

On the stability and stabilizability of a class of continuous-time positive switched systems with rank one difference

Ettore Fornasini and Maria Elena Valcher

Abstract—Given a single-input continuous-time positive system, described by a pair (A, \mathbf{b}) , with A a diagonal matrix, we investigate under what conditions there exist state-feedback laws $u(t) = \mathbf{c}^\top \mathbf{x}(t)$ that make the resulting controlled system positive and asymptotically stable, namely $A + \mathbf{bc}^\top$ Metzler and Hurwitz. In the second part of the paper we assume that the state-space model switches among different state-feedback laws $\mathbf{c}_i^\top, i = 1, 2, \dots, p$, each of them ensuring the positivity, and show that the asymptotic stability of the switched system is equivalent to the asymptotic stability of all the subsystems, while its stabilizability is equivalent to the existence of an asymptotically stable subsystem.

I. INTRODUCTION

Recent years have seen a growing interest in systems that are subject to a positivity constraint on their dynamical variables. There are several motivations for this interest, coming from different domains of science and technology. In fact, the positivity assumption is a natural one when describing physical, biological or economical processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. [3].

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of continuous-time positive state-space models and a switching law, specifying when and how the switching takes place. Switching among different positive subsystems naturally arises as a way to formalize the fact that the behavior of a positive system changes under different operating conditions, and is therefore represented by different mathematical structures.

Recently, CPSSs have been the object of an intense research activity, mainly focused on stability [4], [7], [9], [11], [12], [13] and stabilizability [1], [2], [19]. Special attention has been devoted to the class of CPSSs that switch among subsystems whose matrices differ by a rank one matrix [8], [12], [13], [14], [15], [16]. The reason for the interest in these systems is twofold. On the one hand, they can be thought of as the possible configurations one obtains from a given single-input system, when applying different state-feedback laws that ensure the positivity of the resulting closed-loop system. For this reason, the subsystem matrices can be denoted by $A + \mathbf{bc}_i^\top, i \in \{1, 2, \dots, p\}$. On the other hand, interesting connections have been highlighted [15] between the quadratic stability of CPSSs, switching between two subsystems of matrices A and $A + \mathbf{bc}^\top$, and the SISO circle criterion for the transfer function $\mathbf{c}^\top (sI_n - A)^{-1} \mathbf{b}$.

Ettore Fornasini and Maria Elena Valcher are with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy, {fornasini, memo}@dei.unipd.it.

In this paper we investigate stability and stabilizability of this class of CPSSs, under the additional assumption that the matrix A is a diagonal one. These dynamic models arise when investigating the behavior of non-homogeneous multi-agent systems, each of them described by a scalar system, evolving under the action of a unique input signal, that coordinates their behavior. If we assume that different state-feedback strategies may be employed to control the overall agent behavior, we naturally end up with this class of rank one CPSSs, having a diagonal system matrix.

We show that the asymptotic stability of these switched systems is equivalent to the asymptotic stability of all their subsystems (i.e., the Hurwitz property of all the matrices $A + \mathbf{bc}_i^\top$). On the other hand, stabilizability is equivalent to the asymptotic stability of at least one subsystem (i.e., existence of an index i such that $A + \mathbf{bc}_i^\top$ is Hurwitz). As a preliminary step, in section II, we consider a continuous-time single-input state space model with diagonal state space matrix A , and address the problem of ensuring the positivity and the asymptotic stability of the resulting system $\dot{\mathbf{x}}(t) = (A + \mathbf{bc}^\top) \mathbf{x}(t)$.

It is worth remarking that when A is not diagonal, but the system matrices differ by a rank one matrix, the characterization provided in this paper for the stability property does not hold true, except for two-dimensional systems (i.e., $n = 2$) [13]. On the other hand, stabilizability of CPSSs that switch among subsystems whose matrices differ by a rank one matrix has not been addressed before in the literature, except in [5], where the main focus, however, was on convex combinations of the subsystem matrices.

Even if the class of CPSSs considered in this paper is definitely a particular one, the obtained characterizations are of interest, and not trivial at all. This is one of the few classes of CPSSs for which the standard necessary condition for stability becomes sufficient, and, conversely, the classical sufficient condition for stabilizability is also necessary. This makes stability and stabilizability analyses quite elementary, as they can be performed by means a simple Routh criterion applied to the characteristic polynomials of the subsystem matrices. It is expected that the results of this analysis lead the way to more general characterizations of stability and stabilizability for rank one CPSSs.

Notation. \mathbb{R}_+ is the semiring of nonnegative real numbers and, for any pair of positive integers k, n with $k \leq n$, $[k, n]$ is the set of integers $\{k, k + 1, \dots, n\}$. The i th entry of a vector \mathbf{v} is denoted by $[\mathbf{v}]_i$. We denote by $\mathbf{1}_n$ the n -dimensional vector with all unitary entries, and by \mathbf{e}_i the

i th canonical vector in \mathbb{R}^n (n being clear from the context). A matrix (in particular, a vector) A with entries in \mathbb{R}_+ is called *nonnegative*, and if so we adopt the notation $A \geq 0$. If, in addition, A has at least one positive entry, the matrix is *positive* ($A > 0$), while if all its entries are positive, it is *strictly positive* ($A \gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. A square matrix A is *Hurwitz* if all its eigenvalues have negative real part. In particular, a Metzler matrix is Hurwitz if and only if the coefficients of its characteristic polynomial are all positive [3].

II. DIAGONAL SYSTEMS AND POSITIVITY PRESERVING STABILIZING STATE-FEEDBACK LAWS

Consider a single-input state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t), \quad t \in \mathbb{R}_+, \quad (1)$$

where $\mathbf{x}(t)$ and $u(t)$ are the n -dimensional state variable and the scalar input, respectively, at time t . We assume

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \lambda_i \in \mathbb{R}.$$

Our first goal is to investigate under what conditions the state feedback law

$$u(t) = \mathbf{c}^\top \mathbf{x}(t)$$

makes the resulting autonomous system positive, by this meaning that the matrix $A + \mathbf{b}\mathbf{c}^\top$ is Metzler. To this end, it is convenient to assume that

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_+ \\ \mathbf{b}_- \\ \mathbf{0} \end{bmatrix}, \quad \text{with } \mathbf{b}_+ \gg 0 \text{ and } \mathbf{b}_- \ll 0. \quad (2)$$

The matrix A and the vector \mathbf{c} can be accordingly partitioned as follows:

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} \quad \mathbf{c}^\top = [\mathbf{c}_1^\top \quad \mathbf{c}_2^\top \quad \mathbf{c}_3^\top]. \quad (3)$$

Notice that we can always reduce ourselves to this situation by resorting to a suitable permutation of the state components. We set $n_+ := \dim \mathbf{b}_+$ and $n_- := \dim \mathbf{b}_-$. Clearly $n_+ + n_- \leq n$.

Since A is diagonal and the Metzler property is a constraint on the off-diagonal entries, $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{b}\mathbf{c}^\top$ is.

Lemma 1: Given two vectors \mathbf{b} and $\mathbf{c} \in \mathbb{R}^n$, partitioned as in (2) and (3), the matrix $\mathbf{b}\mathbf{c}^\top$ is Metzler if and only if the following conditions hold:

Case 1: [$n_+ \geq 1$ and $n_- \geq 1$]

- $\mathbf{c}_1 \leq 0$ if $n_+ = 1$, $\mathbf{c}_1 = 0$ if $n_+ > 1$;
- $\mathbf{c}_2 \geq 0$ if $n_- = 1$, $\mathbf{c}_2 = 0$ if $n_- > 1$;
- if $n_+ + n_- < n$ then $\mathbf{c}_3 = 0$.

Case 2: [$n_+ \geq 1$ and $n_- = 0$]

- \mathbf{c}_1 is arbitrary if $n_+ = 1$, $\mathbf{c}_1 \geq 0$ if $n_+ > 1$;
- if $n_+ < n$ then $\mathbf{c}_3 \geq 0$.

Case 3: [$n_+ = 0$ and $n_- \geq 1$]

- \mathbf{c}_2 is arbitrary if $n_- = 1$, $\mathbf{c}_2 \leq 0$ if $n_- > 1$;
- if $n_- < n$ then $\mathbf{c}_3 \leq 0$.

Proof: We only prove Case 1, as the others are obvious.

[Necessity] Consider the matrix

$$\mathbf{b}\mathbf{c}^\top = \begin{bmatrix} \mathbf{b}_+\mathbf{c}_1^\top & \mathbf{b}_+\mathbf{c}_2^\top & \mathbf{b}_+\mathbf{c}_3^\top \\ \mathbf{b}_-\mathbf{c}_1^\top & \mathbf{b}_-\mathbf{c}_2^\top & \mathbf{b}_-\mathbf{c}_3^\top \\ 0 & 0 & 0 \end{bmatrix}.$$

In order for $\mathbf{b}\mathbf{c}^\top$ to be Metzler, it is necessary that $\mathbf{c}_1 \leq 0, \mathbf{c}_2 \geq 0$. Moreover, the following conditions must simultaneously hold: $\mathbf{c}_3 \geq 0$ and $\mathbf{c}_3 \leq 0$, which implies $\mathbf{c}_3 = 0$. On the other hand, $\mathbf{b}_+\mathbf{c}_1^\top \leq 0$ must be Metzler in turn. So, if $n_+ > 1$ this is possible only if $\mathbf{c}_1 = 0$. By the same reasoning, if $n_- > 1$, then $\mathbf{b}_-\mathbf{c}_2^\top \leq 0$ can be Metzler only if $\mathbf{c}_2 = 0$.

[Sufficiency] The proof is obvious. ■

We now investigate under what conditions the state-feedback law makes the resulting system not only positive but also asymptotically stable, which means that $A + \mathbf{b}\mathbf{c}^\top$ is both Metzler and Hurwitz. To this end, we first address Case 1 of Lemma 1. We distinguish the following subcases:

- a) [$n_+ > 1$ and $n_- > 1$]. If so, $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{c} = 0$, and hence $A + \mathbf{b}\mathbf{c}^\top$ coincides with A . This implies that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler Hurwitz if and only if A is Hurwitz.
- b) [$n_+ > 1$ and $n_- = 1$] or [$n_+ = 1$ and $n_- > 1$]. In the first case, $\mathbf{c}_1 = 0, \mathbf{c}_3 = 0$ and c_2 is a nonnegative scalar. Consequently,

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} A_1 & \mathbf{b}_+c_2 & 0 \\ 0 & A_2 + b_-c_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

is a upper triangular matrix, which is Hurwitz if and only if $A_1, A_2 + b_-c_2$ and A_3 are Hurwitz. The second case is symmetric and leads to a lower triangular matrix

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} A_1 + b_+c_1 & 0 & 0 \\ \mathbf{b}_-c_1 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

that is Hurwitz if and only if $A_1 + b_+c_1, A_2$ and A_3 are Hurwitz.

- c) [$n_+ = 1$ and $n_- = 1$]. In this case, $A + \mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $A + \mathbf{b}\mathbf{c}^\top$ takes the form

$$A + \mathbf{b}\mathbf{c}^\top = \begin{bmatrix} \lambda_1 + b_+c_1 & b_+c_2 & 0 \\ b_-c_1 & \lambda_2 + b_-c_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

with $b_+ > 0, b_- < 0, c_1 \leq 0$ and $c_2 \geq 0$. This matrix is Hurwitz if and only if both A_3 and the 2×2 matrix

$$M := \begin{bmatrix} \lambda_1 + b_+c_1 & b_+c_2 \\ b_-c_1 & \lambda_2 + b_-c_2 \end{bmatrix}$$

are Hurwitz.

Remark 1: In case c) of the previous discussion, if the diagonal matrix A is Hurwitz, then every vector $\mathbf{c} \in \mathbb{R}_+^n$ such that $A + \mathbf{bc}^\top$ is Metzler makes $A + \mathbf{bc}^\top$ Hurwitz. Indeed, the matrix $A + \mathbf{bc}^\top$ takes the form

$$A + \mathbf{bc}^\top = \begin{bmatrix} M & 0 \\ 0 & A_3 \end{bmatrix},$$

where A_3 is Hurwitz diagonal by the assumption on A . On the other hand, if we constrain M to be Metzler, condition $\lambda_1 < 0$ and $\lambda_2 < 0$ ensures that $\text{tr}M = (\lambda_1 + b_+c_1) + (\lambda_2 + b_-c_2) \leq \lambda_1 + \lambda_2 < 0$, while $\det M = \lambda_1\lambda_2 + \lambda_1b_-c_2 + \lambda_2b_+c_1 \geq \lambda_1\lambda_2 > 0$. Consequently, M is Hurwitz.

In order to investigate Cases 2 and 3 of Lemma 1, we first address the case when \mathbf{b} is a strictly positive vector of size $n > 1$.

Proposition 1: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, with $\mathbf{b} \gg 0$, the matrix $A + \mathbf{bc}^\top$ is Metzler and Hurwitz if and only if

- i) $\mathbf{c} \geq 0$;
- ii) A is Hurwitz;
- iii) the following condition holds:

$$1 + \mathbf{c}^\top A^{-1} \mathbf{b} > 0. \quad (4)$$

Proof: If $A + \mathbf{bc}^\top$ is Metzler, then \mathbf{bc}^\top is Metzler and hence, by Lemma 1, $\mathbf{c} \geq 0$. But this implies that $A \leq A + \mathbf{bc}^\top$ and hence, as $A + \mathbf{bc}^\top$ is Hurwitz, then A is Hurwitz, too. Finally, if $A + \mathbf{bc}^\top$ and A are Hurwitz, both $\det(sI - A - \mathbf{bc}^\top)$ and $\det(sI - A)$ are positive for $s = 0$, and condition iii) follows from the identity

$$\det(sI - A - \mathbf{bc}^\top) = \det(sI - A)(1 - \mathbf{c}^\top (sI - A)^{-1} \mathbf{b}).$$

Viceversa, assume that i), ii), and iii) hold. Condition i) ensures that \mathbf{bc}^\top , and hence $A + \mathbf{bc}^\top$ are Metzler. On the other hand, condition ii) implies that the diagonal elements λ_i are negative and, upon setting

$$\alpha_i := -\frac{[\mathbf{c}]_i [\mathbf{b}]_i}{\lambda_i} \geq 0$$

condition iii) is equivalent to saying that $1 - \sum_{i=1}^n \alpha_i > 0$. We therefore have

$$\begin{aligned} & \det(sI_n - A - \mathbf{bc}^\top) \\ &= \det(sI_n - A)(1 - \mathbf{c}^\top (sI_n - A)^{-1} \mathbf{b}) \\ &= \prod_{j=1}^n (s - \lambda_j) \left[1 + \sum_{i=1}^n \frac{\lambda_i \alpha_i}{s - \lambda_i} \right] \\ &= (1 - \sum_{i=1}^n \alpha_i) \prod_{j=1}^n (s - \lambda_j) + \sum_{i=1}^n \alpha_i \prod_{j=1, j \neq i}^n (s - \lambda_j) \\ &+ \sum_{i=1}^n \lambda_i \alpha_i \prod_{j \neq i} (s - \lambda_j) \\ &= (1 - \sum_{i=1}^n \alpha_i) \prod_{j=1}^n (s - \lambda_j) + \sum_{i=1}^n \alpha_i s \prod_{j \neq i} (s - \lambda_j). \end{aligned}$$

So, the coefficients of the characteristic polynomial $\det(sI_n - A - \mathbf{bc}^\top)$ are all positive, and the Metzler matrix $A + \mathbf{bc}^\top$ is necessarily Hurwitz. ■

Remark 2: As proved in Proposition 1, when A is diagonal and $\mathbf{b} \gg 0$, the Hurwitz property of $A + \mathbf{bc}^\top$ implies that A is Hurwitz, too. So, the diagonal entries λ_i of A are all negative and condition iii) can be rewritten as

$$\prod_{i=1}^n (-\lambda_i) > \sum_{i=1}^n [\mathbf{b}]_i [\mathbf{c}]_i \prod_{j \neq i} (-\lambda_j). \quad (5)$$

Remark 3: From the proof of Proposition 1 we can deduce that if A is diagonal Hurwitz, \mathbf{b} is strictly positive and \mathbf{c} is a positive vector such that $A + \mathbf{bc}^\top$ is Metzler Hurwitz, then $A + \mathbf{bc}^\top$ has all negative real eigenvalues. Indeed, from the identity

$$\det(sI_n - A - \mathbf{bc}^\top) = \det(sI_n - A)(1 - \mathbf{c}^\top (sI_n - A)^{-1} \mathbf{b})$$

it is clear that $\lambda \in \sigma(A + \mathbf{bc}^\top)$ if and only if $\lambda \in \sigma(A)$ or $(1 - \mathbf{c}^\top (\lambda I_n - A)^{-1} \mathbf{b}) = 1 - \sum_{i=1}^n \frac{[\mathbf{c}]_i [\mathbf{b}]_i}{\lambda - \lambda_i} = 0$. Clearly, all the eigenvalues of A are real. So, we just need to prove that

$$\sum_{i=1}^n \frac{[\mathbf{c}]_i [\mathbf{b}]_i}{\lambda - \lambda_i} = 1 \quad (6)$$

implies $\lambda \in \mathbb{R}$. Assume that $\lambda = \alpha + j\beta$, $\beta \neq 0$, satisfies (6). Then also $\bar{\lambda} = \alpha - j\beta$ satisfies (6), and hence

$$\sum_{i=1}^n \frac{[\mathbf{c}]_i [\mathbf{b}]_i}{(\alpha - \lambda_i) + j\beta} = 1 = \sum_{i=1}^n \frac{[\mathbf{c}]_i [\mathbf{b}]_i}{(\alpha - \lambda_i) - j\beta}.$$

This implies

$$(-2j\beta) \sum_{i=1}^n \frac{[\mathbf{c}]_i [\mathbf{b}]_i}{(\alpha - \lambda_i)^2 + \beta^2} = 0.$$

But since all terms in the previous summation are nonnegative and at least one of them is strictly positive, this implies $\beta = 0$, a contradiction.

We complete our analysis by putting together the cases $\mathbf{b} > 0$ and $\mathbf{b} < 0$. The derivation of these characterizations is straightforward from the previous proposition and Lemma 1, and hence we omit the proof.

Corollary 1: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}$, $n > 1$, and vectors $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, assume w.l.o.g. that they are described as in (2) and (3), and that either n_+ or n_- is zero (namely, all nonzero entries of \mathbf{b} have the same sign). The matrix $A + \mathbf{bc}^\top$ is Metzler and Hurwitz if and only if

- i) if either $n_+ > 1$ or $n_- > 1$, then the nonzero entries of \mathbf{c} have the same sign as the nonzero entries of \mathbf{b} (namely $\mathbf{bc}^\top \geq 0$), while if \mathbf{b} has a unique nonzero entry then the nonzero entries of \mathbf{c}_3 have the same sign as the nonzero entry of \mathbf{b} ;
- ii) A is Hurwitz;
- iii) condition (4) holds.

III. CONTINUOUS-TIME POSITIVE SWITCHED SYSTEMS WITH RANK ONE DIFFERENCE: STABILITY ANALYSIS

In the rest of the paper we consider continuous-time positive switched systems (CPSSs) described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad t \in \mathbb{R}_+, \quad (7)$$

where $\mathbf{x}(t)$ is the n -dimensional state variable and $\sigma(t)$ the switching sequence at time t . At every time $t \in \mathbb{R}_+$,

$$A_{\sigma(t)} \in \{A + \mathbf{b}\mathbf{c}_1^\top, A + \mathbf{b}\mathbf{c}_2^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\}, \quad (8)$$

with $A \in \mathbb{R}^{n \times n}$, while $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n$, for every $i \in [1, p]$. We assume that A is a diagonal matrix, and that for every index $i \in [1, p]$, the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler. This latter condition ensures that the switched system (7) is positive, by this meaning that if the initial state $\mathbf{x}(0)$ is positive then the whole state trajectory remains in the positive orthant \mathbb{R}_+^n for every choice of the switching sequence. For this class of systems, we want to define and characterize asymptotic stability.

Definition 1: The CPSS (7) is *asymptotically stable* if for every initial state $\mathbf{x}(0) \geq 0$ and every switching sequence $\sigma(t), t \in \mathbb{R}_+$, the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

Clearly, if the stability problem is solvable, then all the system matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, must be (Metzler and) Hurwitz. In this section we want to prove that, when A is a diagonal matrix, the fact that all matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are Hurwitz is, in fact, also sufficient for asymptotic stability.

As a first case, we address the situation when the vector \mathbf{b} has nonzero entries of different signs. We have the following result.

Proposition 2: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, assume w.l.o.g. that A and \mathbf{b} are described as in (2) and each \mathbf{c}_i can be accordingly partitioned as

$$\mathbf{c}_i^\top = [\mathbf{c}_1^{(i)\top} \quad \mathbf{c}_2^{(i)\top} \quad \mathbf{c}_3^{(i)\top}], \quad i \in [1, p]. \quad (9)$$

If $n_+ \geq 1, n_- \geq 1$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are all Metzler and Hurwitz, then the CPSS (7) is asymptotically stable.

Proof: By referring to Lemma 1 and the subsequent discussion, we distinguish the following subcases:

- $[n_+ > 1 \text{ and } n_- > 1]$. If so, we have already seen that each matrix $A + \mathbf{b}\mathbf{c}_i^\top$ can be Metzler Hurwitz if and only if it coincides with A and A is Hurwitz. Therefore it is clear that in this case the Metzler Hurwitz property of the matrices guarantees the asymptotic stability of the CPSS (7).
- $[n_+ > 1 \text{ and } n_- = 1]$ or $[n_+ = 1 \text{ and } n_- > 1]$. We have already noticed that, when the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz, it is also upper triangular (in the first case) or lower triangular (in the second case). But a CPSS whose matrices are all Hurwitz and in the same

kind of triangular form is necessarily asymptotically stable [10].

- $[n_+ = 1 \text{ and } n_- = 1]$. In this case, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $A + \mathbf{b}\mathbf{c}_i^\top$ takes the form

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} M_i & 0 \\ 0 & A_3 \end{bmatrix}, \quad (10)$$

where

$$M_i := \begin{bmatrix} \lambda_1 + b_+c_1^{(i)} & b_+c_2^{(i)} \\ b_-c_1^{(i)} & \lambda_2 + b_-c_2^{(i)} \end{bmatrix}, \quad i \in [1, p], \quad (11)$$

and A_3 are all Hurwitz matrices, $b_+ > 0, b_- < 0, c_1 \leq 0$ and $c_2 \geq 0$. If the matrices $M_i, i \in [1, p]$, are all Metzler Hurwitz, then (see Proposition 3 in [5]) all their convex combinations are Metzler Hurwitz, in turn, and this ensures [7] that the two-dimensional CPSS

$$\dot{\mathbf{z}}(t) = M_{\sigma(t)}\mathbf{z}(t), \quad M_{\sigma(t)} \in \{M_1, M_2, \dots, M_p\}, \quad (12)$$

is asymptotically stable. Therefore, as the two-dimensional system (12) is asymptotically stable and A_3 is Hurwitz, the CPSS system (7) described by the matrices (10) is asymptotically stable. \blacksquare

We now address the case when all the nonzero entries of the vector \mathbf{b} have the same sign.

Proposition 3: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, assume w.l.o.g. that A and \mathbf{b} are described as in (2) and each \mathbf{c}_i is accordingly partitioned as in (9). If either $n_+ = 0$ or $n_- = 0$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are all Metzler and Hurwitz, then the CPSS (7) is asymptotically stable.

Proof: Consider, first, the case when $\mathbf{b} \gg 0$ (namely $n = n_+ > 1$ and $n_- = 0$). By Proposition 1, if $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler Hurwitz and $\mathbf{b} \gg 0$, then $\mathbf{c}_i \geq 0$, A is Hurwitz and condition (4) holds, namely $1 + \mathbf{c}_i^\top A^{-1}\mathbf{b} > 0$. Set $\mathbf{w} := -A^{-1}\mathbf{b}$. It is easy to see that $\mathbf{w} \gg 0$ and that

$$(A + \mathbf{b}\mathbf{c}_i^\top)\mathbf{w} = -(1 + \mathbf{c}_i^\top A^{-1}\mathbf{b})\mathbf{b} \ll 0, \quad \forall i \in [1, p].$$

This ensures [4], [9] that the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}^\top \mathbf{x}(t), \quad A_{\sigma(t)} \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable. But then, for each choice of the switching sequence, the product of the matrix exponentials converges to the zero matrix and so does the transposed system, thus ensuring that also the positive switched system

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad A_{\sigma(t)} \in \{A + \mathbf{b}\mathbf{c}_1^\top, \dots, A + \mathbf{b}\mathbf{c}_p^\top\},$$

is asymptotically stable.

Consider now the case when $\mathbf{b} > 0$ has some zero entries. In this case A, \mathbf{b} and the vectors \mathbf{c}_i become

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{b}_+ \\ 0 \end{bmatrix}, \quad \mathbf{c}_i = \begin{bmatrix} \mathbf{c}_1^{(i)} \\ \mathbf{c}_3^{(i)} \end{bmatrix}, \quad \text{with } \mathbf{b}_+ \gg 0,$$

where $\dim \mathbf{b}_+ = \dim A_1 = \dim \mathbf{c}_1^{(i)} = n_+$. Consequently,

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 + \mathbf{b}_+\mathbf{c}_1^{(i)\top} & \mathbf{b}_+\mathbf{c}_3^{(i)\top} \\ 0 & A_3 \end{bmatrix}.$$

If $n_+ > 1$, set $\mathbf{w}_1 := -A_1^{-1}\mathbf{b}_+$. By the same reasoning as in the first part of the proof, we can claim that $\mathbf{w}_1 \gg 0$. Also, for every $\epsilon > 0$, $\mathbf{w}_2 := \epsilon\mathbf{1}_{n-n_+} \gg 0$. It is easy to see that, for a suitably small value of $\epsilon > 0$, the quantity $\alpha := -(1 + \mathbf{c}_1^{(i)\top}A_1^{-1}\mathbf{b}_+) + \epsilon(\mathbf{c}_2^{(i)\top}\mathbf{1}_{n-n_+})$ is negative, and hence the vector

$$(A + \mathbf{b}\mathbf{c}_i^\top) \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{b}_+ \\ \epsilon(A_2\mathbf{1}_{n-n_+}) \end{bmatrix}$$

is strictly negative. Again, this proves that system (7) is asymptotically stable. If $n_+ = 1$, we can come to the same conclusion by using $\mathbf{w}^\top = [1 \ \epsilon\mathbf{1}_{n-1}^\top]$.

The cases $\mathbf{b} \ll 0$ and $\mathbf{b} < 0$ can be addressed along the same lines. ■

By putting together Propositions 2 and 3, we finally get the following result.

Theorem 1: Let $A \in \mathbb{R}^{n \times n}$ be a diagonal matrix, and let $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, be vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1, p]$. The following facts are equivalent:

- i) $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz for every index $i \in [1, p]$;
- ii) the CPSS (7) is asymptotically stable.

IV. CPPSS WITH RANK ONE DIFFERENCE: STABILIZABILITY ANALYSIS

Definition 2: The CPSS (7) is *stabilizable* if for every positive initial state $\mathbf{x}(0)$ there exists a switching sequence $\sigma(t), t \in \mathbb{R}_+$, such that the state trajectory $\mathbf{x}(t), t \in \mathbb{Z}_+$, converges to zero.

In the general case, the stabilization problem is solvable if at least one of the system matrices is Hurwitz. More generally, if there exist $\alpha_i, i \in [1, p], 0 \leq \alpha_i \leq 1$, such that the convex combination $\sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)$ is (Metzler and) Hurwitz, then the system is stabilizable [18]. For matrices that differ by a rank one matrix, these two sufficient conditions for stabilizability are in fact equivalent.

Proposition 4: [6] Let $A \in \mathbb{R}^{n \times n}$ be a Metzler matrix, and assume that $\mathbf{b}, \mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, are column vectors such that the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are Metzler. There exist $\alpha_i, i \in [1, p], 0 \leq \alpha_i \leq 1$, with $\sum_{i=1}^p \alpha_i = 1$, such that $\sum_{i=1}^p \alpha_i(A + \mathbf{b}\mathbf{c}_i^\top)$ is Hurwitz if and only if there exists $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

By making use of the previous result, we can provide an important characterization of the stabilizability property for the class of CPSSs described as in (7), under the additional assumption that A is a diagonal matrix. As in the previous section, we have to address separately the case when \mathbf{b} has

entries of opposite signs and the case when the nonzero entries have all the same sign.

Proposition 5: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, assume w.l.o.g. that A and \mathbf{b} are described as in (2) and each \mathbf{c}_i is accordingly partitioned as in (9). If $n_+ \geq 1, n_- \geq 1$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are all Metzler, then the CPSS (7) is stabilizable if and only if there exists an index $i \in [1, p]$ such that the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Proof: Sufficiency is obvious, so we only prove necessity. As in the proof of Proposition 2, we proceed by considering all possible subcases of Case 1 enlightened in Lemma 1 and following discussion (see Section II):

- a) $[n_+ > 1 \text{ and } n_- > 1]$. If so, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $\mathbf{c}_i = 0$, and hence all matrices $A + \mathbf{b}\mathbf{c}_i^\top$ coincide with A . So, stabilizability requires that all matrices $A + \mathbf{b}\mathbf{c}_i^\top = A$ are Metzler Hurwitz.
- b) $[n_+ > 1 \text{ and } n_- = 1]$ or $[n_+ = 1 \text{ and } n_- > 1]$. In the first case, $\mathbf{c}_1^{(i)} = 0, \mathbf{c}_3^{(i)} = 0$ and $c_2^{(i)}$ is a nonnegative scalar. Consequently, all matrices

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 & \mathbf{b}_+c_2^{(i)} & 0 \\ 0 & A_2 + b_-c_2^{(i)} & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

are upper triangular, and the system is stabilizable if and only if A_1 and A_3 are Hurwitz and there exists at least one index $i \in [1, p]$ such that $A_2 + b_-c_2^{(i)} < 0$. But this means that there exists an index $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. The second case is symmetric (see the proof of Proposition 2).

- c) $[n_+ = 1 \text{ and } n_- = 1]$. In this case, $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler if and only if $A + \mathbf{b}\mathbf{c}_i^\top$ takes the form (10), with the matrix M_i described as in (11). Clearly, the CPSS system (7) with such matrices is stabilizable if and only if A_3 is Hurwitz and the two-dimensional CPSS

$$\dot{\mathbf{z}}(t) = M_{\sigma(t)}\mathbf{z}(t), \quad M_{\sigma(t)} \in \{M_1, M_2, \dots, M_p\},$$

is stabilizable. But for a two-dimensional CPSS, stabilizability is equivalent [1], [2] to the existence of a Hurwitz convex combination of the matrices. By Proposition 4, this implies that at least one of M_i 's is Hurwitz. Therefore there exists an index $i \in [1, p]$ such that the matrix $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz. ■

Proposition 6: Given a diagonal matrix $A \in \mathbb{R}^{n \times n}, n > 1$, and vectors $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, assume w.l.o.g. that A and \mathbf{b} are described as in (2) and each \mathbf{c}_i is accordingly partitioned as in (9). If either $n_+ = 0$ or $n_- = 0$, and the matrices $A + \mathbf{b}\mathbf{c}_i^\top, i \in [1, p]$, are all Metzler, then the CPSS (7) is asymptotically stable if and only if there exists an index $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Proof: Again, we only need to prove the necessity. Consider, first, the case when $\mathbf{b} \gg 0$. We preliminary notice that, by Lemma 1, if $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler and $\mathbf{b} \gg 0$, then

$\mathbf{c}_i \geq 0$, and this ensures that $A + \mathbf{b}\mathbf{c}_i^\top \geq A$. If the system is stabilizable then [2] it is consistently stabilizable, by this meaning that there exists a switching sequence $\bar{\sigma} : \mathbb{R}_+ \rightarrow [1, p]$ that asymptotically drives to zero every initial state. This implies [17] that there exists a periodic switching signal that asymptotically stabilizes the switched system, namely that there exist $r \in \mathbb{Z}_+$, $i_1, i_2, \dots, i_r \in [1, p]$, $\tau_1, \tau_2, \dots, \tau_r \in \mathbb{R}_+$, such that

$$Z := e^{(A+\mathbf{b}\mathbf{c}_{i_1}^\top)\tau_1} e^{(A+\mathbf{b}\mathbf{c}_{i_2}^\top)\tau_2} \dots e^{(A+\mathbf{b}\mathbf{c}_{i_r}^\top)\tau_r}$$

is Schur. Set

$$Z_1 := e^{A\tau_1} e^{A\tau_2} \dots e^{A\tau_r}.$$

It is not difficult to prove that $Z \geq Z_1$. This implies that Z_1 must be Schur, too, and hence the diagonal matrix A must be Hurwitz. Since A is diagonal Hurwitz and $\mathbf{b} \gg 0$, by Proposition 1, if each $A + \mathbf{b}\mathbf{c}_i^\top$ were not Hurwitz then it should be $1 + \mathbf{c}_i^\top A^{-1} \mathbf{b} \leq 0, \forall i \in [1, p]$. Set $\mathbf{w} := -A^{-1} \mathbf{b} \gg 0$ and note, again, that

$$(A + \mathbf{b}\mathbf{c}_i^\top) \mathbf{w} = -(1 + \mathbf{c}_i^\top A^{-1} \mathbf{b}) \mathbf{b}.$$

So, if none of the system matrices were Hurwitz, there would be a (strictly) positive vector \mathbf{w} such that

$$(A + \mathbf{b}\mathbf{c}_i^\top) \mathbf{w} \geq 0, \quad \forall i \in [1, p],$$

thus preventing stabilizability [6]. So, there must be an index $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz.

Consider now the case when $\mathbf{b} > 0$ has some zero entries, and hence

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} \mathbf{b}_+ \\ 0 \end{bmatrix}, \mathbf{c}_i = \begin{bmatrix} \mathbf{c}_1^{(i)} \\ \mathbf{c}_3^{(i)} \end{bmatrix}, \quad \text{with } \mathbf{b}_+ \gg 0,$$

where $\dim A_1 = \dim \mathbf{b}_+ = \dim \mathbf{c}_1^{(i)} = n_+$. Consequently,

$$A + \mathbf{b}\mathbf{c}_i^\top = \begin{bmatrix} A_1 + \mathbf{b}_+ \mathbf{c}_1^{(i)\top} & \mathbf{b}_+ \mathbf{c}_3^{(i)\top} \\ 0 & A_3 \end{bmatrix}.$$

If A_3 is not Hurwitz, clearly the system cannot be stabilizable. So, suppose that A_3 is Hurwitz but all the matrices $A_1 + \mathbf{b}_+ \mathbf{c}_1^{(i)\top}, i \in [1, p]$, are not. If $n_+ > 1$, set $\mathbf{w}_1 := -A_1^{-1} \mathbf{b}_+$. By the same reasoning as in the first part of the proof, we can claim that in this case it would be

$$(A + \mathbf{b}\mathbf{c}_i^\top) \begin{bmatrix} \mathbf{w}_1 \\ 0 \end{bmatrix} \geq 0, \quad \forall i \in [1, p],$$

thus preventing stabilizability. If $n_+ = 1$, we can come to the same conclusion by using $\mathbf{w}^\top = \mathbf{e}_1$. ■

By putting together Propositions 5 and 6, we finally get the following result.

Theorem 2: Let $A \in \mathbb{R}^{n \times n}$ be a diagonal matrix, and let $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}^n, i \in [1, p]$, be vectors such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Metzler for every index $i \in [1, p]$. The following facts are equivalent:

- i) there exists $i \in [1, p]$ such that $A + \mathbf{b}\mathbf{c}_i^\top$ is Hurwitz;
- ii) the CPSS (7) is stabilizable.

REFERENCES

- [1] F. Blanchini, P. Colaneri, and M.E. Valcher. Is stabilization of switched positive linear systems equivalent to the existence of a Hurwitz convex combination of the system matrices? In *Proceedings of the 50th IEEE CDC*, pages 1834 – 1839, Orlando, FL, 2011.
- [2] F. Blanchini, P. Colaneri, and M.E. Valcher. Co-positive Lyapunov functions for the stabilization of positive switched systems. *IEEE Trans. Aut. Contr.*, page DOI 10.1109/TAC.2012.2199169, 2012.
- [3] L. Farina and S. Rinaldi. *Positive linear systems: theory and applications*. Wiley-Interscience, Series on Pure and Applied Mathematics, New York, 2000.
- [4] E. Fornasini and M.E. Valcher. Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Trans. Aut. Contr.*, 55 (8):1933–1937, 2010.
- [5] E. Fornasini and M.E. Valcher. On the stability of continuous-time positive switched systems with rank one difference. *Control & Cybernetics, (special issue for the 80th birthday of T. Kaczorek)*, vol. 42, no. 1, 2013.
- [6] E. Fornasini and M.E. Valcher. Structural properties of continuous-time positive switched systems with rank one difference. in *preparation*, 2012.
- [7] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52 (6):1099–1103, 2007.
- [8] C. King and M. Nathanson. On the existence of common quadratic Lyapunov functions for a rank one difference. *Linear Algebra and its Appl.*, 419:400–416, 2006.
- [9] F. Knorn, O. Mason, and R.N. Shorten. On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45, Issue 8:1943–1947, August 2009.
- [10] D. Liberzon, J.P. Hespanha, and A.S. Morse. Stability of switched systems: a lie-algebraic condition. *Systems & Control Letters*, 37:117–122, 1999.
- [11] O. Mason and R.N. Shorten. Some results on the stability of positive switched linear systems. In *Proceedings of the 43rd Conference on Decision and Control (CDC 2004)*, pages 4601–4606, Paradise Island, Bahamas, 2004.
- [12] O. Mason and R.N. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52, no. 7:1346 – 1349, 2007.
- [13] O. Mason and R.N. Shorten. Quadratic and copositive Lyapunov functions and the stability of positive switched linear systems. In *Proceedings of the American Control Conference (ACC 2007)*, pages 657–662, New York, 2007.
- [14] R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular SISO switching systems. *IEEE Trans. Aut. Contr.*, 54, no. 11:2714–2718, 2009.
- [15] R. Shorten, O. Mason, F. O’Cairbre, and P. Curran. A unifying framework for the SISO circle criterion and other quadratic stability criteria. *Int. J. Contr.*, 77, no.1:1–8, 2004.
- [16] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review*, 49:545–592, 2007.
- [17] Z. Sun and S.S. Ge. *Stability Theory of Switched Dynamical Systems*. Volume in Series Communications and Control Engineering, Springer Verlag, London, 2011.
- [18] M.A. Wicks, P. Peleties, and R.A. De Carlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *European J. of Control*, 4, no. 2:140–147, 1998.
- [19] A. Zappavigna, P. Colaneri, J.C. Jeromel, and R. Middleton. Stabilization of continuous-time switched linear positive systems. In *Proceedings of the 2010 American Control Conference*, pages 3275–3280, Baltimore, MD, 2010.