

A distributed randomized algorithm for relative localization in sensor networks

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Abstract—This paper regards the relative localization problem in sensor networks. We propose for its solution a distributed randomized algorithm, which is based on input-driven consensus dynamics and features pairwise “gossip” communications and updates. Due to the randomness of the updates, the state of this algorithm oscillates in time around a certain limit value. We show that the time-average of the state asymptotically converges, in the mean-square sense, to the least-squares solution of the localization problem. Furthermore, we describe an update scheme ensuring that the time-averaging process is accomplished in a fully distributed way.

I. INTRODUCTION

Problems of networked estimation and control have attracted much interest in recent years, due to their potential applications and theoretical challenges. Indeed, such problems require solutions which involve the least amount of coordination and communication between the nodes.

This paper proposes a new distributed and randomized algorithm to solve the *relative localization* problem, which is now defined. We assume to have a group of agents, representing the nodes of a graph, and a vector, indexed over the agents and unknown to them. The agents are allowed to take relative noisy measurements of their vector entries with respect to their neighbors in the graph. The estimation problem consists in reconstructing the original vector, up to an additive constant. While an optimal solution can be easily found by a centralized least-squares approach, in this paper we are interested in finding effective distributed solutions. More precisely, a solution is said to be distributed if it requires each node to use information which is available at the node itself or from its immediate neighbors. Distributed solutions to this problem are deemed to have many potential applications, ranging from clock synchronization [1], to camera network calibration [2], and to formation control in robotic networks [3].

The proposed algorithm involves, at each time step, the activation of a randomly chosen pair of neighboring nodes. This algorithm, which is inspired by a gradient descent approach, requires minimal coordination among the nodes to

be implemented. Our main contribution is to prove that the algorithm converges, in a suitable sense, to the optimal least-squares solution. More precisely, we show that the mean-square error between the time-average of the states and the optimal solution goes asymptotically to zero.

Literature review

The relative localization problem is formulated in [4]–[6]. The paper [4] also presents an overview of potential applications, including spacial localization of robots and time localization of clocks. Regarding applications, we also mention synchronization of camera networks [7], [8] and sensor calibration in wireless sensor networks [9].

As we mentioned, recent research has focused on distributed algorithms to solve this problem. For instance, an algorithm based on Jacobi iterations is proposed in [4]. In the present work, we instead take a gradient descent approach, as in the algorithm studied in [10]. These two algorithms are synchronous, meaning that all nodes are simultaneously activated. However, this synchrony requirement is hard to satisfy in networked applications. A natural way to relax this assumption amounts to introducing randomization in the nodes’ updates.

In the last decade, randomized algorithms have gained attention in the control community [11]. In recent years, many randomized algorithms have been proposed to solve a wide range of problems in networked control and estimation. A significant example is the consensus problem, whose randomized solutions are studied, for instance, in [12]–[17]. Randomized algorithms involving quantization have also been considered, see [18]–[22]. Moreover, consensus-based randomized algorithms have been shown effective to solve more complex problems [23], such as simultaneous estimation and classification [24], and sensor calibration [9]. Other randomized algorithms have been proposed to solve distributed control problems such as optimal deployment and coverage [25], distributed PageRank computation [26], [27] and web-aggregation [28].

More specifically, randomized algorithms for relative localization are studied in two very recent papers [29], [30]. In the former paper, the algorithm we propose is independently derived as a Randomized Kaczmarz Smoothing algorithm: the resulting oscillations are coped with by introducing suitable receding step-sizes. The latter paper instead provides a randomized algorithm which converges exponentially fast and does not oscillate, at the price for the nodes of using a certain amount of memory.

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Paper structure and summary

In Section II, we formally define the relative localization problem and recall a gradient descent algorithm, which is our starting point. In Section III, we propose a pairwise “gossip” algorithm, and state its convergence properties. In Section IV, we study the dynamics of the proposed algorithm, and prove its convergence. In Section V, we present a variation of the algorithm, which avoids using any global clock or other global information in the computation of time-averages, and is thus amenable for implementation.

II. PROBLEM STATEMENT

Let \mathcal{V} be a set of nodes of cardinality N , and a vector $\tilde{x} \in \mathbb{R}^{\mathcal{V}}$ be given. The relative localization problem consists for each node $u \in \mathcal{V}$ in estimating the scalar value \tilde{x}_u , starting from noisy measurements of differences $\tilde{x}_u - \tilde{x}_v$ with certain neighbors v . The available measurements are represented as the edges \mathcal{E} of an oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The orientation of the pairs is conventionally assumed to be such that $(v, u) \in \mathcal{E}$ only if $u < v$. The graph topology is encoded in the incidence matrix $A \in \{0, \pm 1\}^{\mathcal{E} \times \mathcal{V}}$ defined by

$$A_{ew} = \begin{cases} +1 & \text{if } e = (v, w) \\ -1 & \text{if } e = (w, v) \\ 0 & \text{otherwise} \end{cases}$$

for every $e \in \mathcal{E}$. We let $b \in \mathbb{R}^{\mathcal{E}}$ be the vector collecting the measurements:

$$b = A\tilde{x} + \eta,$$

where $\eta \in \mathbb{R}^{\mathcal{E}}$ is random noise with $\mathbb{E}[\eta] = 0$ and $\mathbb{E}[\eta\eta^T] = \sigma^2 I$, and I is the identity matrix. The measurements can also be conveniently arranged in a matrix $B \in \mathbb{R}^{\mathcal{E} \times \mathcal{V}}$ defined as

$$B_{uv} = \begin{cases} b_{(u,v)} & \text{if } u < v \\ -b_{(u,v)} & \text{otherwise.} \end{cases}$$

Note that $B^T = -B$ and $A^T b = B\mathbf{1}$, where $\mathbf{1}$ a vector of 1's of length N . Moreover, $A\mathbf{1} = 0$, which also implies $\mathbf{1}^T B\mathbf{1} = 0$.

We take a least-squares approach for estimating the state x starting from measurements b . That is, we define the following unconstrained quadratic optimization problem

$$\min_z \Psi(z), \quad (1)$$

with $\Psi(z) = \|Az - b\|^2$, where $\|\cdot\|$ is the Euclidean norm. The solution of this problem is summarized in the following well-known lemma [4].

Lemma 1 (Centralized solution): Given a connected graph \mathcal{G} with incidence matrix A , let \mathcal{X} be the set of solutions of (1) and let $L := A^T A$. The following facts hold:

- 1) $x \in \mathcal{X}$ if and only if $A^T Ax = A^T b$;
- 2) there exists a unique $x^* \in \mathcal{X}$ such that $\|x^*\| = \min_{z \in \mathcal{X}} \|z\|$;
- 3) $x^* = L^\dagger A^T b$, where L^\dagger denotes the Moore-Penrose pseudo-inverse of the Laplacian L .

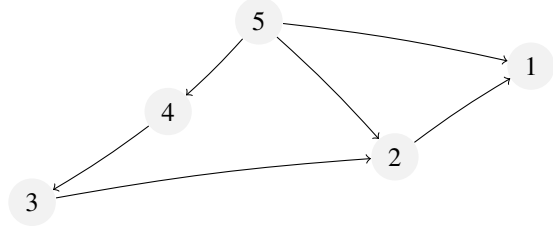


Fig. 1. The network of 5 nodes considered in Example 1.

Note that x^* is the minimum-norm element of the affine space of solutions of (1). Indeed, $Ax^* = A(x^* + c\mathbf{1})$ for all scalar c . We next provide a simple example for illustration.

Example 1: Consider the connected network with $N = 5$ nodes shown in Figure 1. Let $\tilde{x} = [0.1780, 0.5519, 0.1671, 0.4122, 0.9614]^T$ and $\eta = [-0.0155, 0.1005, -0.0865, 0.0470, -0.0278, 0.0243]^T$. Then, the incidence matrix A and the measurements matrix B can be easily constructed:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -0.3894 & 0 & 0 & -0.6829 \\ 0.3894 & 0 & 0.2983 & 0 & -0.3625 \\ 0 & -0.2983 & 0 & -0.2728 & 0 \\ 0 & 0 & 0.2728 & 0 & -0.5249 \\ 0.6829 & 0.3625 & 0 & 0.5249 & 0 \end{bmatrix}$$

Then the vector x^* can be computed as $x^* = [-0.2643, 0.0876, -0.2543, -0.0251, 0.4562]^T$. ■

As shown in Lemma 1, the solution to the least-squares relative localization problem (1) is explicitly known and can be easily computed with a centralized approach. A distributed computation, which relies only on locally available information, is also possible: since the gradient of Ψ is $\nabla\Psi(z) = 2Lz - 2A^T b$, we can use the following gradient descent algorithm. Given a parameter $\tau > 0$ and the initial condition $x(0) = 0$, we let

$$\begin{aligned} x(k+1) &= x(k) - \frac{\tau}{2} \nabla\Psi(x(k)) \\ &= [I - \tau L]x(k) + \tau A^T b. \end{aligned}$$

Defining $P = I - \tau L$ and $y = \tau A^T b$, the algorithm can be rewritten as follows

$$x(k+1) = Px(k) + y. \quad (2)$$

Note that P is doubly stochastic. Then, the algorithm can be seen as an average consensus algorithm evolving with a constant input. The convergence properties of this algorithm are summarized in the following simple result. In this result, and in the rest of this paper unless otherwise stated, the graph \mathcal{G} is assumed to be *connected*. In order to avoid trivialities, we also assume that $|\mathcal{E}| \geq 2$.

Proposition 2: The gradient descent algorithm in (2) with $x(0) = 0$ converges to the optimal least-squares solution x^* if $\tau < 1/d_{\max}$, where d_{\max} is the largest degree in \mathcal{G} .

The algorithm in (2) is distributed and synchronous: a performance analysis is provided in [10]. The algorithm is distributed in the sense that each node only needs for its update to know the states of its neighbors. The algorithm is synchronous in the sense that all nodes update their states at the same time. In this work, we are instead interested in randomized asynchronous algorithms.

III. PAIRWISE RANDOMIZED ALGORITHM

In this paper we propose and study a randomized algorithm involving at each time step communication across one edge only. This section is devoted to illustrate our new algorithm. We have noted above that the algorithm (2) is related very closely to a consensus algorithm. On the other hand, we have also mentioned in our literature review that randomized algorithms, which involve communication across one edge per time step, have been used to solve consensus problems and PageRank computation. Inspired by this connection, we propose the following algorithm.

Let us fix a real number $\gamma \in (0, 1)$. At every time instant $t \in \mathbb{Z}_+$, an edge $(u, v) \in \mathcal{E}$ is selected from the set \mathcal{E} and the states are updated according to the following rule

$$\begin{aligned} x_u(k+1) &= (1-\gamma)x_u(k) + \gamma x_v(k) + \gamma b_{(u,v)} \\ x_v(k+1) &= (1-\gamma)x_v(k) + \gamma x_u(k) - \gamma b_{(u,v)} \\ x_w(k+1) &= x_w(k) \quad \text{if } w \notin \{u, v\}. \end{aligned} \quad (3)$$

The equations in (3) can be rewritten as follows

$$x(k+1) = (I - \gamma E^{(u,v)})x(k) - \gamma(E^{(u,v)} \cdot B)\mathbf{1}$$

where $E^{(u,v)} = (e_u - e_v)(e_u - e_v)^\top$ and the symbol “ \cdot ” denotes the entry-wise product. The process of edge selection is ruled by a (discrete-time) random process $\theta(k) \in \mathcal{E}$. Specifically, the sequence $\theta(k)$ is assumed to be i.i.d., and its probability distribution to be uniform, i.e.,

$$\mathbb{P}[\theta(k) = (u, v)] = \frac{1}{|\mathcal{E}|}, \quad \forall k \in \mathbb{Z}_+. \quad (4)$$

Example 2: We consider the same graph as in Example 1 and let $\gamma = 0.5$. If $\theta(k) = (2, 1)$, then the update matrices are given by

$$E^{(2,1)} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E^{(2,1)} \cdot B = - \begin{bmatrix} 0 & -0.3894 & 0 & 0 & 0 \\ 0.3894 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Correspondingly, the states evolve as

$$\begin{aligned} x_1(k+1) &= 0.5x_1(k) + 0.5x_2(k) - 0.1947 \\ x_2(k+1) &= 0.5x_2(k) + 0.5x_1(k) + 0.1947 \\ x_3(k+1) &= x_3(k) \\ x_4(k+1) &= x_4(k) \\ x_5(k+1) &= x_5(k). \end{aligned}$$

Simulations show that $x(k)$ does not converge, but instead each state randomly oscillates. A closer look, however, reveals that these oscillations concentrate about a mean value. In order to highlight this average value, we must average out the oscillations. It is thus useful to define the time-average of the state, denoted by $\bar{x}(k)$, as the average of the sample path $x(0), \dots, x(k)$, i.e.,

$$\bar{x}(k) = \frac{1}{k+1} \sum_{\ell=0}^k x(\ell). \quad (5)$$

We can show that the time-averages of the states converge in a mean-square sense to the solution of the least-squares problem. This fact is formally stated in the following theorem, which is the main result of this paper.

Theorem 3 (MSE convergence): The algorithm in (3), with uniform node selection (4) and $x(0) = 0$, is such that

$$\lim_{k \rightarrow +\infty} \mathbb{E}[|\bar{x}(k) - x^*|^2] = 0.$$

The convergence result is illustrated by simulations, which we have performed on a complete graph, and are reported in Figure 2.

IV. CONVERGENCE ANALYSIS

This section is devoted to the study of the dynamics (3), culminating in the proof of Theorem 1. We start with a simple result about a class of stochastic matrices, defined from graph Laplacians. We refer the reader to [31] or [3] for similar results. For any square matrix M , we denote $\|M\|_{2,\perp\mathbf{1}} := \sup\{\frac{\|Mv\|}{\|v\|} : v^\top \mathbf{1} = 0\}$.

Lemma 4: Let L be the Laplacian of the oriented graph \mathcal{G} and sort its eigenvalues as $0 = \lambda_0 < \lambda_1 \leq \lambda_{i+1} \leq 2d_{\max}$. Then, for every $c \in (0, 1/d_{\max})$, the matrix $I - cL$ is doubly stochastic and such that

$$\|I - cL\|_{2,\perp\mathbf{1}} = \max\{1 - c\lambda_1, -1 + c\lambda_{N-1}\} < 1.$$

If $c < \frac{2}{\lambda_{N-1} + \lambda_1}$, then $\|I - cL\|_{2,\perp\mathbf{1}} = 1 - c\lambda_1$.

Next, we present the following lemma about the edge selection process, which can be proved by direct computation.

Lemma 5: For the distribution (4), $\forall k \in \mathbb{Z}_+$ it holds

$$\begin{aligned} \mathbb{E}[E^{\theta(k)}] &= \frac{L}{|\mathcal{E}|} \\ \mathbb{E}[E^{\theta(k)} \cdot B] &= -\frac{B}{|\mathcal{E}|} \\ \mathbf{1}^\top \mathbb{E}[(E^{\theta(k)} \cdot B)^2] \mathbf{1} &= -\frac{1}{|\mathcal{E}|} \mathbf{1}^\top (B \cdot B) \mathbf{1} \\ \mathbf{1}^\top \mathbb{E}[(E^{\theta(k)} \cdot B)^\top (I - \gamma E^{\theta(k)})] &= -\frac{(1-2\gamma)}{|\mathcal{E}|} \mathbf{1}^\top B^\top. \end{aligned}$$

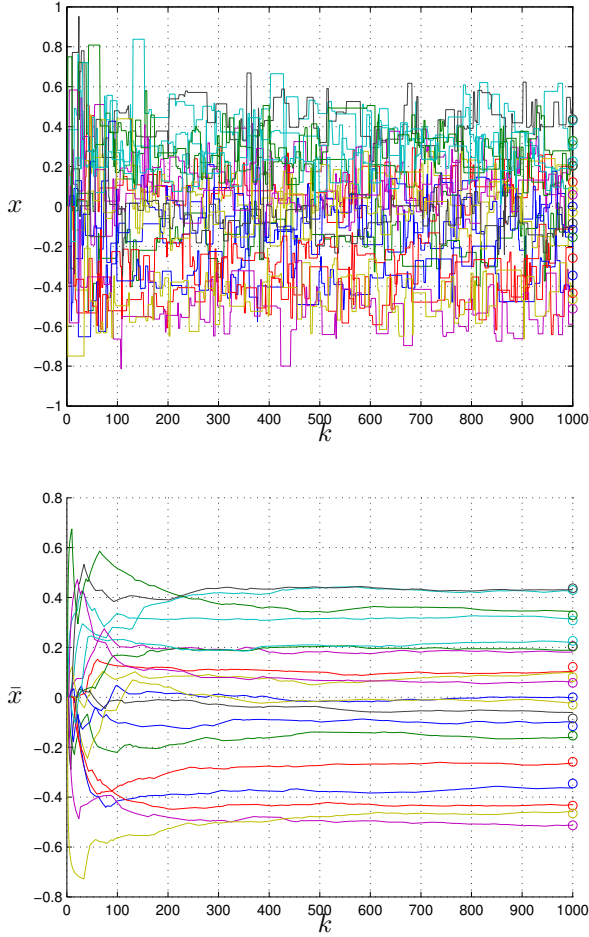


Fig. 2. Complete graph, $N = 20$, zero initial conditions. Upper diagram plots the evolution of algorithm (3) with $\gamma = 0.9$: top plot shows the $x(\cdot)$ variable, bottom plot shows the corresponding time-averages. Circles on the right of each diagram represent x^* for comparison.

These formulas allow us to perform a first order analysis of the system, showing convergence in expectation.

Proposition 6 (Convergence in expectation): Consider the dynamics (3)-(4). Then,

$$\lim_{k \rightarrow +\infty} \mathbb{E}[x(k)] = x^*.$$

Proof: We compute the expected value conditioned upon the previous state as

$$\begin{aligned} \mathbb{E}[x(k+1)|x(k)] &= \mathbb{E} \left[\left(I - \gamma E^{(u,v)} \right) x(k) - \gamma \left(E^{(u,v)} \cdot B \right) \mathbf{1} | x(k) \right] \\ &= \left(I - \mathbb{E} \left[\gamma E^{(u,v)} \right] \right) x(k) - \mathbb{E} \left[\left(\gamma E^{(u,v)} \cdot B \right) \mathbf{1} \right] \\ &= \left(I - \frac{\gamma}{|\mathcal{E}|} L \right) x(k) + \frac{\gamma}{|\mathcal{E}|} B \mathbf{1}, \end{aligned}$$

where the last step follows from the fact that $E^{(u,v)}$ is independent from $x(k)$. We conclude from linearity of ex-

pectation that

$$\mathbb{E}[x(k+1)] = \left(I - \frac{\gamma}{|\mathcal{E}|} L \right) \mathbb{E}[x(k)] + \frac{\gamma}{|\mathcal{E}|} B \mathbf{1}. \quad (6)$$

By solving this recursion, we get that

$$\mathbb{E}[x(k)] = \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^k x(0) + \sum_{\ell=0}^{k-1} \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^\ell \frac{\gamma}{|\mathcal{E}|} B \mathbf{1},$$

and then

$$\lim_{k \rightarrow +\infty} \mathbb{E}[x(k)] = \left(\frac{\gamma}{|\mathcal{E}|} L \right)^\dagger \frac{\gamma}{|\mathcal{E}|} B \mathbf{1} = L^\dagger B \mathbf{1}. \quad \blacksquare$$

Note that by Lemma 4, the proof of Proposition 6 entails a condition $\gamma < \frac{|\mathcal{E}|}{d_{\max}}$. However, the latter inequality is always satisfied as $\gamma < 1$.

Next, we move on to develop a second-order analysis, and we show that the second moment of $x(k)$ is bounded.

Lemma 7: Consider the dynamics (3). There exists a constant $\alpha > 0$ such that

$$\mathbb{E} [\|x(k) - x^*\|^2] \leq \alpha \quad \forall k \in \mathbb{Z}_+.$$

Proof: First, we compute

$$\begin{aligned} \mathbb{E} [\|x(k) - x^*\|^2] &= \mathbb{E} [\|x(k)\|^2] - 2(x^*)^\top \mathbb{E}[x(k)] + \|x^*\|^2 \\ &\leq \mathbb{E} [\|x(k)\|^2] + 2\|x^*\| \mathbb{E} [\|x(k)\|] + \|x^*\|^2. \end{aligned}$$

By Proposition 6, the second term is bounded, so we only have to estimate the first term. To this goal, we compute the expected value of $\|x(k)\|^2$ conditioned upon the previous state

$$\begin{aligned} \mathbb{E} [\|x(k)\|^2 | x(k-1)] &= x(k-1)^\top \mathbb{E} \left[\left(I - \gamma E^{\theta(k-1)} \right)^2 \right] x(k-1) \\ &\quad - \gamma^2 \mathbf{1}^\top \mathbb{E} \left[\left(E^{\theta(k-1)} \cdot B \right)^2 \right] \mathbf{1} \\ &\quad - 2\gamma \mathbb{E} \left[\mathbf{1}^\top \left(E^{\theta(k-1)} \cdot B \right)^\top \left(I - \gamma E^{\theta(k-1)} \right) \right] x(k-1) \\ &= x(k-1)^\top \mathbb{E} \left[I - 2\gamma E^{\theta(k-1)} + 2\gamma^2 E^{\theta(k-1)} \right] x(k-1) \\ &\quad - \gamma^2 \mathbf{1}^\top \mathbb{E} \left[\left(E^{\theta(k-1)} \cdot B \right)^2 \right] \mathbf{1} \\ &\quad - 2\gamma \mathbb{E} \left[\mathbf{1}^\top \left(E^{\theta(k-1)} \cdot B \right)^\top \left(I - \gamma E^{\theta(k-1)} \right) \right] x(k-1) \\ &= x(k-1)^\top \left(I - 2\gamma(1-\gamma) |\mathcal{E}|^{-1} L \right) x(k-1) \\ &\quad + \gamma^2 |\mathcal{E}|^{-1} \mathbf{1}^\top (B \cdot B) \mathbf{1} \\ &\quad + 2\gamma(1-2\gamma) |\mathcal{E}|^{-1} x(k-1)^\top B \mathbf{1}, \end{aligned}$$

where the last equality is obtained from Lemma 5. We deduce that

$$\begin{aligned} \mathbb{E} [\|x(k)\|^2] &= \mathbb{E} [\mathbb{E} [\|x(k)\|^2 | x(k-1)]] \\ &= \mathbb{E} [x(k-1)^\top \left(I - 2\gamma(1-\gamma) |\mathcal{E}|^{-1} L \right) x(k-1)] \\ &\quad + \gamma^2 |\mathcal{E}|^{-1} \mathbf{1}^\top (B \cdot B) \mathbf{1} \\ &\quad + 2\gamma(1-2\gamma) |\mathcal{E}|^{-1} \mathbb{E} [x(k-1)]^\top B \mathbf{1} \\ &\leq \mathbb{E} [x(k-1)^\top \left(I - 2\gamma(1-\gamma) |\mathcal{E}|^{-1} L \right) x(k-1)] \\ &\quad + \gamma^2 |\mathcal{E}|^{-1} \mathbf{1}^\top (B \cdot B) \mathbf{1} \\ &\quad + 2\gamma(1-2\gamma) |\mathcal{E}|^{-1} \| \mathbb{E} [x(k-1)] \| \cdot \| B \mathbf{1} \|, \end{aligned} \quad (7)$$

where the third term of (7) is again bounded thanks to Proposition 6. Since $\frac{2\gamma(1-\gamma)}{|\mathcal{E}|} < \frac{1}{2d_{\max}}$, we argue by Lemma 4 that

$$\begin{aligned} \mathbb{E} [x(k-1)^\top (I - 2\gamma(1-\gamma)|\mathcal{E}|^{-1}L)x(k-1)] \\ \leq (1 - 2\gamma(1-\gamma)\lambda_1|\mathcal{E}|^{-1})\mathbb{E}[\|x(k-1)\|^2], \end{aligned}$$

where $\lambda_1 > 0$ is the second-smallest eigenvalue of L . The statement of the theorem now follows by combining the last inequality with (7). ■

We are now ready to prove our main result.

Proof of Theorem 1: Let the error from the average be $e(k) := x(k) - x^*$. As the update in (3) preserves the average at each iteration, then $e(k)$ is such that $e(k)^\top \mathbf{1} = 0$. Let us observe that

$$\bar{x}(k) - x^* = \frac{1}{k+1} \sum_{\ell=0}^k (x(\ell) - x^*) = \frac{1}{k+1} \sum_{\ell=0}^k e(\ell).$$

We thus have

$$\begin{aligned} \mathbb{E} \|\bar{x}(k) - x^*\|^2 &= \mathbb{E} \left\| \frac{1}{(k+1)} \sum_{\ell=0}^k e(\ell) \right\|^2 \\ &= \frac{1}{(k+1)^2} \sum_{\ell=0}^k \mathbb{E} [e(\ell)^\top e(\ell)] \\ &\quad + 2 \sum_{\ell=0}^k \sum_{r=\ell+1}^k \mathbb{E} [e(\ell)^\top e(r)]. \end{aligned}$$

By Lemma 7 there exists $\alpha \in \mathbb{R}$ such that

$$\frac{1}{(k+1)} \sum_{\ell=0}^k \mathbb{E} [\|e(\ell)\|^2] \leq \alpha \quad \forall k.$$

Then, note that

$$\begin{aligned} \mathbb{E} [e(\ell)^\top e(\ell+r)] &= \mathbb{E} [\mathbb{E} [e(\ell)^\top e(\ell+r) | x(\ell)]] \\ &= \mathbb{E} [e(\ell)^\top \mathbb{E} [e(\ell+r) | x(\ell)]] \quad (8) \\ &= \mathbb{E} [e(\ell)^\top (\mathbb{E} [x(\ell+r) | x(\ell)] - x^*)]. \end{aligned}$$

By repeated conditioning on $x(\ell), \dots, x(\ell+r-1)$ we obtain

$$\begin{aligned} \mathbb{E} [x(\ell+r) | x(\ell)] \\ = \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^r x(\ell) + \frac{\gamma}{|\mathcal{E}|} \sum_{s=0}^{r-1} \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^s B \mathbf{1}, \end{aligned}$$

and by recalling Proposition 6 we see

$$x^* = \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^r x^* + \frac{\gamma}{|\mathcal{E}|} \sum_{s=0}^{r-1} \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^s B \mathbf{1}. \quad (9)$$

From equations (8) and (9) we obtain

$$\mathbb{E} [e(\ell)^\top e(\ell+r)] = \mathbb{E} \left[e(\ell)^\top \left(I - \frac{\gamma}{|\mathcal{E}|} L \right)^r e(\ell) \right] \leq \alpha \rho_{\max}^r,$$

where, by Lemma 4, $\rho_{\max} < 1$. Finally, we have

$$\begin{aligned} \mathbb{E} [\|\bar{x}(k) - x^*\|^2] &\leq \frac{\alpha}{(k+1)^2} \left(k+1 + 2 \sum_{\ell=0}^{k-1} \sum_{r=0}^{k-\ell} \rho_{\max}^r \right) \\ &\leq \frac{\alpha}{(k+1)} \left(1 + \frac{2}{1-\rho_{\max}} \right), \end{aligned}$$

from which we obtain the thesis. ■

V. DISTRIBUTED TIME-AVERAGING

The algorithm in (3) requires time-averaging in order to converge, as stated in Theorem 1. The time-averaging method, as defined in (5), requires each node to know the current discrete time k . Since this time variable actually counts the total number of updates performed so far in the network, we recognize that time-averaging, as defined above, assumes the network to share this global variable. As any global knowledge is hard to maintain in a network, we would prefer to remove such an assumption. This goal is achieved by the following algorithm, in which time-averages are computed according to a *local* counter of the updates. This algorithm thus amounts to a totally distributed and asynchronous implementation of (3) with time-averaging.

Let $\gamma \in (0, 1)$ be fixed, and let the initial condition be a triple of vectors of \mathbb{R}^V , consisting of $x(0)$, $\kappa(0) = \mathbf{1}$ and $\tilde{x}(0) = x(0)$. At every time instant $k \in \mathbb{Z}_+$, an edge $(u, v) \in \mathcal{E}$ is selected from the set \mathcal{E} , according to the law (4). The update is performed according to the following rules: the states update as

$$\begin{aligned} x_u(k+1) &= (1-\gamma)x_u(k) + \gamma x_v(k) + \gamma b_{(u,v)} \\ x_v(k+1) &= (1-\gamma)x_v(k) + \gamma x_u(k) - \gamma b_{(u,v)} \quad (10a) \\ x_w(k+1) &= x_w(k) \quad \text{if } w \notin \{u, v\}; \end{aligned}$$

the local times as

$$\begin{aligned} \kappa_u(k+1) &= \kappa_u(k) + 1 \\ \kappa_v(k+1) &= \kappa_v(k) + 1 \quad (10b) \\ \kappa_w(k+1) &= \kappa_w(k) \quad \text{if } w \notin \{u, v\}; \end{aligned}$$

and the time-averages as

$$\begin{aligned} \tilde{x}_u(k+1) &= \frac{1}{\kappa_u(k+1)} (\kappa_u(k)\tilde{x}_u(k) + x_u(k+1)) \\ \tilde{x}_v(k+1) &= \frac{1}{\kappa_v(k+1)} (\kappa_v(k)\tilde{x}_v(k) + x_v(k+1)) \quad (10c) \\ \tilde{x}_w(k+1) &= \tilde{x}_w(k) \quad \text{if } w \notin \{u, v\}. \end{aligned}$$

Note that in this algorithm, the variable $x(\cdot)$ is just the same as in (3), whereas $\kappa(\cdot)$ and $\tilde{x}(\cdot)$ are new variables: for each $u \in \mathcal{V}$, $\kappa_u(k)$ is the number of updates (among the first k updates), which have involved u , whereas $\tilde{x}_u(k)$ corresponds to an approximation of $\bar{x}_u(k)$. We stress the fact that the nodes need not to know the value of k : the time variable is just introduced for the purpose of a meaningful mathematical definition.

Simulations (see Figure 3) show that the evolution of \tilde{x} approximates \bar{x} well, and lead us to formulate the following conjecture.

Conjecture 1: The algorithm in (10), with uniform node selection (4) and $x(0) = 0$, is such that $\lim_{t \rightarrow +\infty} \mathbb{E}[\|\tilde{x}(k) - x^*\|^2] = 0$.

VI. CONCLUDING REMARKS AND OPEN PROBLEMS

In this paper, we have shown that the relative localization problem can be solved by a distributed ‘‘gossip’’ algorithm, which involves pairwise randomized updates. The states of such an algorithm exhibit oscillations, which can be

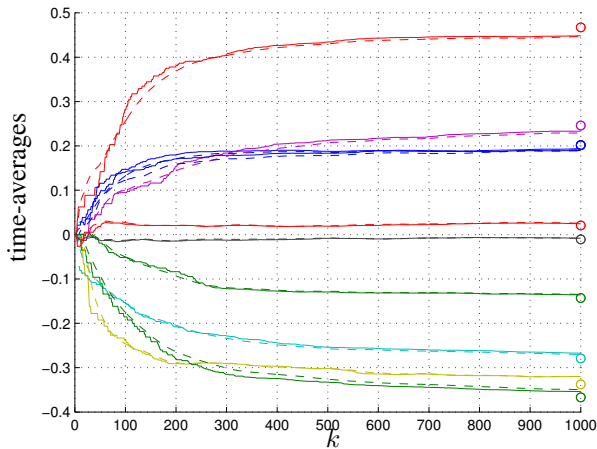


Fig. 3. Complete graph, $N = 20$, $\gamma = 0.1$, zero initial conditions. Dashed lines represent time-averages $\bar{x}(k)$, solid lines represent $\tilde{x}(k)$.

smoothed out by a suitable time-averaging. Indeed, these time-averages converge and can be computed in a completely distributed way. Future work will be devoted to a thorough analysis of the proposed distributed time-averaging method, and to investigate the connections between our localization algorithm and other algorithms which feature ergodic oscillations, such as those for PageRank computation in [26]. Some recent results in the former direction are available in [32].

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