two plots: desired internal and external forces. Bottom two plots: internal force and moment errors. The solid vertical lines at $t = 0.5$ s indicate the time of contact with the environment.

impedance controllers are $S_{i0} = 100I_6$ and $B_{i0} = I_6$ and the gains of the external impedance controller are $D_o = 0.1I_6$, $B_o - \Sigma B_{env} = 6.28I_6$ and $S_o - \Sigma S_{env} = 98.70I_6$. Note that due to the structure of M_i , the matrices D_i , B_i and S_i have nonzero coupling matrices.

The motion errors $\xi_{oca} = \frac{1}{2}(n_o \times n_{oc} + s_o \times s_{oc} + a_o \times a_{oc})$ and $\Delta x_{oca,t} = x_{oc,t} - x_{o,t}$, shown in the bottom plots of Fig. 3, converge to zero. In the bottom plots of Fig. 4, the internal force Δf_I and moment $\Delta \mu_I$ errors are shown. These errors also converge to zero, thus the desired compression of the object is achieved. The small transients at $t = 0.5$ s are due to the damping components in (19): impact of the object with the environment at nonzero velocity results in a discontinuous change of h_{ext} . The commanded and actual external force between object and environment are shown in the second plot of Fig. 4. After contact with the environment is made, the commanded contact force of 5 N is obtained. Note that the contact dynamics are shaped by tuning the impedance parameters in (16).

VII. CONCLUSION

With the proposed control algorithm for cooperative manipulation, the motion, internal and external forces of the object can be controlled. Using impedance relationships, a commanded object trajectory is converted into reference trajectories for the motion controllers of the manipulators such that the desired internal and contact forces can be achieved. In contrast to previously published results, criteria for each controller are determined to guarantee asymptotically stable behavior of the cooperative manipulator system. Guidelines are presented for the internal impedance relationships to compute the control parameters. As a result, all control parameters can be tuned intuitively. The implementation of the control algorithm is illustrated with simulations.

APPENDIX

Proof of Proposition 1. Stability of the controlled system, described by (10) and (12), is investigated with the candidate Lyapunov function

$$V_2 = V_1 + \frac{1}{2} \Delta \dot{\mathbf{x}}^T \mathbf{D} \Delta \dot{\mathbf{x}} + \frac{1}{2} \Delta \tilde{\mathbf{x}}^T \mathbf{S} \Delta \tilde{\mathbf{x}} \quad (25)$$

with $\Delta \tilde{\mathbf{x}} = [\Delta \tilde{\mathbf{x}}_1^T, \dots, \Delta \tilde{\mathbf{x}}_n^T]^T \in \mathbb{R}^{6n}$, and the matrices block diagonal, e.g. $\mathbf{D} = \text{diag}\{\mathbf{D}_1, \dots, \mathbf{D}_n\} \in \mathbb{R}^{6n \times 6n}$. Computing $\dot{V}_2$, using (12), results in

$$\dot{V}_2 = \dot{V}_1 + \Delta \dot{\mathbf{x}}^T [\Delta \mathbf{h}_I - \mathbf{B} \Delta \dot{\mathbf{x}}] \quad (26)$$

Based on (2), the following relation can be obtained

$$\begin{bmatrix} \dot{\mathbf{x}}_{id,t} - \dot{\mathbf{x}}_{ir,t} \\ \boldsymbol{\omega}_{id} - \boldsymbol{\omega}_{ir} \end{bmatrix} = \mathbf{J}_{oi}(\mathbf{p}_{id}) \begin{bmatrix} \dot{\mathbf{x}}_{od,t} \\ \boldsymbol{\omega}_{od} \end{bmatrix} - \mathbf{J}_{oi}(\mathbf{p}_{ir}) \begin{bmatrix} \dot{\mathbf{x}}_{or,t} \\ \boldsymbol{\omega}_{or} \end{bmatrix} \quad (27)$$

with $\mathbf{p}_{ij} = \mathbf{R}_{ij} \mathbf{p}_i^1 \in \mathbb{R}^3$, $j \in \{r, d\}$, the reference and desired virtual sticks. Assuming that the orientation error $\tilde{\xi}_i$ can be kept small ($\mathbf{R}_{id} \approx \mathbf{R}_{ir}$, thus large $\mathbf{D}_i$, $\mathbf{B}_i$ and $\mathbf{S}_i$), (27) can be reduced to (28), and the matrix $\tilde{\mathbf{L}}_i \approx \mathbf{I}$ (see (15)), so that the velocity error $\Delta \dot{\mathbf{x}}_i$ of (14) reduces to (29)

$$\begin{bmatrix} \dot{\mathbf{x}}_{id,t} - \dot{\mathbf{x}}_{ir,t} \\ \boldsymbol{\omega}_{id} - \boldsymbol{\omega}_{ir} \end{bmatrix} = \mathbf{J}_{oi}(\mathbf{p}_{ir}) \begin{bmatrix} \dot{\mathbf{x}}_{od,t} - \dot{\mathbf{x}}_{or,t} \\ \boldsymbol{\omega}_{od} - \boldsymbol{\omega}_{or} \end{bmatrix} \quad (28)$$

$$\Delta \dot{\mathbf{x}}_i = \begin{bmatrix} \dot{\mathbf{x}}_{id,t} - \dot{\mathbf{x}}_{ir,t} \\ \boldsymbol{\omega}_{id} - \boldsymbol{\omega}_{ir} \end{bmatrix}. \quad (29)$$

Combining (28) and (29), and stacking the vectors results in

$$\Delta \dot{\mathbf{x}} = \mathbf{J}_o(\mathbf{p}_r) \begin{bmatrix} \dot{\mathbf{x}}_{od,t} - \dot{\mathbf{x}}_{or,t} \\ \boldsymbol{\omega}_{od} - \boldsymbol{\omega}_{or} \end{bmatrix} \quad (30)$$

where $\mathbf{J}_o(\mathbf{p}_r) = [\mathbf{J}_{o1r}^T(\mathbf{p}_{1r}), \dots, \mathbf{J}_{onr}^T(\mathbf{p}_{nr})]^T \in \mathbb{R}^{6n \times 6}$ and $\mathbf{p}_r = \{\mathbf{p}_{1r}, \dots, \mathbf{p}_{nr}\}$. Substituting (30) in (26) yields

$$\dot{V}_2 = \dot{V}_1 + \begin{bmatrix} \dot{\mathbf{x}}_{od,t} - \dot{\mathbf{x}}_{or,t} \\ \boldsymbol{\omega}_{od} - \boldsymbol{\omega}_{or} \end{bmatrix}^T \mathbf{J}_o^T(\mathbf{p}_r) \Delta \mathbf{h}_I - \Delta \dot{\mathbf{x}}^T \mathbf{B} \Delta \dot{\mathbf{x}}.$$

Since $\Delta \mathbf{h}_I$ lies in the null space of $\mathbf{J}_o^T(\mathbf{p}_r)$ (for small ξ), the second term on the right hand side equals zero and thus

$$\dot{V}_2 = -\Delta \dot{\mathbf{x}}^T \mathbf{K}_v \Delta \dot{\mathbf{x}} - \Delta \dot{\mathbf{x}}^T \mathbf{B} \Delta \dot{\mathbf{x}} \leq 0. \quad (31)$$

Therefore, $\dot{V}_2$ is negative semi-definite, so the controlled system is locally stable, i.e. $\Delta \dot{\mathbf{x}} \rightarrow 0$ for $t \rightarrow \infty$.

Asymptotic stability of the equilibrium $\Delta \tilde{\mathbf{x}} = \mathbf{0}$ of (12) is investigated using LaSalle's invariance principle. Previously it was proven that the equilibrium $\Delta \mathbf{x} = \Delta \dot{\mathbf{x}} = \mathbf{0}$ is asymptotically stable. Premultiplying (12) with $\mathbf{J}_o^T(\mathbf{p}_r)$ and using $\Delta \tilde{\mathbf{x}} = \mathbf{0}$ and $\Delta \dot{\mathbf{x}} = \mathbf{0}$, results in

$$\mathbf{J}_o^T(\mathbf{p}_r) \mathbf{S} \Delta \tilde{\mathbf{x}} = \mathbf{J}_o^T(\mathbf{p}_r) \Delta \mathbf{h}_I = \mathbf{0}, \quad (32)$$

since $\Delta \mathbf{h}_I$ lies in the null space of $\mathbf{J}_o^T(\mathbf{p}_r)$. For convenience, the two-manipulator case is considered below. It is straightforward to extend the results to multiple arms. For the two-manipulator case, (32) reduces to

$$\mathbf{J}_{o1}^T(\mathbf{p}_{1r}) \mathbf{S}_1 \Delta \tilde{\mathbf{x}}_1 + \mathbf{J}_{o2}^T(\mathbf{p}_{2r}) \mathbf{S}_2 \Delta \tilde{\mathbf{x}}_2 = \mathbf{0}. \quad (33)$$

Expression (33) consists of 12 unknowns ($\Delta \tilde{\mathbf{x}}_{1r}$ and $\Delta \tilde{\mathbf{x}}_{2r}$) in 6 equations. Since the manipulators have a tight grasp on the object, there exist kinematic constraints among the

manipulators. From Fig. 1, the relation $\mathbf{x}_{1t} + \mathbf{R}_1 \mathbf{p}_{12}^1 = \mathbf{x}_{2t}$ between the position of the end-effectors $\mathbf{x}_{it}$ can be obtained; $\mathbf{p}_{12}^1 \in \mathbb{R}^3$ denotes a constant vector from frame $\mathcal{T}_1$ to frame $\mathcal{T}_2$, expressed in frame $\mathcal{T}_1$. The same expressions can be obtained for the reference and desired trajectories, leading to the following three position constraints in the error space

$$\Delta \tilde{\mathbf{x}}_{1t} + \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \Delta \tilde{\mathbf{x}}_{2t} \quad (34)$$

with $\Delta \tilde{\mathbf{R}}_1 = \mathbf{R}_{1d} - \mathbf{R}_{1r}$. An important property of $\Delta \tilde{\mathbf{R}}_1$ is that it is not of full rank, but it has rank 2 (see Lemma 1).

The other three constraints can be obtained from the orientations of the end-effectors. Due to the tight grasp of the manipulators on the object (i.e. no DOF exists between object and manipulator), the rotation matrix $\mathbf{R}_2^1 = \mathbf{R}_1^T \mathbf{R}_2$, defining the orientation of frame $\mathcal{T}_2$ w.r.t. frame $\mathcal{T}_1$, is constant and independent of the actual, reference and desired trajectories: $\mathbf{R}_{1r}^T \mathbf{R}_{2r} = \mathbf{R}_2^1 = \mathbf{R}_{1d}^T \mathbf{R}_{2d}$. Following [5], we conclude that the orientation errors of both end-effectors are equal, i.e.

$$\tilde{\xi}_1 = \tilde{\xi}_2, \quad (35)$$

Combining (33), (34) and (35) leads to 12 unknowns in 12 equations

$$\begin{bmatrix} \mathbf{J}_{o1}^T(\mathbf{p}_{1r}) \mathbf{S}_1 & \mathbf{J}_{o2}^T(\mathbf{p}_{2r}) \mathbf{S}_2 \\ \mathbf{I}_6 & -\mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \Delta \tilde{\mathbf{x}}_1 \\ \Delta \tilde{\mathbf{x}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_6 \\ \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 \\ \mathbf{0}_3 \end{bmatrix} = \mathbf{0}_{12}.$$

where the term $\Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1$ is a nonlinear function of the desired and reference orientations of the first end-effector. Due to this nonlinear term, asymptotic stability could not be proven in [5] for an object with nonzero dimensions (i.e. $\mathbf{p}_{12}^1 \neq \mathbf{0}$). To prove asymptotic stability for $\mathbf{p}_{12}^1 \neq \mathbf{0}$, we exploit the result $\Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0}$ of Lemma 3, resulting in

$$\begin{bmatrix} \mathbf{J}_{o1}^T(\mathbf{p}_{1r}) \mathbf{S}_1 & \mathbf{J}_{o2}^T(\mathbf{p}_{2r}) \mathbf{S}_2 \\ \mathbf{I}_6 & -\mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \Delta \tilde{\mathbf{x}}_1 \\ \Delta \tilde{\mathbf{x}}_2 \end{bmatrix} = \mathbf{0}_{12}. \quad (36)$$

The matrix on the left hand side has full rank, since $\mathbf{J}_{oi}$ is positive definite and $\mathbf{S}_i$ is symmetric positive definite. Consequently, the unique solution to (36) is $\Delta \tilde{\mathbf{x}}_i = \mathbf{0}$, such that $\Delta \tilde{\mathbf{x}} = \mathbf{0}$. From LaSalle's invariance principle, asymptotic stability can be concluded: $\Delta \tilde{\mathbf{x}}, \Delta \dot{\mathbf{x}} \rightarrow \mathbf{0}$ for $t \rightarrow \infty$. Furthermore, from (12), it follows that for $t \rightarrow \infty$, also $\Delta \mathbf{h}_I \rightarrow \mathbf{0}$. So, both the desired motion and desired internal force can be achieved. $\square$

A. Lemmas

Lemma 1. Consider two rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, $\mathbf{R}_1 \neq \mathbf{R}_2$. Then, the matrix $\Delta \mathbf{R}_{12} := \mathbf{R}_1 - \mathbf{R}_2 \neq \mathbf{O}$ is always of rank 2.

Proof. By computing the determinant and using scalar products of the columns of $\Delta \mathbf{R}_{12}$, it can be shown that $\Delta \mathbf{R}_{12}$ cannot have rank 3 or 1, and therefore must have rank 2. Due to space limitations, the full proof is omitted. $\square$

Lemma 2. Consider two different rotation matrices $\mathbf{R}_1 = [\mathbf{n}_1, \mathbf{s}_1, \mathbf{a}_1]$ and $\mathbf{R}_2 = [\mathbf{n}_2, \mathbf{s}_2, \mathbf{a}_2]$ and their difference $\Delta \mathbf{R}_{12} := \mathbf{R}_1 - \mathbf{R}_2 = [\mathbf{n}_1 - \mathbf{n}_2, \mathbf{s}_1 - \mathbf{s}_2, \mathbf{a}_1 - \mathbf{a}_2] \neq \mathbf{O}$. Define the orientation error according to the angle/axis representation, $\xi_{12} = \frac{1}{2}(\mathbf{n}_2 \times \mathbf{n}_1 + \mathbf{s}_2 \times \mathbf{s}_1 + \mathbf{a}_2 \times \mathbf{a}_1) \neq \mathbf{0}$. Then, the rotation error ξ_{12} is orthogonal to the columns of $\Delta \mathbf{R}_{12}$ so that $(\Delta \mathbf{R}_{12})^T \xi_{12} = \mathbf{0}$.

Proof. By taking the dot product of the columns of $\Delta \mathbf{R}_{12}$ with ξ_{12} , it can be proven that $(\Delta \mathbf{R}_{12})^T \xi_{12} = \mathbf{0}$. Due to space limitations, the full proof is omitted. $\square$

Property 1. Consider two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and a regular matrix $\mathbf{M} \in \mathbb{R}^{3 \times 3}$. The cross product has the following property under matrix transformations:

$$(\mathbf{M}\mathbf{a}) \times (\mathbf{M}\mathbf{b}) = \det(\mathbf{M})\mathbf{M}^{-T}(\mathbf{a} \times \mathbf{b})$$

Lemma 3. Consider the desired and reference orientation of the end-effector of manipulator 1, defined by $\mathbf{R}_{1d}$ and $\mathbf{R}_{1r}$ respectively. Let the constant vector from frame $\mathcal{T}_1$ to frame $\mathcal{T}_2$, expressed in frame $\mathcal{T}_1$ (see Fig. 1), be denoted by $\mathbf{p}_{12}^1 \neq \mathbf{0}$. Then, the vector $\mathbf{p}_{12}^1$ is projected onto the null space of $\Delta \tilde{\mathbf{R}}_1 = \mathbf{R}_{1d} - \mathbf{R}_{1r}$, so that $\Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0}$.

Proof. Using (2), (13) and

$$\mathbf{S}_i = \begin{bmatrix} \mathbf{S}_{it} & \mathbf{S}_{ic} \\ (\mathbf{S}_{ic})^T & \mathbf{S}_{i\alpha} \end{bmatrix}$$

where $\mathbf{S}_{it}, \mathbf{S}_{i\alpha}, \mathbf{S}_{ic} \in \mathbb{R}^{3 \times 3}$ are the translational, rotational and coupling stiffness matrices, rewrite (33) to

$$\mathbf{S}_{1t} \Delta \tilde{\mathbf{x}}_{1t} + \mathbf{S}_{1c} \tilde{\xi}_1 + \mathbf{S}_{2t} \Delta \tilde{\mathbf{x}}_{2t} + \mathbf{S}_{2c} \tilde{\xi}_2 = \mathbf{0} \quad (37)$$

$$\begin{aligned} & ((\mathbf{S}_{1c})^T - \Lambda(\mathbf{p}_{1r})\mathbf{S}_{1t})\Delta \tilde{\mathbf{x}}_{1t} + (\mathbf{S}_{1\alpha} - \Lambda(\mathbf{p}_{1r})\mathbf{S}_{1c})\tilde{\xi}_1 \\ & + ((\mathbf{S}_{2c})^T - \Lambda(\mathbf{p}_{2r})\mathbf{S}_{2t})\Delta \tilde{\mathbf{x}}_{2t} + (\mathbf{S}_{2\alpha} - \Lambda(\mathbf{p}_{2r})\mathbf{S}_{2c})\tilde{\xi}_2 = \mathbf{0}. \end{aligned} \quad (38)$$

Expressions (37) and (38) denote 6 equations with 12 unknowns. In [6], asymptotic stability was concluded from (37) and (38) for the case $\mathbf{S}_{ic} = \mathbf{0}$, but it can easily be seen that this does not yield the unique solution.

Now, substitute (37) into (38) to obtain an expression with 12 unknowns in 3 equations

$$\begin{aligned} & ((\mathbf{S}_{1c})^T - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1t})\Delta \tilde{\mathbf{x}}_{1t} + (\mathbf{S}_{2c})^T \Delta \tilde{\mathbf{x}}_{2t} \\ & + (\mathbf{S}_{1\alpha} - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1c})\tilde{\xi}_1 + \mathbf{S}_{2\alpha}\tilde{\xi}_2 = \mathbf{0} \end{aligned} \quad (39)$$

where $\Lambda(\mathbf{p}_{1r}) - \Lambda(\mathbf{p}_{2r}) = \Lambda(\mathbf{p}_{1r} - \mathbf{p}_{2r}) = \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)$, since $\mathbf{p}_{1r} = \mathbf{R}_{1r}\mathbf{p}_1^1$ and $\mathbf{p}_{2r} = \mathbf{R}_{1r}\mathbf{p}_2^1$. From (37) follows

$$\Delta \tilde{\mathbf{x}}_{2t} = -(\mathbf{S}_{2t})^{-1}(\mathbf{S}_{1t}\Delta \tilde{\mathbf{x}}_{1t} + \mathbf{S}_{1c}\tilde{\xi}_1 + \mathbf{S}_{2c}\tilde{\xi}_2) \quad (40)$$

Substituting (40), together with $\tilde{\xi}_1 = \tilde{\xi}_2$ from (35), into (39), eliminates the unknowns of $\Delta \tilde{\mathbf{x}}_{2t}$ and $\tilde{\xi}_2$ and results in 6 unknowns ($\Delta \tilde{\mathbf{x}}_{1t}$ and $\tilde{\xi}_1$) in 3 equations

$$\begin{aligned} & ((\mathbf{S}_{1c})^T - (\mathbf{S}_{2c})^T(\mathbf{S}_{2t})^{-1}\mathbf{S}_{1t} \\ & - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1t})\Delta \tilde{\mathbf{x}}_{1t} + (-\Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1c})^T \\ & + \mathbf{S}_{1\alpha} + \mathbf{S}_{2\alpha} - \mathbf{S}_{2c}(\mathbf{S}_{2t})^{-1}(\mathbf{S}_{1c} + \mathbf{S}_{2c})\tilde{\xi}_1 = \mathbf{0}. \end{aligned} \quad (41)$$

Substituting (40) into (34) and rewriting, yields

$$\Delta \tilde{\mathbf{x}}_{1t} = -(\mathbf{S}_{1t} + \mathbf{S}_{2t})^{-1}(\mathbf{S}_{2t}\Delta \tilde{\mathbf{R}}_1\mathbf{p}_{12}^1 + (\mathbf{S}_{1c} + \mathbf{S}_{2c})\tilde{\xi}_1). \quad (42)$$

Substitution of (42) into (41) to eliminate $\Delta \tilde{\mathbf{x}}_{1t}$, results in

$$\begin{aligned} & \Gamma \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 + \Psi \tilde{\xi}_1 = \mathbf{0} \\ \Rightarrow & \tilde{\xi}_1 + \Psi^{-1} \Gamma \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0} \end{aligned} \quad (43)$$

with

$$\begin{aligned} \Gamma = & -((\mathbf{S}_{1c})^T - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1t} \\ & - (\mathbf{S}_{2c})^T(\mathbf{S}_{2t})^{-1}\mathbf{S}_{1t})(\mathbf{S}_{1t} + \mathbf{S}_{2t})^{-1}\mathbf{S}_{2t} \end{aligned} \quad (44)$$

$$\begin{aligned} \Psi = & [\mathbf{S}_{1\alpha} + \mathbf{S}_{2\alpha} - (\mathbf{S}_{2c})^T(\mathbf{S}_{2t})^{-1}(\mathbf{S}_{1c} + \mathbf{S}_{2c}) \\ & - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1c} - ((\mathbf{S}_{1c})^T - \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1)\mathbf{S}_{1t} \\ & - (\mathbf{S}_{2c})^T(\mathbf{S}_{2t})^{-1}\mathbf{S}_{1t})(\mathbf{S}_{1t} + \mathbf{S}_{2t})^{-1}(\mathbf{S}_{1c} + \mathbf{S}_{2c})]. \end{aligned} \quad (45)$$

Expression (43) has 3 unknowns ($\tilde{\xi}_1$) in 3 equations, since $\Delta \tilde{\mathbf{R}}_1 = \Delta \tilde{\mathbf{R}}_1(\tilde{\xi}_1)$. Note that it is always possible to find a combination of $\mathbf{S}_{it}$, $\mathbf{S}_{i\alpha}$ and $\mathbf{S}_{ic}$ that results in the full rank of the square matrices Γ and Ψ . Multiplying (43) with $(\Delta \tilde{\mathbf{R}}_1)^T$ and using Lemma 2, results in

$$(\Delta \tilde{\mathbf{R}}_1)^T \Psi^{-1} \Gamma \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0}. \quad (46)$$

Since $\Delta \tilde{\mathbf{R}}_1$ has rank 2 (see Lemma 1), (46) consists of only two independent equations. Therefore, $\tilde{\xi}_1 = \mathbf{0}$ is not the unique solution to (46). A third independent equation can be obtained, using the well-known property of a skew symmetric matrix $\Lambda(\cdot) \in \mathbb{R}^{3 \times 3}$, i.e. $\mathbf{a}^T \Lambda(\cdot) \mathbf{a} = 0$, where $\mathbf{a} \in \mathbb{R}^3$ is an arbitrary vector. So, the third independent equation reads

$$\begin{aligned} & (\mathbf{p}_{12}^1)^T (\Delta \tilde{\mathbf{R}}_1)^T \Lambda(\mathbf{R}_{1r}\mathbf{p}_{12}^1) \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = 0 \\ \Rightarrow & (\Lambda(\mathbf{R}_{1d}\mathbf{p}_{12}^1)(\mathbf{R}_{1r}\mathbf{p}_{12}^1))^T \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = 0. \end{aligned} \quad (47)$$

The two independent rows of (46) and (47) are combined to

$$\underbrace{\begin{bmatrix} (\mathbf{n}_{1d} - \mathbf{n}_{1r})^T \Psi^{-1} \Gamma \\ (\mathbf{s}_{1d} - \mathbf{s}_{1r})^T \Psi^{-1} \Gamma \\ (\Lambda(\mathbf{R}_{1d}\mathbf{p}_{12}^1)(\mathbf{R}_{1r}\mathbf{p}_{12}^1))^T \end{bmatrix}}_{\mathbf{W}} \Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0}. \quad (48)$$

The rank of $\mathbf{W}$ for $\Delta \tilde{\mathbf{R}}_1 \neq \mathbf{0}$ is investigated by computing the determinant and using Property 1:

$$\begin{aligned} \det(\mathbf{W}) = & -\det(\Psi^{-1} \Gamma) (\mathbf{s}_{1d} - \mathbf{s}_{1r})^T \Lambda(\mathbf{n}_{1d} - \mathbf{n}_{1r}) \\ & (\Gamma^T \Psi^{-T})^{-1} \Lambda(\mathbf{R}_{1d}\mathbf{p}_{12}^1) \mathbf{R}_{1r} \mathbf{p}_{12}^1. \end{aligned}$$

For a proper choice of Γ and Ψ , the terms $(\mathbf{s}_{1d} - \mathbf{s}_{1r})^T \Lambda(\mathbf{n}_{1d} - \mathbf{n}_{1r})$ and $(\Gamma^T \Psi^{-T})^{-1} \Lambda(\mathbf{R}_{1d}\mathbf{p}_{12}^1) \mathbf{R}_{1r} \mathbf{p}_{12}^1$ are linearly independent, so that $\det(\mathbf{W}) \neq 0$. Thus, $\mathbf{W}$ has full rank and is therefore invertible. Consequently, the unique solution of (48) reads $\Delta \tilde{\mathbf{R}}_1 \mathbf{p}_{12}^1 = \mathbf{0}$. $\square$

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