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A New Observer-Based Stabilization Method for Linear Systems with Uncertain Parameters

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***Abstract*— This paper deals with the problem of observer-based stabilization for linear systems with parameter uncertainties. A new design methodology is proposed thanks to a judicious use of the famous Young relation. This latter leads to a less restrictive synthesis condition, expressed in term of Linear Matrix Inequality (LMI), than those available in the literature. A numerical example is provided in order to show the validity and superiority of the proposed method.**

***Index Terms*— Observer-based control; Linear matrix inequalities (LMIs); Lyapunov function; Uncertain linear system.**

I. INTRODUCTION

It is well known that in many practical control systems, the system almost presents some uncertainties and perturbations (see e.g. [1], [2], [3], [4]) because it is very difficult to obtain an exact mathematical model due to environmental noises, data errors, ageing of systems, uncertain or slowly varying parameters, etc. The presence of uncertainties may cause instability and bad performances on a controlled system. Therefore, considerable efforts have been assigned to the robust stability and stabilization of linear systems with parameter uncertainties. For recent works, we refer the readers to [5], [6], [3], [7], [1].

In many real models, state feedback control might fail to guarantee the stabilizability when some of the system states are not measurable. This is why a state observer is required and included in the feedback control [8], [7], [9], [10], [11], [12]. Observer-based controllers are often used to stabilize unstable systems or to improve the system performances. Observer based stabilization problem for both deterministic and stochastic linear systems is well characterized in the pioneer work [13] and [14]. An optimal observer-based control strategy is given in both cases. Nevertheless, for uncertain systems, there is no generic algorithm. Tremendous research activities in the last recent years have been developed for both linear and nonlinear systems with uncertain parameters [5], [15], [16], [17], [18], [19], [4], [20]. However, the resulted methods remain conservatives [21], [22], [3].

Motivated by the above discussions, and in order to derive less conservative results than those obtained in [3], a new design methodology is proposed. By using the Lyapunov function approach combined with a judicious use of the Young's relation, we get a new LMI synthesis methodology. This leads to a quite simple LMI condition that is numerically tractable with any LMI software. It is important to underline that the proposed LMI condition is derived without any additional restrictive conditions, namely the *a priori* choice of the Lyapunov matrix and the equality constraint [3], [2].

Our approach is applied on a single-link flexible robot manipulator in the goal to provide comparisons and to show the superiority of the proposed new design methodology.

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Notations : Throughout this paper, we use the following notations :

- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- $\mathbb{R}^{n\times m}$ is the set of all real n by m matrices;
- I is an identity matrix with approximate dimension and I_r denotes an identity matrix with dimension r;
- The value 0 denotes a zero matrix with approximate dimension, and $\mathbb{O}_{\mathbb{R}^{n\times m}}$ denotes a zero matrix with the dimension n by m;
- for a square matrix S, $S > 0$ $(S < 0)$ means that this matrix is positive definite (negative definite);
- $A \leq B$ means that the matrix $B - A$ is symmetric positive semi definite.

Before giving the formulation of our problem, recall the following basic results [23] which we will used in the proof of our main results :

Lemma 1 (Young's inequality): For given matrices X and Y with appropriate dimensions, we have for any invertible matrix S and scalar $\varepsilon > 0$,

$$X^TY + Y^TX \leq \varepsilon X^TSX + \frac{1}{\varepsilon}Y^TS^{-1}Y. \tag{1}$$

Lemma 2 (Schur Lemma): Let Q_1, Q_2 and Q_3 be three matrices of appropriate dimensions such that $Q_1 = Q_1^T$ and $Q_3 = Q_3^T$. Then,

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} < 0$$

if and only if $Q_3 < 0$ and $Q_1 - Q_2Q_3^{-1}Q_2^T < 0$.

II. PROBLEM FORMULATION

In this section, we introduce the class of linear system with parameters uncertainties to be studied and the proposed observer-based control. Consider a continuous uncertain linear system of the form

$$\dot{x} = \Big(A + \Delta A(t)\Big)x + Bu \tag{2a}$$

$$y = \Big(C + \Delta C(t)\Big)x + Du \tag{2b}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output measurement and $u \in \mathbb{R}^m$ is the control input vector. A, B, C and D are constant matrices of adequate dimensions.

First, we consider the following assumptions :

Assumption 1: Suppose that :

- the pairs (A, B) and (A, C) are respectively stabilizable and detectable;
- there exist matrices $M_i, N_i, F_i(t), i = 1, 2$, of appropriate dimensions so that

$$\Delta A(t) = M_1 F_1(t) N_1, \quad \Delta C(t) = M_2 F_2(t) N_2 \quad (3)$$

where the unknown matrices $F_i(t)$ satisfy the condition

$$F_i^T(t) F_i(t) \leq I, \quad \text{for} \quad i = 1, 2. \quad (4)$$

The observer-based controller that we consider in this paper is under the form :

$$\dot{\hat{x}} = A\hat{x} + Bu + L\Big(y - C\hat{x}\Big) \quad (5a)$$

$$u = -K\hat{x} \quad (5b)$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x, $K \in \mathbb{R}^{m \times n}$ is the control gain, $L \in \mathbb{R}^{n \times p}$ is the observer gain. Hence, we can write

$$\dot{x} = \Big(A - BK + \Delta A(t)\Big)x + BK\varepsilon \quad (6a)$$

$$\dot{\varepsilon} = \Big(\Delta A(t) - \Delta C(t)\Big)x + \Big(A - LC\Big)\varepsilon \quad (6b)$$

or equivalently

$$\overset{\cdot}{\overbrace{\begin{bmatrix} x \\ \varepsilon \end{bmatrix}}} = \begin{bmatrix} \Big(A - BK + \Delta A(t)\Big) & BK \\ \Big(\Delta A(t) - \Delta C(t)\Big) & \Big(A - LC\Big) \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} \quad (7)$$

where $\varepsilon = x - \hat{x}$ represents the estimation error of the system.

Now, consider the Lyapunov function candidate

$$V\left(\begin{bmatrix} x \\ \varepsilon \end{bmatrix}\right) = \begin{bmatrix} x \\ \varepsilon \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} = x^T P x + \varepsilon^T R \varepsilon. \quad (8)$$

Notice that the Lyapunov function (8) is well known in the literature for this problem, especially in [3] which is the main motivation of this paper. Indeed, the main contribution of this note consists to develop a new design methodology that we compare efficiently to the design methods provided in [3]. Now, after calculating the derivative of V along the trajectories of (7), we have:

$$\begin{aligned} \dot{V} \leq\; & x^T \left[\Big(A - BK\Big)^T P + P\Big(A - BK\Big)\right] x \\ & + \varepsilon^T \left[\Big(A - LC\Big)^T R + R\Big(A - LC\Big)\right] \varepsilon + 2x^T PBK\varepsilon \\ & + x^T \left[(\epsilon_1 + \epsilon_2) N_1^T N_1 + \epsilon_3 N_2^T N_2 + \frac{1}{\epsilon_1} P M_1 M_1^T P\right] x \\ & + \varepsilon^T \left[\frac{1}{\epsilon_2} R M_1 M_1^T R + \frac{1}{\epsilon_3} R L M_2 M_2^T L^T R\right] \varepsilon \\ =\; & \begin{bmatrix} x \\ \varepsilon \end{bmatrix}^T \begin{bmatrix} \sum_{11} & PBK \\ (\star) & \sum_{22} \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} \end{aligned} \quad (9)$$

where

$$\begin{aligned} \sum_{11} =\; & \left[\Big(A - BK\Big)^T P + P\Big(A - BK\Big)\right] \\ & + \left[(\epsilon_1 + \epsilon_2) N_1^T N_1 + \epsilon_3 N_2^T N_2 + \frac{1}{\epsilon_1} P M_1 M_1^T P\right] \end{aligned} \quad (10)$$

$$\begin{aligned} \sum_{22} =\; & \left[\Big(A - LC\Big)^T R + R\Big(A - LC\Big)\right] \\ & + \left[\frac{1}{\epsilon_2} R M_1 M_1^T R + \frac{1}{\epsilon_3} R L M_2 M_2^T L^T R\right] \end{aligned} \quad (11)$$

and $\epsilon_1, \epsilon_2, \epsilon_3$ are positive real constants (for more details, see [3], eq. (7)).

Notice that $\dot{V} < 0, \forall \begin{bmatrix} x \\ \varepsilon \end{bmatrix} \neq 0$ if the matrix inequality

$$\begin{bmatrix} \sum_{11} & PBK \\ (\star) & \sum_{22} \end{bmatrix} < 0 \quad (12)$$

holds. However, the matrix inequality (12) is a Bilinear Matrix Inequality (BMI), which is not exploitable numerically. On the other hand, linearizing the BMI (12) is a very difficult task because of the presence of the coupling term PBK. Many researchers in this field have attempt to solve this problem but the resulted methods remain conservatives [3], [2], [16], [18], [5], [1]. In the next section, we recall some available results in the literature and we describe, with details, especially the results of [3] which constitutes the main motivation of this paper.

III. Background Results

Here, we summarize the LMI results given in [3] for asymptotic stability of the system (7).

Theorem 1: *([3])* System (2a) is asymptotically stabilizable by (5) if there exist some positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, a positive definite matrix $R \in \mathbb{R}^{n \times n}$, and $K \in \mathbb{R}^{m \times n}$, $\hat{L} \in \mathbb{R}^{n \times p}$ such that

$$\begin{bmatrix} X_{11} & BK & M_1 & 0 & 0 \\ (\star) & X_{22} & 0 & RM_1 & \hat{L}M_2 \\ (\star) & (\star) & -\epsilon_1 I & 0 & 0 \\ (\star) & (\star) & (\star) & -\epsilon_2 I & 0 \\ (\star) & (\star) & (\star) & (\star) & -\epsilon_3 I \end{bmatrix} < 0 \quad (13)$$

where

$$X_{11} = A^T + A - K^T B^T - BK + (\epsilon_1 + \epsilon_2) N_1^T N_1 + \epsilon_3 N_2^T N_2$$

$$X_{22} = A^T R + RA - \hat{L}C - C^T \hat{L}^T.$$

The stabilizing observer-based control gains are given by K and $L = \hat{R}^{-1}\hat{L}$.

In view of the prove of this theorem, the authors made the particular choice of P, that is $P = I$, in order to linearize the BMI (12). Moreover, to remedy this inconvenience and overcome the obstacle $P = I$ in the linearization of the BMI (12), the author has introduced a new invertible matrix $\hat{P}$ satisfying the condition $PB = B\hat{P}$. By putting $\hat{K} = \hat{P}K$ in (12), he obtained the following theorem.

Theorem 2: ([3]) System (2a) is asymptotically stabilizable by (5) if there exist some positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, two positive definite matrices $P, R \in \mathbb{R}^{n\times n}$, and matrices $\hat{K} \in \mathbb{R}^{m\times n}$, $\hat{L} \in \mathbb{R}^{n\times p}$, $\hat{P} \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} Y_{11} & B\hat{K} & PM_1 & 0 & 0 \\ (\star) & Y_{22} & 0 & RM_1 & \hat{L}M_2 \\ (\star) & (\star) & -\epsilon_1 I & 0 & 0 \\ (\star) & (\star) & (\star) & -\epsilon_2 I & 0 \\ (\star) & (\star) & (\star) & (\star) & -\epsilon_3 I \end{bmatrix} < 0 \tag{14}$$

$$PB = B\hat{P} \tag{15}$$

where

$$Y_{11} = A^T P + PA - \hat{K}^T B^T - B\hat{K} + (\epsilon_1+\epsilon_2)N_1^T N_1 + \epsilon_3 N_2^T N_2$$

$$Y_{22} = A^T R + RA - \hat{L}C - C^T \hat{L}^T.$$

The stabilizing observer-based control gains are given by $K = \hat{P}^{-1}\hat{K}$ and $L = \hat{R}^{-1}\hat{L}$.

Even if Theorem 2 provides less restrictive synthesis conditions than that of Theorem 1, it remains conservative because of the presence of the equality constraint (15). To overcome this difficulty, many research activities have been recently proposed in the literature, but this concern systems in discrete-time case [2], [16], [18], [5], [1]. In the next section, we present a new design methodology for continuous-time systems. We propose a novel manner to overcome the obstacle of the coupling PBK without any equality constraint.

IV. Main Result : New Design Methodology

A. Systems Without Parameter Uncertainties

For simplicity of the presentation and to understand easily the proposed main result, we first consider linear systems without parameter uncertainties. That is $\Delta A(t) = \mathbb{O}_{\mathbb{R}^{n\times n}}$ and $\Delta C(t) = \mathbb{O}_{\mathbb{R}^{p\times n}}$. The technique is based on the use of the Young's relation with an elegant manner in order to obtain a new LMI synthesis methodology. At this stage, we can state the following main theorem.

Theorem 3: System (2a) is asymptotically stabilizable by (5) if for a fixed scalar $\epsilon > 0$, there exist two positive definite matrices $\mathcal{P} \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, and matrices $\hat{K} \in \mathbb{R}^{n\times m}$, $\hat{L} \in \mathbb{R}^{p\times n}$ so that the following LMI condition is feasible:

$$\begin{bmatrix} \overbrace{\begin{bmatrix} \mathbb{P}_1(\mathcal{P},\hat{K}) & 0 \\ 0 & \mathbb{P}_2(R,\hat{L}) \end{bmatrix}}^{\mathcal{Q}_1} & \overbrace{\begin{bmatrix} B\hat{K}^T & 0 \\ 0 & I \end{bmatrix}}^{\mathcal{Q}_2^T} \\ \overbrace{\begin{bmatrix} \hat{K}B^T & 0 \\ 0 & I \end{bmatrix}}^{\mathcal{Q}_2} & \overbrace{\begin{bmatrix} -\frac{1}{\epsilon}\mathcal{P} & 0 \\ 0 & -\epsilon\mathcal{P} \end{bmatrix}}^{\mathcal{Q}_3} \end{bmatrix} < 0 \tag{16}$$

where

$$\mathbb{P}_1(\mathcal{P},\hat{K}) = \mathcal{P}A^T - \hat{K}B^T + A\mathcal{P} - B\hat{K}^T$$

$$\mathbb{P}_2(R,\hat{L}) = A^T R - C^T\hat{L} + RA - \hat{L}^T C.$$

Hence, the stabilizing observer-based control gains are given by $K = \hat{K}^T\mathcal{P}^{-1}$ and $L = \hat{R}^{-1}\hat{L}^T$.

Proof: The aim consists to linearize the bilinear inequality (12). First, notice that from the congruence technique, we have:

$$\Sigma = \begin{bmatrix} \sum_{11} & PBK \\ (\star) & \sum_{22} \end{bmatrix} < 0 \tag{17}$$

is equivalent to

$$\overbrace{\begin{bmatrix} P^{-1} & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} \sum_{11} & PBK \\ (\star) & \sum_{22} \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & I_n \end{bmatrix}}^{\Sigma_{\text{equiv}}} < 0. \tag{18}$$

Detailing the calculations and letting $\mathcal{P} = P^{-1}$, we obtain

$$\Sigma_{\text{equiv}} = \begin{bmatrix} \Sigma_{\text{equiv}}^{11} & BK \\ (BK)^T & \Sigma_{\text{equiv}}^{22} \end{bmatrix},$$

$$\Sigma_{\text{equiv}}^{11} = \mathcal{P}\Big(A - BK\Big)^T + \Big(A - BK\Big)\mathcal{P},$$

$$\Sigma_{\text{equiv}}^{22} = \Big(A - LC\Big)^T R + R\Big(A - LC\Big)$$

which can be rewritten under the following suitable form:

$$\mathcal{Q}_1 + \overbrace{\begin{bmatrix} BK \\ 0 \end{bmatrix}}^{X^T}\overbrace{\begin{bmatrix} 0 & I \end{bmatrix}}^{Y} + \overbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}^{Y^T}\overbrace{\begin{bmatrix} (BK)^T & 0 \end{bmatrix}}^{X} \tag{19}$$

with

$$\mathcal{Q}_1 = \begin{bmatrix} \Sigma_{\text{equiv}}^{11} & 0 \\ 0 & \Sigma_{\text{equiv}}^{22} \end{bmatrix}.$$

From Young's relation, we deduce that

$$\Sigma_{\text{equiv}} \leq \mathcal{Q}_1 + \epsilon X^T S X + \frac{1}{\epsilon} Y^T S^{-1} Y. \tag{20}$$

The key idea to retrieve the variable $\hat{K} = \mathcal{P}K^T$ and eliminating the isolated variable K consists to replace in the Young's relation the matrix S by $\mathcal{P}$. Hence, from the Schur lemma and $S = \mathcal{P}$, the inequality $\Sigma_{\text{equiv}} < 0$ holds if the following one is fulfilled.

$$\begin{bmatrix} \mathcal{Q}_1 & \overbrace{\begin{bmatrix} B(K\mathcal{P}) & 0 \\ 0 & I \end{bmatrix}}^{\mathcal{Q}_2^T} \\ \overbrace{\begin{bmatrix} (K\mathcal{P})^T B^T & 0 \\ 0 & I \end{bmatrix}}^{\mathcal{Q}_2} & \overbrace{\begin{bmatrix} -\frac{1}{\epsilon}\mathcal{P} & 0 \\ 0 & -\epsilon\mathcal{P} \end{bmatrix}}^{\mathcal{Q}_3} \end{bmatrix} < 0. \tag{21}$$

Notice that (21) is identical to (16) using the change of variables $\hat{K} = \mathcal{P}K^T, \hat{L} = L^T R$. That is, under the LMI condition (16) of theorem 3, we have $\Sigma < 0$ which means that the vector $\begin{bmatrix} x \\ \varepsilon \end{bmatrix}$ is asymptotically stable. This ends the proof. ∎

$$
\begin{bmatrix}
\overbrace{\begin{bmatrix} \mathcal{P}A^T - \hat{K}B^T + A\mathcal{P} - B\hat{K}^T & 0 \\ 0 & A^T R - C^T \hat{L} + RA - \hat{L}^T C \end{bmatrix}}^{\mathcal{Q}_1} & \overbrace{\begin{bmatrix} \begin{bmatrix} B\hat{K}^T & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} \mathcal{P}N_1^T & \mathcal{P}N_2^T & M_1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ RM_1 & \hat{L}^T M_2 \end{bmatrix} \end{bmatrix}}^{\mathcal{Q}_2^T} \\
\overbrace{\begin{bmatrix} \begin{bmatrix} \hat{K}B^T & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} N_1\mathcal{P} & 0 \\ N_2\mathcal{P} & 0 \\ M_1^T & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & M_1^T R \\ 0 & M_2^T \hat{L} \end{bmatrix} \end{bmatrix}}^{\mathcal{Q}_2} & \overbrace{\begin{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon_4}\mathcal{P} & 0 \\ 0 & -\epsilon_4 \mathcal{P} \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} -\frac{1}{\epsilon_1+\epsilon_2} I & 0 & 0 \\ 0 & -\frac{1}{\epsilon_3} I & 0 \\ 0 & 0 & -\epsilon_1 I \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -\epsilon_2 I & 0 \\ 0 & -\epsilon_3 I \end{bmatrix} \end{bmatrix}}^{\mathcal{Q}_3}
\end{bmatrix} < 0 \tag{22}
$$

B. Systems With Uncertainties

Theorem 4: System (2a) is asymptotically stabilizable by (5) if for fixed scalars $\epsilon_1 > 0, \epsilon_2 > 0$ and $\epsilon_4 > 0$, there exist two positive definite matrices $\mathcal{P} \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, two matrices $\hat{K} \in \mathbb{R}^{n\times m}$, $\hat{L} \in \mathbb{R}^{p\times n}$ and a positive scalar ϵ_3 so that the LMI condition (22) is feasible.

Hence, the stabilizing observer-based control gains are given by $K = \hat{K}^T \mathcal{P}^{-1}$ and $L = \hat{R}^{-1}\hat{L}^T$,

Proof: We exploit the Young's relation exactly as in the proof of theorem 3. Therefore, we have

$$
\begin{aligned}
\Sigma_{\text{equiv}} &= \begin{bmatrix} \mathcal{P}\sum_{11}\mathcal{P} & BK \\ (\star) & \sum_{22} \end{bmatrix} \\
&\le \begin{bmatrix} \mathcal{P}\sum_{11}\mathcal{P} & 0 \\ (\star) & \sum_{22} \end{bmatrix} + \epsilon_4 \begin{bmatrix} BK \\ 0 \end{bmatrix} \mathcal{P} \begin{bmatrix} BK \\ 0 \end{bmatrix}^T \\
&\quad + \frac{1}{\epsilon_4} \begin{bmatrix} 0 \\ I \end{bmatrix} \mathcal{P}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}^T
\end{aligned} \tag{23}
$$

where $\sum_{11}$ and $\sum_{22}$ are defined in (10) and (11), respectively. Now, using the change of variables $\hat{K} = \mathcal{P}K^T, \hat{L} = L^T R$, the right hand side of (23) can be rewritten under the form

$$\mathcal{Q}_1 - \mathcal{Q}_2\mathcal{Q}_3^{-1}\mathcal{Q}_2^T$$

where $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_3$ are given in (22). Consequently, from Schur lemma, the inequality $\Sigma_{\text{equiv}} < 0$ holds if the LMI condition (22) is feasible.

■

C. On the Necessary Conditions for the Feasibility of (22)

This part is devoted to some remarks about the feasibility of the proposed sufficient LMI condition. A discussion on the necessary conditions for the feasibility of (22) is provided. It should be noticed that the necessary condition for the feasibility of the LMI (22) is $\mathcal{Q}_1 < 0$, which is equivalent to the stabilizability and detectability of the system (2). However, in theorem 1 and theorem 2, the necessary condition for the feasibility of (13) and (14) is more strong than the stabilizability of (A, B). Indeed, to linearize the BMI (12) the authors in [3] chose to take particular solutions, namely $P = I$ in (13) and the additional strong equality constraint (15) in theorem 2.

Indeed, let us consider the following example :

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

$$M_1 = 0,\ M_2 = 0,\ N_1 = 0,\ N_2 = 0$$

If the equality constraint $PB = B\hat{P}$ holds, then we will get $B^{\perp}PB = 0$, where $B^{\perp} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is the orthogonal matrix of B. Let us putting $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ and $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Since $B^{\perp}PB = p_{12} = 0$, then the Lyapunov matrix P must be diagonal.

Now, assume that LMI (14) is satisfied. This leads obviously to the inequality $Y_{11} < 0$, from which we deduce that

$$\Big(A - BK\Big)^T P + P\Big(A - BK\Big) < 0 \tag{24}$$

By developing the calculation, we obtain

$$P\Big(A - BK\Big) = \begin{bmatrix} -p_{11}(1 + k_1) & p_{11}(2 - k_2) \\ 2p_{22} & 3p_{22} \end{bmatrix}.$$

Hence,

$$\Big(A - BK\Big)^T P + P\Big(A - BK\Big) = \begin{bmatrix} -2p_{11}(1 + k_1) & \mathbf{(1.2)} \\ \mathbf{(1.2)}^T & 6p_{22} \end{bmatrix} < 0$$

where

$$\mathbf{(1.2)} = 2p_{22} + p_{11}(2 - k_2)$$

and consequently, $p_{22} < 0$, which contradicts the definition of $P > 0$. On the other hand, the LMI (13) is not solvable since the Lyapunov matrix chosen by the author is $P = I_n$, which is diagonal.

Otherwise, by using Matlab LMI toolbox, our LMI (22) is solvable by choosing $\epsilon_1 = \epsilon_2 = 10$ and $\epsilon_4 = 100$, and we obtain the following solutions :

$$\mathcal{P} = \begin{bmatrix} 2.2124 & -0.2815 \\ -0.2815 & 0.0951 \end{bmatrix}, \quad R = \begin{bmatrix} 25.7343 & -7.6906 \\ -7.6906 & 0.0951 \end{bmatrix},$$

$$\hat{K} = \begin{bmatrix} 53.7083 \\ 1.7287 \end{bmatrix}, \quad \hat{L} = \begin{bmatrix} 35.68 & 12.60 \end{bmatrix}, \quad \epsilon_3 = 134.62,$$

The gains K and L are respectively given by

$$K = \begin{bmatrix} 42.65 & 144.48 \end{bmatrix}, \quad L = \begin{bmatrix} 12.69 \\ 37.84 \end{bmatrix}.$$

Now, consider a more general case. One takes a system so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

The equality constraint $PB = B\hat{P}$ leads to $P_{12} = B^{\perp}PB = 0$ with $B^{\perp} = \begin{bmatrix} 0 & I \end{bmatrix}$, which means that the Lyapunov matrix P is diagonal. On the other hand, the LMI (14) means that

$$\Big(A - BK\Big)^T P + P\Big(A - BK\Big) = \begin{bmatrix} \textbf{(1.1)} & \textbf{(1.2)} \\ & \\ \textbf{(1.2)}^T & \textbf{(2.2)} \end{bmatrix} < 0$$

where

$$\textbf{(1.1)} = P_{11}(A_{11} - B_1K_1) + (A_{11} - B_1K_1)^T P_{11},$$

$$\textbf{(1.2)} = A_{21}^T P_{22} + P_{11}(A_{12} - B_1K_2),$$

$$\textbf{(2.2)} = A_{22}^T P_{22} + P_{22}A_{22}.$$

Hence, A_{22} must be Schur stable to guarantee

$$A_{22}^T P_{22} + P_{22}A_{22} < 0. \tag{25}$$

Notice also that the feasibility of the LMI (13) requires

$$A_{22}^T + A_{22} < 0 \tag{26}$$

since, in particular, we have $P_{22} = I$.

To resume, for this type of systems, in addition to the stabilizability and the detectability, the necessary condition for the feasibility of (13) and (14)-(15) is the Schur stability of A_{22} in the sense of (25) and (26), respectively. This shows the superiority of the proposed design methodology.

V. Application to one link robot Manipulator

Here, let us consider, as real example, the flexible link robot. The dynamic model is nonlinear. The nonlinearity is considered as structured uncertainties. The dynamic of the system is described by the following differential equations :

$$\begin{cases} \dot{\theta}_m = \omega_m \\ \dot{\omega}_m = \frac{\tau}{J_m}(\theta_l - \theta_m) - \frac{b}{J_m}\omega_m + \frac{K_\tau}{J_m}u \\ \dot{\theta}_l = \omega_l \\ \dot{\omega}_l = -\frac{\tau}{J_l}(\theta_l - \theta_m) - \frac{Mgh}{J_l}\sin(\theta_l) \end{cases} \tag{27}$$

where $\theta_m, \omega_m, \theta_l$ and ω_l are the motor and link positions and velocities respectively. J_m and J_l are the motor and link inertia, $2h$ and M are the length and mass of the link, b is the viscous friction and K_τ is the amplifier gain. The measurements are the position and velocity of the motor.

From the mean value theorem, there exists $0 < \eta < 1$ so that the nonlinearity $\sin(\theta_l)$ can be rewritten under the form

$$\sin(\theta_l) = \cos(\eta\theta_l)\theta_l.$$

Therefore, considering the nonlinearity as uncertainty and by setting $x = \begin{bmatrix} \theta_m & \omega_m & \theta_l & \omega_l \end{bmatrix}^T$, the set of equations (27) can be rewritten under the form (2), with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\tau}{J_m} & -\frac{b}{J_m} & \frac{\tau}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\tau}{J_l} & 0 & -\frac{\tau}{J_l} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0$$

$$\Delta A(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{Mgh}{J_l}\cos(\eta\theta_l) & 0 \end{bmatrix}, \quad \Delta C(t) = 0_{\mathbb{R}^{2\times 4}}.$$

The uncertainties can be rewritten under the form (3)-(4) with

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\eta\theta_l) \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{Mgh}{J_l} & 0 \end{bmatrix}$$

$$F_2(t) = 1, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We show through this example that Theorems 1 and 2 cannot provide solutions. Indeed, as shown in section IV-C (equation (25)), if the LMI (14) under the equality constraint (15) is feasible, then the matrix bloc $A_{22} = \begin{bmatrix} 0 & 1.0000 \\ -1.9355 & 0 \end{bmatrix}$ of A must be *Schur* stable. That is, there exists $P_{22} = P_{22}^T = \begin{bmatrix} P_{22}^{11} & P_{22}^{12} \\ P_{22}^{12} & P_{22}^{22} \end{bmatrix} > 0$ so that

$$A_{22}^T P_{22} + P_{22}A_{22} = \begin{bmatrix} -3.87P_{22}^{12} & -1.93P_{22}^{22} + P_{22}^{11} \\ -1.93P_{22}^{22} + P_{22}^{11} & 2P_{22}^{12} \end{bmatrix} < 0$$

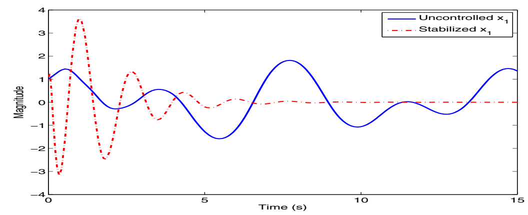
which leads (if satisfied) to $P_{22}^{12} > 0$ and $P_{22}^{12} < 0$, which is contradictory. Notice also that LMI (13) cannot be applied to this real example because we have $P = I$, which is diagonal.

On the other hand, applying our design methodology, after solving the LMI (22), we find the following gains:

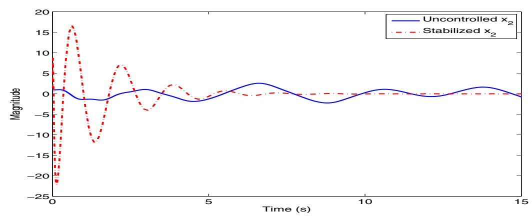
$$K = \begin{bmatrix} 9.8591 & 0.8716 & 173.1857 & 47.4371 \end{bmatrix}$$

$$L = 10^4 \times \begin{bmatrix} 0.0000 & -0.0000 \\ 0.0004 & 0.0049 \\ 0.0060 & 0.0521 \\ 1.4625 & 9.9618 \end{bmatrix}.$$

with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.0101$ and $\epsilon_4 = 0.0638$. The simulation results are given in Figures 1 and 2.

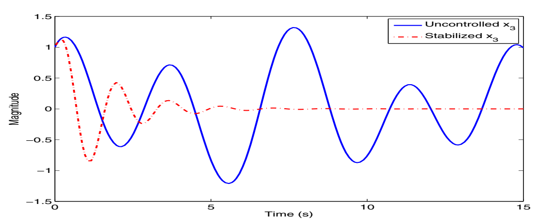


(a) The motor position θ_m

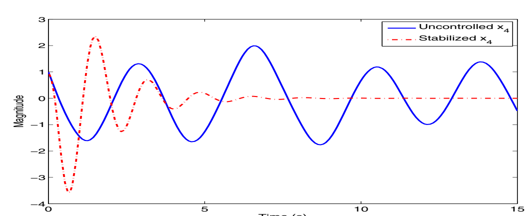


(b) The motor velocity ω_m

Fig. 1. The motor behavior: stabilized states vs uncontrolled states



(a) The link position θ_l



(b) The link velocity ω_l

Fig. 2. The link behavior: stabilized states vs uncontrolled states

VI. Conclusion

In this paper, a linear matrix inequality approach to design observer-based controllers for uncertain linear systems is addressed. We have shown that a judicious use of Young's relation led to a less restrictive LMI condition. A comparison study of the results derived in this work with respect to those given in [3] and [22] shows the superiority of the proposed design methodology.

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