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Stability Analysis of Time-Delay Systems Based on a Power of the Monodromy Operator

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Abstract— **This paper studies stability analysis of linear time-invariant time-delay feedback systems consisting of a finite-dimensional system and a pure delay. The continuous-time state transition is described by the monodromy operator in discrete-time through the lifting technique, and a stability condition in terms of its spectral radius is first restated in terms of the norm of a power of the operator. Quasi-finite-rank approximation is then applied on the power of the operator, and by evaluating the effect of the approximation error on the norm, a new necessary and sufficient stability condition is given finally.**

I. Introduction

Extensive studies have been conducted on time-delay systems and their stability problems, e.g., [1]–[3], to mention just a few of relatively new references. On the other hand, a new approach to time-delay systems was initiated in [4] by applying to time-delay systems the continuous-time lifting technique developed in the study of sampled-data systems [5]. Such a research direction was further sophisticated in [6] and [7], and the monodromy operator approach to time-delay systems was developed. This approach views the continuous-time state transition of time-delay systems in discrete-time, and thus is very consistent with the use of a discrete-time controller. Hence, it is expected to be a fundamental approach also for dealing with the analysis and synthesis of continuous-time time-delay systems with a discrete-time controller.

A continuous-time time-delay system is stable if and only if the spectral radius of its monodromy operator $\mathbf{T}$ is less than 1. In each of [6] and [7], a method was given for constructing a sequence of approximate values of the spectral radius, and it was established that the limit of the sequence coincides with the exact value of the spectral radius. Even though the effectiveness of these stability analysis methods have been confirmed numerically, it is hard to determine stability without taking the limit of the sequence (i.e., only with finite truncations of the sequence), rigorously speaking. This is because the sequence is constructed by introducing a sequence of approximations of the monodromy operator $\mathbf{T}$, where the spectral radius of each approximation can be computed exactly. A difficulty in this direction is to evaluate how far the spectral radius of the resulting approximate operator can be from that of the original $\mathbf{T}$. In contrast to deficiency in effective and tight inequality for perturbation analysis of spectral radius, we can apply the triangular inequality if we can restate the stability condition in terms of the norm of an operator.

As in the study of [8], this paper makes use of the well-known relation between the spectral radius of an operator and the norms of the powers $\mathbf{T}^\eta$ of the operator, and restates the stability condition in terms of the norm $\|\mathbf{T}^\eta\|$. We further show that a closed-form representation can be obtained for $\mathbf{T}^\eta$, and that it can be approximated easily as in the preceding studies [6],[7]. This leads to a sequence in the exponent η of the monodromy operator. Through perturbation analysis in terms of norm, we finally give a necessary and sufficient condition for stability. It turns out that, as long as the time-delay system is stable, we can rigorously determine its stability at a finite exponent η of the monodromy operator.

The contents of this paper are as follows. Section II reviews the monodromy operator $\mathbf{T}$ of a time-delay system. Section III reviews its stability condition in terms of the spectral radius of $\mathbf{T}$, reviews the stability analysis methods in [6] and [7], and motivates the present study based on the power $\mathbf{T}^\eta$. Section IV provides a closed-form representation of $\mathbf{T}^\eta$, which will be approximated with a more tractable operator in Section V through what we call quasi-finite-rank approximation. Section V finally gives a necessary and sufficient condition for stability of time-delay systems through the perturbation analysis, in terms of norm, associated with quasi-finite-rank approximation. Section VI studies a numerical example, and demonstrates the effectiveness of the developed stability analysis method.

The notations used in this paper are as follows. $\mathbb{R}$ denotes the set of real numbers while $\mathbb{N}$ denotes the set of positive integers. The shorthand notation $\mathcal{K}_m$ is used to denote the Hilbert space $(L_2([0,h);\mathbb{R}))^m$, and $\|\cdot\|$ denotes the induced norm of a matrix or an operator. The spectral radius of a matrix or an operator is denoted by $\rho(\cdot)$.

II. Monodromy Operator of Time-Delay System Σ

This paper deals with the linear time-invariant (LTI) time-delay feedback system Σ shown in Fig. 1 consisting of the finite-dimensional system F and the pure delay H. We assume that F is described by the state equation

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du \tag{1}$$

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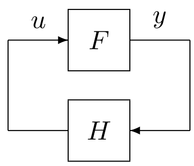

Fig. 1. Feedback system Σ with delay H.

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times\mu}$, $C \in \mathbb{R}^{\mu\times n}$, $D \in \mathbb{R}^{\mu\times\mu}$, and that the input-output relation of H is given by

$$u(t) = y(t-h) \tag{2}$$

where $h > 0$ denotes the length of the delay. We take $t = 0$ as the initial time instant, regard the relation (2) to be valid only for $t - h \geq 0$, and denote the initial condition of Σ by $x(0) = x_0$ and $u(t) = \widehat{u}_0(t)$ ($0 \leq t < h$, $\widehat{u}_0 \in \mathcal{K}_\mu$). Note that the notation $\widehat{u}_0$ here is consistent with the lifted representation of u given by $\{\widehat{u}_k\}_{k=0}^{\infty}$, $\widehat{u}_k(\theta) = u(kh+\theta)$; following the preceding studies [6],[7], we apply the lifting treatment [5] to deal with the time-delay system Σ. More precisely, we deal with Σ through its monodromy operator $\mathbf{T}$, i.e.,

$$\begin{bmatrix} x_{k+1} \\ \widehat{u}_{k+1} \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_k \\ \widehat{u}_k \end{bmatrix}, \quad \mathbf{T} := \begin{bmatrix} A_d & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{3}$$

where $x_k := x(kh)$, and the matrix $A_d \in \mathbb{R}^{n\times n}$ and the operators $\mathbf{B} : \mathcal{K}_\mu \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathcal{K}_\mu$ and $\mathbf{D} : \mathcal{K}_\mu \to \mathcal{K}_\mu$ are defined as follows.

$$A_d = \exp(Ah) \tag{4}$$

$$\mathbf{B}f = \int_0^h \exp(A(h-\tau))Bf(\tau)d\tau \tag{5}$$

$$(\mathbf{C}v)(\theta) = C\exp(A\theta)v \tag{6}$$

$$(\mathbf{D}f)(\theta) = \int_0^\theta C\exp(A(\theta-\tau))Bf(\tau)d\tau + Df(\theta) \tag{7}$$

III. Basic Stability Condition of Σ

It has been shown in [6] that Σ is exponentially stable if and only if $\rho(\mathbf{T}) < 1$. This motivates us to compute $\rho(\mathbf{T})$ for stability analysis of Σ, but since $\mathbf{T}$ is an infinite-rank operator, exact computation of $\rho(\mathbf{T})$ is hard.

To circumvent the difficulty, the monodromy operator $\mathbf{T}$ was approximated by more tractable operators in [6] and [7] by applying the (quasi-)finite-rank approximation technique developed in [9]. More precisely, the case with $D = 0$ (which corresponds to retarded time-delay systems) was dealt with in [6], and finite-rank approximation of $\mathbf{T}$ was applied, while the case with $D \neq 0$ (which corresponds to neutral time-delay systems, in general) was dealt with in [7], and quasi-finite-rank approximation of $\mathbf{T}$ was applied[1]. Quasi-finite-rank approximation of $\mathbf{T}$ is associated with a positive integer parameter N, and it was established theoretically (and confirmed numerically with a satisfactory convergence property) in each of these preceding studies that the estimation of $\rho(\mathbf{T})$ through the resulting tractable operators tends to the exact value of $\rho(\mathbf{T})$ as $N \to \infty$.

Even though quasi-finite-rank approximation is effective in the sense described above, a drawback of the methods in [6] and [7] is that it is hard to derive a sufficient condition for stability of Σ for one fixed (and thus finite) N: even if the estimation of $\rho(\mathbf{T})$ in these methods were apparently small enough compared with 1 for some N, it does not immediately ensure that Σ is stable, rigorously speaking. Moreover, it is not trivial if evaluating the size of the gap between the estimated value of $\rho(\mathbf{T})$ and 1 could somehow ensure stability of Σ in a rigorous way. This is because the perturbation of the spectral radius under a bounded perturbation of an operator is generally hard to evaluate in an explicit and tight way.

The purpose of this paper is to circumvent the drawback mentioned above, and give such a stability analysis method that can determine, even with a fixed parameter N for quasi-finite-rank approximation and whenever Σ is stable, that Σ is indeed stable. As one can easily expect, giving such a method is closely related to establishing, for each fixed N, a sufficient condition for stability of Σ, and we indeed give such a condition. What should be stressed here, however, is that the condition we will establish in the remainder of this paper is in fact a necessary and sufficient condition for stability of Σ for each fixed N. In this sense, the role of the parameter N in the present study is different from that in the preceding studies [6] and [7]; in those studies, it was necessary that we make $N \to \infty$, while in the present study, it suffices for us to take a fixed N for stability analysis of Σ. Hence, even $N = 1$ could suffice as an extreme choice, theoretically speaking, but it will turn out that taking $N > 1$ can improve the overall performance of stability analysis.

A key towards the direction to such arguments is to use the norm-based stability condition as opposed to the one based on the spectral radius that we have stated above (see [8] for similar treatment of the stability condition of time-delay systems in this sense). In fact, we use the following lemma.

Lemma 1 Given $r > 0$, the following conditions are equivalent to each other.

(i) $\rho(\mathbf{T}) < r$.
(ii) There exists $\eta_0 \in \mathbb{N}$ such that $\|\mathbf{T}^\eta\| < r^\eta$, $\forall \eta \geq \eta_0$.
(iii) There exists $\eta \in \mathbb{N}$ such that $\|\mathbf{T}^\eta\| < r^\eta$.

Once we employ a norm-based equivalent stability condition, we can easily evaluate the effect of quasi-finite-rank approximation error on the associated stability analysis by simply applying the triangle inequality on norm. However, in order for such an idea to work in a theoretically sound way, quasi-finite-rank approximation of $\mathbf{T}^\eta$ should be dealt with directly, rather than first applying quasi-finite-rank approximation to $\mathbf{T}$ and then taking the η-th power of the resulting operator. Hence, it is crucial to give an explicit representation of $\mathbf{T}^\eta$ so that quasi-finite-rank approximation

[1]From now on, we refrain from distinguishing finite-rank approximation and quasi-finite-rank approximation by regarding the former as a special case of the latter (which is indeed the case), to simplify the descriptions.

can easily be applied on it. Such a representation of $\mathbf{T}^\eta$ will be given in the following section.

IV. Closed-Form Representation of $\mathbf{T}^\eta$

This section aims at giving a closed-form representation of $\mathbf{T}^\eta$, which is obviously an operator from $[x_k^T,\ \widehat{u}_k^T]^T$ to $[x_{k+\eta}^T,\ \widehat{u}_{k+\eta}^T]^T$:

$$\begin{bmatrix} x_{k+\eta} \\ \widehat{u}_{k+\eta} \end{bmatrix} = \mathbf{T}^\eta \begin{bmatrix} x_k \\ \widehat{u}_k \end{bmatrix} \quad (8)$$

We derive such a representation by introducing F^η, i.e., the series connection of η copies of F, just for the sake of convenience (even though Σ does not contain F^η but it only contains F).

We first note that F^η is described by

$$\frac{d\chi}{dt} = A_\eta \chi + B_\eta w, \quad z = C_\eta \chi + D_\eta w \quad (9)$$

where

$$A_\eta := \begin{bmatrix} A & 0 & \cdots & 0 \\ BC & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ BD^{\eta-2}C & \cdots & BC & A \end{bmatrix},\ B_\eta := \begin{bmatrix} B \\ BD \\ \vdots \\ BD^{\eta-1} \end{bmatrix},$$

$$C_\eta := \begin{bmatrix} D^{\eta-1}C & \cdots & DC & C \end{bmatrix}, \quad D_\eta := D^\eta \quad (10)$$

Let us denote the lifted representation of (9) by

$$\chi_{k+1} = A_{\eta d}\chi_k + \mathbf{B}_\eta \widehat{w}_k \quad (11)$$

$$\widehat{z}_k = \mathbf{C}_\eta \chi_k + \mathbf{D}_\eta \widehat{w}_k \quad (12)$$

Here, the matrix $A_{\eta d}$ and the operators $\mathbf{B}_\eta$, $\mathbf{C}_\eta$ and $\mathbf{D}_\eta$ are given respectively by (4), (5), (6) and (7) with A, B, C and D replaced by A_η, B_η, C_η and D_η, respectively.

We skip the details because of limited space, but to obtain a closed-form representation of $\mathbf{T}^\eta$, a similar representation for $\mathbf{T}^i$, which represents the mapping from $[x_k, \widehat{u}_k]$ to $[x_{k+i}, \widehat{u}_{k+i}]$, should naturally be considered at the same time. Hence, let us define

$$\mathbf{T}^i =: \begin{bmatrix} A_d^{(i)} & \mathbf{B}^{(i)} \\ \mathbf{C}^{(i)} & \mathbf{D}^{(i)} \end{bmatrix} \quad (i = 1, \cdots, \eta) \quad (13)$$

Then, we can derive a recursive relation among $A_d^{(i)}$ $(i = 1, \cdots, \eta)$, by which $\mathbf{B}^{(\eta)}$, $\mathbf{C}^{(\eta)}$ and $\mathbf{D}^{(\eta)}$ can also be represented easily, as shown in the following result.

Lemma 2 Let

$$E_i := [\overbrace{0_n \cdots 0_n}^{i-1}\ I_n\ \overbrace{0_n \cdots 0_n}^{\eta-i}] \in \mathbb{R}^{n\times\eta n} \quad (14)$$

$$\mathcal{A}_i := E_i A_{\eta d} \begin{bmatrix} I_{in} \\ 0 \end{bmatrix} \in \mathbb{R}^{n\times in} \quad (15)$$

and let us define $A_d^{(i)}$ $(i = 1, \cdots, \eta)$ recursively by

$$A_d^{(1)} := \mathcal{A}_1 \in \mathbb{R}^{n\times n},$$

$$A_d^{(i)} := \mathcal{A}_i \begin{bmatrix} I \\ A_d^{(1)} \\ \vdots \\ A_d^{(i-1)} \end{bmatrix} \in \mathbb{R}^{n\times n}\ (i = 2, \cdots, \eta) \quad (16)$$

Then,

$$\mathbf{B}^{(\eta)} = B_d^{(\eta)}\mathbf{B}_\eta \quad (17)$$

$$\mathbf{C}^{(\eta)} = \mathbf{C}_\eta C_d^{(\eta)} \quad (18)$$

$$\mathbf{D}^{(\eta)} = \mathbf{C}_\eta D_d^{(\eta)}\mathbf{B}_\eta + \mathbf{D}_\eta \quad (19)$$

where

$$B_d^{(\eta)} := \begin{bmatrix} A_d^{(\eta-1)} & \cdots & A_d^{(1)} & I \end{bmatrix} \in \mathbb{R}^{n\times\eta n} \quad (20)$$

$$C_d^{(\eta)} := \begin{bmatrix} I \\ A_d^{(1)} \\ \vdots \\ A_d^{(\eta-1)} \end{bmatrix} \in \mathbb{R}^{\eta n\times n} \quad (21)$$

$$D_d^{(\eta)} := \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ I & \ddots & & & \vdots \\ A_d^{(1)} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ A_d^{(\eta-2)} & \cdots & A_d^{(1)} & I & 0 \end{bmatrix} \in \mathbb{R}^{\eta n\times\eta n} \quad (22)$$

We have thus arrived at the following closed-form representation of $\mathbf{T}^\eta$.

$$\mathbf{T}^\eta = \begin{bmatrix} A_d^{(\eta)} & B_d^{(\eta)}\mathbf{B}_\eta \\ \mathbf{C}_\eta C_d^{(\eta)} & \mathbf{C}_\eta D_d^{(\eta)}\mathbf{B}_\eta + \mathbf{D}_\eta \end{bmatrix} \quad (23)$$

V. Stability Analysis Based on Quasi-Finite-Rank Approximation of $\mathbf{T}^\eta$

We have stated in Section III the motivation of the present study, in which a key idea was to deal with the norm $\|\mathbf{T}^\eta\|$ rather than the spectral radius $\rho(\mathbf{T})$ so that perturbation analysis may be applied when we approximate an operator with a tractable one. We have then given a closed-form representation of $\mathbf{T}^\eta$ in Section IV. This section is devoted to presenting the main results of the present study, in which we apply quasi-finite-rank approximation on $\mathbf{T}^\eta$, takes into account the error introduced into the estimation of $\|\mathbf{T}^\eta\|$ through such approximation, and then give an alternative form of necessary and sufficient condition for stability of Σ. We then show that this condition leads to a numerically tractable (and stable) stability analysis method of Σ.

A. Quasi-Finite-Rank Approximation

Among the operators in the representation of $\mathbf{T}^\eta$ given in (23), the only infinite-rank operator is $\mathbf{D}_\eta$, and this is precisely the one that makes the stability analysis of Σ hard. Hence, we approximate $\mathbf{D}_\eta$ with another more tractable operator by applying quasi-finite-rank approximation on $\mathbf{D}_\eta$, and this treatment immediately leads to the associated quasi-finite-rank approximation of $\mathbf{T}^\eta$. In the method of quasi-finite-rank approximation developed in [9], what is called fast-lifting with the parameter $N \in \mathbb{N}$ is first applied on $\mathbf{D}_\eta$ so that the associated error in the approximation can be reduced as N is made larger. Hence, we first review fast-lifting.

Given a vector function f defined on the interval $[0, h)$ and $N \in \mathbb{N}$, let us define $h' := h/N$ and

$$\check{f} := [(f^{(1)})^T \ \cdots \ (f^{(N)})^T]^T, \quad f^{(i)} \in \mathcal{K}'_\mu \tag{24}$$

$$f^{(i)}(\theta') := f((i-1)h' + \theta') \ (0 \le \theta' < h') \tag{25}$$

where $\mathcal{K}'_\mu$ denotes $\mathcal{K}_\mu$ with the underlying interval $[0, h)$ replaced by $[0, h')$. The mapping from f to $\check{f}$ is called fast-lifting, and is denoted by $\check{f} = \mathbf{L}_N f$.

We now introduce $\mathbf{I}(\mathbf{L}_N) := \mathrm{diag}[I, \mathbf{L}_N]$ and then the fast-lifted counterpart of $\mathbf{T}^\eta$, i.e.,

$$\mathbf{T}_N^\eta = \mathbf{I}(\mathbf{L}_N)\mathbf{T}^\eta \mathbf{I}(\mathbf{L}_N)^{-1} \tag{26}$$

which obviously equals the η-th power of the fast-lifted counterpart of $\mathbf{T}$, i.e., $\mathbf{T}_N := \mathbf{I}(\mathbf{L}_N)\mathbf{T}\mathbf{I}(\mathbf{L}_N)^{-1}$. Since $\mathbf{L}_N$ and thus $\mathbf{I}(\mathbf{L}_N)$ is unitary, we have $\|\mathbf{T}^\eta\| = \|\mathbf{T}_N^\eta\|$. In view of this equality, we rather consider approximating $\mathbf{T}_N^\eta$ instead of doing the same directly on $\mathbf{T}^\eta$. Even though such approximation will eventually be applied solely on the right-lower block of $\mathbf{T}_N^\eta$, or more precisely $\mathbf{L}_N \mathbf{D}_\eta \mathbf{L}_N^{-1}$, which is the part involving the troublesome operator $\mathbf{D}_\eta$, we are now interested in giving an explicit representation of the whole $\mathbf{T}_N^\eta$ because we do need it eventually. To this end, let us introduce $A'_{\eta d}$, $\mathbf{B}'_\eta$, $\mathbf{C}'_\eta$ and $\mathbf{D}'_\eta$, which are defined as $A_{\eta d}$, $\mathbf{B}_\eta$, $\mathbf{C}_\eta$ and $\mathbf{D}_\eta$, respectively, with the underlying interval $[0, h)$ replaced by $[0, h')$. Then, it follows immediately (as in [6]) from the property of $\mathbf{L}_N$ that

$$\mathbf{B}_\eta \mathbf{L}_N^{-1} = \left[\, (A'_{\eta d})^{N-1}\mathbf{B}'_\eta \ \cdots \ A'_{\eta d}\mathbf{B}'_\eta \ \mathbf{B}'_\eta \, \right] \tag{27}$$

$$\mathbf{L}_N \mathbf{C}_\eta = \begin{bmatrix} \mathbf{C}'_\eta \\ \mathbf{C}'_\eta A'_{\eta d} \\ \vdots \\ \mathbf{C}'_\eta (A'_{\eta d})^{N-1} \end{bmatrix} \tag{28}$$

$$\mathbf{L}_N \mathbf{D}_\eta \mathbf{L}_N^{-1} = \begin{bmatrix} \mathbf{D}'_\eta & 0 & \cdots & 0 \\ \mathbf{C}'_\eta \mathbf{B}'_\eta & \mathbf{D}'_\eta & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{C}'_\eta (A'_{\eta d})^{N-2}\mathbf{B}'_\eta & \cdots & \mathbf{C}'_\eta \mathbf{B}'_\eta & \mathbf{D}'_\eta \end{bmatrix} \tag{29}$$

By introducing

$$B_{\eta N} := \left[\ (A'_{\eta d})^{N-1} \quad \cdots \quad A'_{\eta d} \quad I \ \right] \tag{30}$$

$$C_{\eta N} := \begin{bmatrix} I \\ A'_{\eta d} \\ \vdots \\ (A'_{\eta d})^{N-1} \end{bmatrix} \tag{31}$$

we see that (27) and (28) can be rewritten as

$$\mathbf{B}_\eta \mathbf{L}_N^{-1} = B_{\eta N} \overline{\mathbf{B}'_\eta} \tag{32}$$

$$\mathbf{L}_N \mathbf{C}_\eta = \overline{\mathbf{C}'_\eta} C_{\eta N} \tag{33}$$

where $\overline{(\cdot)}$ denotes $\mathrm{diag}[(\cdot), \cdots, (\cdot)]$ consisting of N copies of $(\cdot)$. Hence, in view of the representation of $\mathbf{T}^\eta$ given by (23), we are led to

$$\mathbf{T}_N^\eta = \begin{bmatrix} A_d^{(\eta)} & B_{dN}^{(\eta)} \overline{\mathbf{B}'_\eta} \\ \overline{\mathbf{C}'_\eta} C_{dN}^{(\eta)} & \overline{\mathbf{C}'_\eta} D_{dN}^{(\eta)} \overline{\mathbf{B}'_\eta} + \mathbf{L}_N \mathbf{D}_\eta \mathbf{L}_N^{-1} \end{bmatrix} \tag{34}$$

where

$$B_{dN}^{(\eta)} = B_d^{(\eta)} B_{\eta N}, \quad C_{dN}^{(\eta)} = C_{\eta N} C_d^{(\eta)},$$
$$D_{dN}^{(\eta)} = C_{\eta N} D_d^{(\eta)} B_{\eta N} \tag{35}$$

If we substitute (29) into (34), we see that the only infinite-rank operator on the right-hand side is $\mathbf{D}'_\eta$. Recalling that the only infinite-rank operator on the right-hand side of (23) was $\mathbf{D}_\eta$, we see that we have successfully reduced by the factor of $1/N$ the size of the interval on which the infinite-rank operators are defined. Hence, we can have a better chance for having more effective approximation of such troublesome operators by taking $N > 1$ (for each fixed η), and this is precisely the reason why we have applied fast-lifting and introduced $\mathbf{T}_N^\eta$ instead of working directly on $\mathbf{T}^\eta$.

Now, what remains is to approximate the infinite-rank operator $\mathbf{D}'_\eta$ with a tractable operator, preferably with a finite-rank operator. In view of (7), however, we see that $\mathbf{D}'_\eta$ is the sum of the operator of multiplication by the matrix D_η $(= D^\eta)$ and the compact operator $\mathbf{D}'_{\eta 0} := \mathbf{D}'_\eta - D_\eta$, and a finite-rank operator can reasonably approximate only the compact operator $\mathbf{D}'_{\eta 0}$. Since $\mathbf{B}'_\eta$ and $\mathbf{C}'_\eta$ are involved in $\mathbf{T}_N^\eta$ (see (34)) and since they should also be 'discretized' to have a finite-dimensional computation method, we consider approximating $\mathbf{D}'_{\eta 0}$ using these operators, more precisely by $\mathbf{C}'_\eta X \mathbf{B}'_\eta$ with a matrix $X \in \mathbb{R}^{\eta n \times \eta n}$ (while D_η is left unapproximated). This leads to approximating $\mathbf{D}'_\eta$ with $\mathbf{C}'_\eta X \mathbf{B}'_\eta + D_\eta$, which is called quasi-finite-rank approximation of $\mathbf{D}'_\eta$ (when $D_\eta = 0$, it is particularly called finite-rank approximation of $\mathbf{D}'_\eta$). An optimal X minimizing the $\mathcal{K}'_\mu$-induced norm of the approximation error $\mathbf{E}'_{\eta X} := \mathbf{D}'_\eta - (\mathbf{C}'_\eta X \mathbf{B}'_\eta + D_\eta)$ $(= \mathbf{D}'_{\eta 0} - \mathbf{C}'_\eta X \mathbf{B}'_\eta)$ can be obtained (approximately) by the method in [9], where it has been shown that $\|\mathbf{E}'_{\eta X}\|$ under fixed $\eta \in \mathbb{N}$ can be made arbitrarily small by taking a large enough N.

In fact, however, we will see that the standpoint of the present study is rather that N is fixed first, and then η is made larger until we arrive at a conclusion on the stability analysis of Σ. Even in such a situation, the optimal value of $\|\mathbf{E}'_{\eta X}\|$ can also be made arbitrarily small by taking a large enough η (provided that Σ is stable), as we shall establish later in (42). Having said this, it might sound that taking $N = 1$ suffices, but in fact taking $N > 1$ is helpful for some reasons. Taking a modestly large N is first useful to benefit from the property mentioned in the two paragraphs above. Furthermore, since $X \in \mathbb{R}^{\eta n \times \eta n}$, its size grows with η, and this might possibly lead to a numerical problem in determining X minimizing $\|\mathbf{E}'_{\eta X}\|$, when η is very large. Taking a modestly large N can be a useful alternative to avoid such an issue while reducing approximation errors, because the size of X is independent of N.

By introducing the quasi-finite-rank approximation of $\mathbf{D}'_\eta$, it follows from (34) together with (29) that

$$\mathbf{T}_N^\eta = (\mathbf{T}_N^\eta)_X + \mathbf{O}(\overline{D_\eta}) + \mathbf{O}(\overline{\mathbf{E}'_{\eta X}}) \tag{36}$$

where $\mathbf{O}(\cdot) := \mathrm{diag}[0, (\cdot)]$, and $(\mathbf{T}_N^\eta)_X$ approximating the 'compact part' of $\mathbf{T}_N^\eta$ is defined by

$$(\mathbf{T}_N^\eta)_X = \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{C}'_\eta} \end{bmatrix} \begin{bmatrix} A_d^{(\eta)} & B_{dN}^{(\eta)} \\ C_{dN}^{(\eta)} & D_{dNX}^{(\eta)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \overline{\mathbf{B}'_\eta} \end{bmatrix} \tag{37}$$

$$D_{dNX}^{(\eta)} = D_{dN}^{(\eta)} + \begin{bmatrix} X & 0 & \cdots & 0 \\ I & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (A'_{\eta d})^{N-2} & \cdots & I & X \end{bmatrix} \tag{38}$$

We will give a stability analysis method for Σ in the following subsection, which is based on a necessary and sufficient condition for stability of Σ that can be derived from the fundamental relation (36) together with Lemma 1.

B. Numerically Tractable Stability Condition for Σ

We are now in a position to prove the following theorem giving an alternative necessary and sufficient condition for stability of Σ.

Theorem 1 Given any $N \in \mathbb{N}$, the time-delay feedback system Σ is exponentially stable if and only if there exists $\eta \in \mathbb{N}$ such that

$$\gamma_{\eta N} := \|(\mathbf{T}_N^\eta)_X\| + \|D^\eta\| + \|\mathbf{E}'_{\eta X}\| < 1 \tag{39}$$

where X is a matrix minimizing $\|\mathbf{E}'_{\eta X}\|$.

Proof: Sufficiency: If we apply the triangle inequality to (36), it readily follows from (39) that $\|\mathbf{T}_N^\eta\| < 1$, or equivalently $\|\mathbf{T}^\eta\| < 1$, since $D_\eta = D^\eta$. Hence $\rho(\mathbf{T}) < 1$ by Lemma 1, and thus Σ is exponentially stable.

Necessity: Solving (36) for $(\mathbf{T}_N^\eta)_X$ and taking its norm leads to

$$\|(\mathbf{T}_N^\eta)_X\| \le \|\mathbf{T}_N^\eta\| + \|D^\eta\| + \|\mathbf{E}'_{\eta X}\| \tag{40}$$

since $D_\eta = D^\eta$. Since X minimizes $\|\mathbf{E}'_{\eta X}\|$, it follows that

$$\|\mathbf{E}'_{\eta X}\| \le \|\mathbf{D}'_{\eta 0} - \mathbf{C}'_\eta X_0 \mathbf{B}'_\eta\|_{X_0 = -(A'_{\eta d})^{N-1} D_d^{(\eta)}} \tag{41}$$

where the operator on the right hand side is nothing but the one on the right-lower corner in $\mathbf{T}_N^\eta - \mathbf{O}(\overline{D_\eta})$, as can be seen from (29) and (34). Hence, we have

$$\|\mathbf{E}'_{\eta X}\| \le \|\mathbf{T}_N^\eta - \mathbf{O}(\overline{D_\eta})\| \le \|\mathbf{T}^\eta\| + \|D^\eta\| \tag{42}$$

Substituting this and (40) into (39) leads to

$$\gamma_{\eta N} \le 3\|\mathbf{T}^\eta\| + 4\|D^\eta\| \tag{43}$$

Hence, the assertion follows immediately. ■

Theorem 1 can be interpreted as giving a numerically tractable necessary and sufficient condition for stability of Σ for the following reasons. First, as discussed in [6], $\|\mathbf{E}'_{\eta X}\|$ can also be computed for the optimal X. Furthermore, $(\mathbf{T}_N^\eta)_X$ is obviously a finite-rank operator as can easily be seen from (37), and thus its norm can also be computed readily. More precisely, we have

$$\|(\mathbf{T}_N^\eta)_X\| = \left\| \begin{bmatrix} I & 0 \\ 0 & \overline{V'_\eta} \end{bmatrix} \begin{bmatrix} A_d^{(\eta)} & B_{dN}^{(\eta)} \\ C_{dN}^{(\eta)} & D_{dNX}^{(\eta)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \overline{W'_\eta} \end{bmatrix} \right\| \tag{44}$$

where V'_η and W'_η are arbitrary matrices satisfying $(V'_\eta)^T(V'_\eta) = (\mathbf{C}'_\eta)^*\mathbf{C}'_\eta$ and $W'_\eta(W'_\eta)^T = \mathbf{B}'_\eta(\mathbf{B}'_\eta)^*$, respectively, which can easily be computed as a matrix square root of observability and controllability Gramians. Hence, for fixed N and η, the quantity $\gamma_{\eta N}$ in (39) can be computed only with finite-dimensional computations (including the computation of X; see [6]), and once we have $\gamma_{\eta N} < 1$ for some (finite) N and η, we can rigorously conclude that Σ is exponentially stable. This is in sharp contrast with similar methods in [6] and [7] based on the monodromy operator approach, and is expected to be a solid basis on which this approach can be developed further to encompass such an advanced issue of designing stabilizing discrete-time controllers.

Remark 1 We can easily see from the necessity proof of Theorem 1 that, whenever Σ is stable, $\gamma_{\eta N}$ (> 0) becomes arbitrarily small as η becomes arbitrarily large (for each fixed N, or more strongly, uniformly with respect to N). In particular, $\gamma_{\eta N} < 1$ holds for any sufficiently large η. This suggests that the stability analysis based on the (sufficient) condition $\gamma_{\eta N} < 1$ would be numerically stable (except for the cases when Σ is so close to marginal stability that $\gamma_{\eta N} < 1$ can hold only for extremely large η).

Remark 2 Since $\rho(\mathbf{T})^\eta \le \|\mathbf{T}^\eta\| = \|\mathbf{T}_N^\eta\|$ for any N and η, it follows from (36) that $\rho(\mathbf{T}) \le \gamma_{\eta N}^{1/\eta}$. That is, $\gamma_{\eta N}^{1/\eta}$ is an upper bound of $\rho(\mathbf{T})$ for any N and η. On the other hand, we can show that $\gamma_{\eta N} < r^\eta$ for any sufficiently large η whenever $\rho(\mathbf{T}) < r$. In other words, $\gamma_{\eta N}^{1/\eta} < r$ for any sufficiently large η whenever $\rho(\mathbf{T}) < r$. This implies that $\gamma_{\eta N}^{1/\eta}$, which is an upper bound of $\rho(\mathbf{T})$, can be made arbitrarily close to $\rho(\mathbf{T})$ by taking a sufficiently large η. More precisely, we have $\rho(\mathbf{T}) = \lim_{\eta\to\infty} \gamma_{\eta N}^{1/\eta}$.

VI. Numerical Example

As an example of Σ in Fig. 1, let us consider F given by $A = -1/2$, $B = 1$, $C = -1$ and $D = -1/2$. We can show (by the Nyquist criterion) that the maximal value of $\overline{h}$ such that Σ is stable for all $h \in (0, \overline{h})$ is given by

$$\overline{h}^\star = \frac{\pi + \arg(-F(j\omega_0))}{\omega_0} \approx 1.828 \tag{45}$$

where $\omega_0 = \sqrt{7}/2$ and $-\pi < \arg(z) \le \pi$ denotes the argument of the complex number z.

By fixing η and N, and computing $\gamma_{\eta N}$ for each h, we obtained the maximal $\overline{h}$, denoted by $\overline{h}_{\eta N}$, such that $\gamma_{\eta N} < 1$ (and thus Σ with h is ensured to be stable by Theorem 1) for all $h \in (0, \overline{h})$. The results are shown in Table I. We can confirm that each value of $\gamma_{\eta N}$ in this table is in fact a lower bound of $\overline{h}^\star$, and thus confirm the validity of the sufficiency assertion of Theorem 1. Furthermore, for each N, we can see that $\gamma_{\eta N}$ comes closer to $\overline{h}^\star$ as η becomes larger, which corresponds to the necessity assertion (although this property is not monotonic with respect to η, in general). For reference, we can have $\overline{h}_{\eta N} = 1.827$ (which is rather close to $h^\star$) for $\eta = 360$ and $N = 1$. For each η, on the other hand, we can see that $\overline{h}_{\eta N}$ with $N = 30$ gives a better (not worse at least) lower bound of $\overline{h}^\star$ than that with $N = 1$, and this suggests that taking a modestly large N can indeed contribute to improving the results of stability analysis (for fixed η). Such improvement, however, becomes ineffective when η is larger than a relatively small value.

To confirm the effectiveness of the stability analysis method based on Theorem 1, we also apply the method developed in [7], which is also based on the monodromy operator approach and approximately computes $\rho(\mathbf{T})$ by applying quasi-finite-rank approximation directly on $\mathbf{T}$ (or $\mathbf{D}$ contained in it). As the parameter N in quasi-finite-rank approximation tends to ∞, it has been shown that the computed estimation of $\rho(\mathbf{T})$ tends to the exact value of $\rho(\mathbf{T})$, and thus this method can also be used as a stability analysis method of Σ, provided that a sufficiently large N is used. Let us denote by $\overline{h}_N$ the maximal value of $\overline{h}$ such that the estimated value of $\rho(\mathbf{T})$ under the parameter N is less than 1 (so that Σ would virtually be considered to be stable if N were qualified as being sufficiently large) for all $h \in (0, \overline{h})$. The results are shown in Table II. For those values of N in this table, we see that all the values of $\overline{h}_N$ exceed $\overline{h}^\star$, which implies that these values of N are not sufficiently large for valid stability analysis of Σ in this example. For the sake of fair comparison, it would be important for us to remark that when h is taken, for example, to be $h = 1.829$, which slightly exceeds $\overline{h}^\star$, the estimated values of $\rho(\mathbf{T})$ by the method in [7] actually all failed to be less than 1 (as we wish since Σ is unstable) whenever $N \geq 51$. However, a difficulty is that there is no clear criterion about how large N we should take to apply this method without arriving at a wrong conclusion. In contrast, the method based on Theorem 1 is free from such a difficulty, and once we obtain $\gamma_{\eta N} < 1$ for some η and N, then we can rigorously conclude that Σ is stable. This feature is attractive especially when the analysis method is to be extended to encompass a stabilizing controller design problem. Note that the nature of the monodromy operator $\mathbf{T}$, which views the continuous-time state transition of Σ in discrete-time, is very consistent with the use of a discrete-time controller, and that the monodromy operator can readily be extended to the case with such a controller. Hence, the arguments in the present study are expected to be a basis for discrete-time stabilizing controller design, which will be an interesting further topic in the monodromy operator approach.

TABLE I

GUARANTEED STABILITY REGION $\overline{h}_{\eta N}$ ABOUT h.

$N \setminus \eta$	5	10	15	20	25
1	1.200	1.654	1.733	1.775	1.801
30	1.516	1.684	1.740	1.775	1.801

TABLE II

ESTIMATED STABILITY REGION ABOUT h BY THE METHOD IN [7].

N	10	20	30	40	50
$\overline{h}_N$	1.841	1.831	1.829	1.829	1.829

VII. Conclusion

This paper gave a stability analysis method of time-delay systems based on the norm of the power $\mathbf{T}^\eta$ of the monodromy operator $\mathbf{T}$. It is based on a necessary and sufficient condition for stability, and has an advantage over the preceding studies [6] and [7] in the sense that, whenever a time-delay system is stable, its stability can always be determined rigorously in a non-asymptotic sense (i.e., we need not consider taking a limit of an infinite sequence, but stability can be determined rigorously at a finite η). It is expected that this feature can be exploited in the future in the discrete-time stabilizing controller design for continuous-time time-delay systems. It is also shown that the involved computations are numerically tractable and are expected to give a numerically stable stability analysis method. The effectiveness of this new method was demonstrated with a numerical example.

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