

Consistency Analysis of Orthogonal Projection Based Closed-Loop Subspace Identification Methods*

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Abstract—Based on the orthogonal projection approach, a few closed-loop subspace identification methods have been developed recently in the literature. To guarantee consistent estimation of these methods against measurement noise, this paper analyzes the rank condition for performing the corresponding singular value decomposition (SVD) involved in deriving the extended observability matrix and the lower triangular block-Toeplitz matrix of the plant state-space model, together with the requirement on the input excitation signal, such that no mistake will be made in picking up the corresponding eigenvalues for numerical computation. Also, no knowledge of the closed-loop controller is required for the computation of the model state matrices. An example from the literature is used to demonstrate the proposed conditions for consistent estimation.

I. INTRODUCTION

Closed-loop identification methods have been increasingly explored in the past decade, owing to the requirements on closed-loop system operation and safety reasons for more and more industrial and chemical processes [1]. As surveyed in the recent literature [2], most of existing references have been devoted to closed-loop identification methods in terms of the continuous- or discrete-time transfer function models or frequency domain models. By comparison, subspace identification methods (SIMs) have been increasingly developed in the past years [3], owing to the fact that state-space models have been widely used for model predictive control (MPC) of various industrial processes. For closed-loop SIM, there are a few methods developed based on the orthogonal projection approach (OPA): Huang et al [4] combined the principal component analysis (PCA) with the OPA to determine the orthogonal complement of the extended observability matrix; By introducing a column weighting matrix to perform the OPA for model identification, an alternative closed-loop SIM was given in [5]; In contrast, a joint input-output space was specified to perform the OPA

such that the deterministic component of the joint space can be derived for simultaneous identification of both the plant state-space model and the close-loop controller [6], and this idea was further extended for better accuracy in terms of using a finite data length in the later paper [7]. Although these OPA based SIMs have demonstrated through benchmark examples the computation efficiency and enhanced identification accuracy in comparison with other closed-loop SIMs. e.g. [8-10], the consistency analysis remains open for the presence of measurement noise. As far as we know, little result has been reported for such study, besides the exceptional work in [11] studying a few specific closed-loop SIMs like [12] and [13].

In this paper, we analyze the rank condition for performing the OPA to eliminate the noise influence as developed in [4] and [5], through which the PCA-based singular value decomposition (SVD) can be precisely conducted to warrant consistent estimation of the extended observability matrix and lower triangular Toeplitz matrix of the plant state-space model. Accordingly, the state matrices can be retrieved from the above matrices directly or using the standard least-squares (LS) algorithm. Meanwhile, the requirement on the input excitation signal is made clear.

Throughout this paper, the following notations are used: $\mathfrak{R}^{n \times m}$ denotes a $n \times m$ real matrix space. For any matrix $P \in \mathfrak{R}^{m \times m}$, $P > 0$ (or $P \geq 0$) means P is a positive (or semipositive) definite symmetric matrix, in which the symmetric elements are indicated by ‘*’. Denote by P^T the transpose of P . For $P \in \mathfrak{R}^{m \times m}$ of full rank, P^{-1} denotes the inverse of P ; for $P \in \mathfrak{R}^{m \times n}$ of full row (or column) rank, P^\dagger denotes the Moore-Penrose pseudo-inverse of P . The identity matrix and the zero matrix with appropriate dimensions are denoted by I and $\mathbf{0}$, respectively. Note that $I_{m \times m}$ (or $\mathbf{0}_{m \times m}$) means $I_{m \times m} \in \mathfrak{R}^{m \times m}$ ($\mathbf{0}_{m \times m} \in \mathfrak{R}^{m \times m}$). Denote by $\text{rank}(P)$ the rank number of P . Denote by $E\{\cdot\}$ the statistical expectation operator. Denote by Λ/V an orthogonal projection of the row space of $\Lambda \in \mathfrak{R}^{m \times j}$ onto the row space of $V \in \mathfrak{R}^{n \times j}$, which may be computed through $\Lambda/V = \Lambda V^T (V V^T)^{-1} V$. The orthogonal complement of the row space Λ is denoted by $\Lambda^\perp \in \mathfrak{R}^{(j-m) \times j}$, and correspondingly, there is $\Lambda/V^\perp = \Lambda - \Lambda/V$. Denote by $\delta_{i,j}$ the Kronecker delta function, $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. Denote by $PE(u)$ the persistent excitation order of the input excitation signal for identification.

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For the rank analysis of matrix multiplication, the following lemma is needed here:

Lemma 1 [14]: Let $A \in \mathfrak{R}^{m \times n}$ and $B \in \mathfrak{R}^{n \times l}$. Then, the following statements hold:

- (i) $\text{rank}(A) \leq \min\{m, n\}$;
- (ii) $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(A^T A) = \text{rank}(A A^T)$;
- (iii) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$;
- (iv) $\text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n$;
- (v) $\text{rank}(AB) = n$ if $\text{rank}(A) = \text{rank}(B) = n$.

II. A PRELIMINARY REVIEW OF THE OPA BASED CLOSED-LOOP SIMS

Consider a closed-loop system where the controlled plant is generally described in the innovation form,

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) + Ke(t) \\ y(t) = Cx(t) + Du(t) + e(t) \end{cases} \quad (1)$$

where t denotes the discrete time step for implementation, $x(t) \in \mathfrak{R}^{n_x}$, $y(t) \in \mathfrak{R}^{n_y}$, $u(t) \in \mathfrak{R}^{n_u}$, and $e(t) \in \mathfrak{R}^{n_x}$ denotes the noise innovation sequence with unknown variance σ_e^2 . The closed-loop controller is described by

$$\begin{cases} x_c(t+1) = A_c x_c(t) + B_c[r(t) - y(t)] \\ u(t) = C_c x_c(t) + D_c[r(t) - y(t)] \end{cases} \quad (2)$$

where $r(t) \in \mathfrak{R}^{n_r}$ is the setpoint input, and the controller state is $x_c(t) \in \mathfrak{R}^{n_{c-x}}$.

Regarding the time step, $t_0 + i$, in an identification test, the past and future input block-Hankel matrices are denoted by:

$$U_p = \begin{bmatrix} u(t_0) & u(t_0+1) & \cdots & u(t_0+j-1) \\ u(t_0+1) & u(t_0+2) & \cdots & u(t_0+j) \\ \vdots & \vdots & \cdots & \vdots \\ u(t_0+i-1) & u(t_0+i) & \cdots & u(t_0+i+j-2) \end{bmatrix} \in \mathfrak{R}^{i n_u \times j} \quad (3)$$

$$U_f = \begin{bmatrix} u(t_0+i) & u(t_0+i+1) & \cdots & u(t_0+i+j-1) \\ u(t_0+i+1) & u(t_0+i+2) & \cdots & u(t_0+i+j) \\ \vdots & \vdots & \cdots & \vdots \\ u(t_0+2i-1) & u(t_0+2i) & \cdots & u(t_0+2i+j-2) \end{bmatrix} \in \mathfrak{R}^{i n_u \times j} \quad (4)$$

where $j > i > n_x$ is a precondition adopted for data collection and computation.

Similarly, the block-Hankel matrices of the setpoint input, system output, and noise innovation are defined by R_p , $R_f \in \mathfrak{R}^{i n_y \times j}$, Y_p , $Y_f \in \mathfrak{R}^{i n_y \times j}$, and ξ_p , $\xi_f \in \mathfrak{R}^{i n_x \times j}$, respectively. Correspondingly, the past and future state vectors are denoted by

$$X_p \triangleq [x(t_0), x(t_0+1), \cdots, x(t_0+j-1)] \in \mathfrak{R}^{n_x \times j} \quad (5)$$

$$X_f \triangleq [x(t_0+i), x(t_0+i+1), \cdots, x(t_0+i+j-1)] \in \mathfrak{R}^{n_x \times j} \quad (6)$$

$$X_p^c \triangleq [x_c(t_0), x_c(t_0+1), \cdots, x_c(t_0+j-1)] \in \mathfrak{R}^{n_{c-x} \times j} \quad (7)$$

$$X_f^c \triangleq [x_c(t_0+i), x_c(t_0+i+1), \cdots, x_c(t_0+i+j-1)] \in \mathfrak{R}^{n_{c-x} \times j} \quad (8)$$

Iterating (1) and (2) in terms of the past and future time sequences with respect to $t_0 + i$, yields

$$Y_p = \Gamma_i X_p + H_i U_p + \Phi_i \xi_p \quad (9)$$

$$Y_f = \Gamma_i X_f + H_i U_f + \Phi_i \xi_f \quad (10)$$

$$U_p = \Gamma_i^c X_p^c + H_i^c (R_p - Y_p) \quad (11)$$

$$U_f = \Gamma_i^c X_f^c + H_i^c (R_f - Y_f) \quad (12)$$

where

$$\Gamma_i = [C^T, A^T C^T, \cdots, (A^{i-1})^T C^T]^T \in \mathfrak{R}^{i n_y \times n_x} \quad (13)$$

$$H_i = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & D & 0 \\ CA^{i-2}B & CA^{i-3}B & \cdots & CB & D \end{bmatrix} \in \mathfrak{R}^{i n_y \times i n_u} \quad (14)$$

$$\Phi_i = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ CK & I & 0 & \cdots & 0 \\ CAK & CK & I & \cdots & 0 \\ \vdots & \vdots & \vdots & I & 0 \\ CA^{i-2}K & CA^{i-3}K & \cdots & CK & I \end{bmatrix} \in \mathfrak{R}^{i n_y \times i n_y} \quad (15)$$

$$\Gamma_i^c = [C_c^T, A_c^T C_c^T, \cdots, (A_c^{i-1})^T C_c^T]^T \in \mathfrak{R}^{i n_y \times n_{c-x}} \quad (16)$$

$$H_i^c = \begin{bmatrix} D_c & 0 & 0 & \cdots & 0 \\ C_c B_c & D_c & 0 & \cdots & 0 \\ C_c A_c B_c & C_c B_c & D_c & \cdots & 0 \\ \vdots & \vdots & \vdots & D_c & 0 \\ C_c A_c^{i-2} B_c & C_c A_c^{i-3} B_c & \cdots & C_c B_c & D_c \end{bmatrix} \in \mathfrak{R}^{i n_y \times i n_y} \quad (17)$$

From (9) or (10) it can be seen that the state-space model identification depends on the estimation of the extended observability matrix Γ_i and the block Toeplitz matrix H_i . Denote two short-hands, $W_p = [Y_p^T, U_p^T]^T \in \mathfrak{R}^{(i n_y + i n_u) \times j}$ and $W_f = [Y_f^T, U_f^T]^T \in \mathfrak{R}^{(i n_y + i n_u) \times j}$. It follows from (10) and (12) that

$$\begin{bmatrix} I & -H_i \\ H_i^c & I \end{bmatrix} W_f = \begin{bmatrix} \Gamma_i & 0 \\ 0 & \Gamma_i^c \end{bmatrix} \begin{bmatrix} X_f \\ X_f^c \end{bmatrix} + \begin{bmatrix} \Phi_i & 0 \\ 0 & H_i^c \end{bmatrix} \begin{bmatrix} \xi_f \\ R_f \end{bmatrix} \quad (18)$$

To separately identify Γ_i and H_i of the plant from those of the closed-loop controller, by considering that W_p is uncorrelated with ξ_f , an instrumental matrix variable, $\hat{W} = [W_p^T, R_f^T]^T$, was chosen in [4] to perform an orthogonal projection of (18) onto the row space of \hat{W} in order to eliminate the noise influence, obtaining

$$\begin{bmatrix} I & -H_i \\ H_i^c & I \end{bmatrix} W_f / \hat{W} = \begin{bmatrix} \Gamma_i & 0 \\ 0 & \Gamma_i^c \end{bmatrix} \begin{bmatrix} X_f / \hat{W} \\ X_f^c / \hat{W} \end{bmatrix} + \begin{bmatrix} \Phi_i & 0 \\ 0 & H_i^c \end{bmatrix} \begin{bmatrix} 0 \\ R_f / \hat{W} \end{bmatrix} \quad (19)$$

It can be seen from (19) that

$$[I \quad -H_i] W_f / \hat{W} = \Gamma_i X_f / \hat{W} \quad (20)$$

Then, premultiplying both sides of (20) by the orthogonal complement of Γ_i , denoted by Γ_i^\perp , yields

$$(\Gamma_i^\perp)^T [I \quad -H_i] W_f / \hat{W} = 0 \quad (21)$$

Performing a PCA based SVD on W_f / \hat{W} gives

$$W_f / \hat{W} = [P \quad \tilde{P}] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_p^T \\ V_p^T \end{bmatrix} \quad (22)$$

It can be seen from (21) that the orthogonal column space of W_f / \hat{W} is equal to the column space of $(\Gamma_i^\perp)^T [\mathbf{I} \ -H_i]^T$. Hence, it follows that

$$(\Gamma_i^\perp)^T [\mathbf{I} \ -H_i]^T = \tilde{\mathbf{P}}M \quad (23)$$

where M is any matrix of full rank.

By partitioning

$$\tilde{\mathbf{P}}M = \begin{bmatrix} \tilde{\mathbf{P}}_1 \\ \tilde{\mathbf{P}}_2 \end{bmatrix} \quad (24)$$

it can be solved from (23) that

$$\Gamma_i = \tilde{\mathbf{P}}_1^\perp \quad (25)$$

$$H_i = -(\tilde{\mathbf{P}}_1^T)^\dagger \tilde{\mathbf{P}}_2^T \quad (26)$$

Consequently, the state matrices (A, B, C, D) can be retrieved from an analytical shift-invariant approach as developed in [15] and [16].

In [5], it was proposed to add a column weighting matrix, $W_c = [(1/N)\hat{W}\hat{W}^T]^{-1/2}$, to perform the projection of (18) onto the row space of \hat{W} , i.e.

$$(\Gamma_i^\perp)^T [\mathbf{I} \ -H_i] W_f \hat{W}^T W_c = \mathbf{0} \quad (27)$$

Since it was clarified in [5] that the above weighting projection is essentially equivalent to the orthogonal projection of (21) given in [4], the achievable identification accuracy is the same with each other.

It is obvious that for the deterministic case (noise-free), i.e., $\xi_f = \mathbf{0}$, consistent estimation can be guaranteed by the above PCA-based SVD approach. In the presence of measurement noise, additional nonzero eigenvalue(s) will occur in the above SVD of (22), which may confuse the choice of the orthogonal column space of W_f / \hat{W} for computation. To overcome the deficiency, we analyze the rank condition for consistent estimation against measurement noise in the following section.

III. CONSISTENCY ANALYSIS

Consider the form of $\hat{W} = [W_p^T \ R_f^T]^T$, it can be easily verified that if the setpoint input excitation satisfies $PE(r) \geq 2i$ ($i > n_x$), \hat{W} is guaranteed full row rank and therefore, its SVD can be obtained as

$$\hat{W} = U_{\hat{W}} \begin{bmatrix} \Lambda_{\hat{W}} & \mathbf{0} \\ & V_{\hat{W}_2}^T \end{bmatrix} \quad (28)$$

where $U_{\hat{W}} \in \mathfrak{R}^{i(n_u+n_y+n_r) \times i(n_u+n_y+n_r)}$ is a unitary matrix, $\Lambda_{\hat{W}} \in \mathfrak{R}^{i(n_u+n_y+n_r) \times i(n_u+n_y+n_r)}$ is a diagonal matrix with nonnegative entries, $V_{\hat{W}_1} \in \mathfrak{R}^{j \times i(n_u+n_y+n_r)}$ and $V_{\hat{W}_2} \in \mathfrak{R}^{j \times (j-i n_u - i n_y - i n_r)}$ are of full column rank matrix composed of unitary orthogonal column vectors.

Substituting (28) into the orthogonal projection of W_f / \hat{W} , we obtain

$$W_f / \hat{W} = W_f V_{\hat{W}_1} V_{\hat{W}_1}^T \quad (29)$$

Note that

$$\text{rank}(U_{\hat{W}} \Lambda_{\hat{W}}) = \text{rank}(\Lambda_{\hat{W}}^T U_{\hat{W}}^T) = i(n_u + n_y + n_r) \quad (30)$$

According to the statements (iii) and (iv) in Lemma 1, we have

$$\text{rank}[\hat{E}(W_f \hat{W}^T)] = \text{rank}[\hat{E}(W_f V_{\hat{W}_1} \Lambda_{\hat{W}}^T U_{\hat{W}}^T)] \leq \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] \quad (31)$$

$$\text{rank}[\hat{E}(W_f \hat{W}^T)] \geq \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] + \text{rank}(\Lambda_{\hat{W}}^T U_{\hat{W}}^T) - i(n_u + n_y + n_r) \quad (32)$$

Substituting (30) into (32), one can see that

$$\text{rank}[\hat{E}(W_f \hat{W}^T)] = \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] \quad (33)$$

Similarly, it follows that

$$\text{rank}[\hat{E}(W_f V_{\hat{W}_1} V_{\hat{W}_1}^T)] \leq \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] \quad (34)$$

$$\text{rank}[\hat{E}(W_f V_{\hat{W}_1} V_{\hat{W}_1}^T)] \geq \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] + \text{rank}(V_{\hat{W}_1}^T) - i(n_u + n_y + n_r) \quad (35)$$

In view of $\text{rank}(V_{\hat{W}_1}^T) = i(n_u + n_y + n_r)$, there exists

$$\text{rank}[\hat{E}(W_f V_{\hat{W}_1} V_{\hat{W}_1}^T)] = \text{rank}[\hat{E}(W_f V_{\hat{W}_1})] \quad (36)$$

Therefore, it follows from (29) and (33) that

$$\text{rank}[\hat{E}(W_f / \hat{W})] = \text{rank}[\hat{E}(W_f \hat{W}^T)] \quad (37)$$

For using the column weighting matrix [5], it can be easily verified that

$$(W_f / \hat{W})(W_f / \hat{W})^T = \frac{1}{N} (W_f \hat{W}^T W_c)(W_f \hat{W}^T W_c)^T \quad (38)$$

According to the statements (ii) in Lemma 1, we have

$$\text{rank}[\hat{E}(W_f / \hat{W})] = \text{rank}[\hat{E}(W_f \hat{W}^T W_c)] \quad (39)$$

For consistent estimation of the extended observability matrix Γ_i and the block Toeplitz matrix H_i against the measurement noise, the following theorem is given:

Theorem 1 Given white measurement noise in an identification test, if the input excitation sequence satisfies $PE(u) \geq 2i$, there follows

$$\text{rank}[\lim_{N \rightarrow \infty} \hat{E}(W_f V_{\hat{W}_1} V_{\hat{W}_1}^T)] = n_x + i n_u \quad (40)$$

and for $N \rightarrow \infty$,

$$\begin{bmatrix} \Gamma_i^\perp \\ -H_i^T \Gamma_i^\perp \end{bmatrix} = \tilde{\mathbf{P}}\Phi \quad (41)$$

where $\tilde{\mathbf{P}}$ is obtained from the SVD of

$$W_f V_{\hat{W}_1} V_{\hat{W}_1}^T = \begin{bmatrix} \mathbf{P} & \tilde{\mathbf{P}} \end{bmatrix} \begin{bmatrix} \Lambda_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} V_p^T \\ V_{\tilde{\mathbf{P}}}^T \end{bmatrix} \quad (42)$$

and $\Phi \in \mathfrak{R}^{(i n_y - n_x) \times (i n_y - n_x)}$ is any matrix of full rank.

Proof: It follows from (9) and (11) that

$$\hat{W} = \begin{bmatrix} Y_p \\ U_p \\ U_f \end{bmatrix} = \begin{bmatrix} \Gamma_i & H_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix} + \begin{bmatrix} \Phi_i \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \xi_p \quad (43)$$

$$W_f = \begin{bmatrix} Y_f \\ U_f \end{bmatrix} = \begin{bmatrix} \Gamma_i & H_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} X_f \\ U_f \end{bmatrix} + \begin{bmatrix} \Phi_i \\ \mathbf{0} \end{bmatrix} \xi_f \quad (44)$$

Postmultiplying both sides of (44) by \hat{W}^T and taking the statistical expectation to eliminate the noise term yield

$$\lim_{N \rightarrow \infty} \hat{E}(W_f \hat{W}^T) = \lim_{N \rightarrow \infty} \hat{E} \left(\begin{bmatrix} \Gamma_i & H_i \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} X_f \\ U_f \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix}^T \begin{bmatrix} \Gamma_i^T & \mathbf{0} & \mathbf{0} \\ H_i^T & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \right) \quad (45)$$

In consideration of the plant description in (1) is minimal in the sense that (A, B) is reachable and (A, C) is observable, it follows that

$$\text{rank}[B, AB, \dots, A^{n_x-1}B] = n_x \quad (46)$$

$$\text{rank}[C^T, A^T C^T, \dots, (A^T)^{n_x-1} C^T]^T = n_x \quad (47)$$

Therefore,

$$\text{rank}(\Gamma_i) = n_x \quad (48)$$

$$\text{rank} \begin{pmatrix} \Gamma_i & H_i \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = n_x + in_u \quad (49)$$

Similarly, it can be verified that the rightmost matrix at the right-hand side of (45) is full row rank.

According to the statements (ii)-(iv) in Lemma 1, if

$$\text{rank} \left\{ \lim_{N \rightarrow \infty} \hat{E} \left(\begin{bmatrix} X_f \\ U_f \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix}^T \right) \right\} = n_x + in_u \quad (50)$$

there will be

$$\text{rank} \left\{ \lim_{N \rightarrow \infty} \hat{E}(W_f \hat{W}^T) \right\} = \text{rank} \left\{ \lim_{N \rightarrow \infty} \hat{E} \left(\begin{bmatrix} X_f \\ U_f \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix}^T \right) \right\} \quad (51)$$

Since it can be derived from (1),

$$x(t_0 + i) = A^i x(t_0) + \sum_{k=1}^i A^{i-k} B u(t_0 + i - k) \quad (52)$$

we formulate

$$\begin{bmatrix} X_f \\ U_f \end{bmatrix} = \begin{bmatrix} A^i & A^{i-1}B & \dots & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix} + \begin{bmatrix} \mathbf{0} & A^{i-1}K & A^{i-2}K & \dots & K \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \xi_p \quad (53)$$

Note that the coefficient matrix of the first term at the right-hand side of (53) is full row rank of $n_x + in_u$ owing to (46). Moreover, it follows from the fact that the white noise innovation, $e(t)$, is independent of the plant state sequence $x(t)$, and the excitation sequence $u(t)$ used in an identification test, i.e.

$$E \left\{ \zeta(t) \begin{bmatrix} x(t) & u(k) \end{bmatrix} \right\} = \mathbf{0}, \quad \forall t, k \quad (54)$$

Therefore, we can establish a sufficient condition to guarantee (50), i.e.

$$\begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix} \begin{bmatrix} X_p \\ U_p \\ U_f \end{bmatrix}^T > \mathbf{0} \quad (55)$$

Owing to the fact from (1) that

$$X_p = (z\mathbf{I} - A)^{-1} \{ B[u(t_0), u(t_0+1), \dots, u(t_0+j-1)] + K[e(t_0), e(t_0+1), \dots, e(t_0+j-1)] \} \quad (56)$$

there follows for $PE(u) \geq 2i$ that

$$E \begin{pmatrix} X_p \\ U_p \end{pmatrix} \begin{pmatrix} X_p \\ U_p \end{pmatrix}^T = \mathbf{0} \quad (57)$$

Then, using a proved fact for $PE(u) > i$ (see [17]) that

$$\begin{bmatrix} X_p \\ U_p \end{bmatrix} \begin{bmatrix} X_p \\ U_p \end{bmatrix}^T > \mathbf{0} \quad (58)$$

it is then straightforward to prove that (55) holds true.

Hence, we obtain from (51) that

$$\text{rank} \left[\lim_{N \rightarrow \infty} \hat{E}(W_f \hat{W}^T) \right] = n_x + in_u \quad (59)$$

According to (29) and (37), the rank condition in (40) follows directly.

Correspondingly, in the SVD of $W_f V_{\hat{w}_1} V_{\hat{w}_1}^T$ as shown in (42), there should be

$$\text{rank}(\mathbf{P}) = n_x + in_u, \quad \mathbf{P} \in \mathfrak{R}^{[i(n_u+n_y)+q] \times (n_x+q+in_u)} \quad (60)$$

$$\text{rank}(\tilde{\mathbf{P}}) = in_y - n_x, \quad \tilde{\mathbf{P}} \in \mathfrak{R}^{[i(n_u+n_y)+q] \times (in_y-n_x)} \quad (61)$$

It is obvious that the orthogonal column space of $W_f V_{\hat{w}_1} V_{\hat{w}_1}^T$ is spanned by $\tilde{\mathbf{P}}$. Meanwhile, it can be seen from (21) that the orthogonal column space of $W_f V_{\hat{w}_1} V_{\hat{w}_1}^T$ is equivalent to the column space of $((\Gamma_i^\perp)^T [\mathbf{I} \quad -H_i])^T$. Therefore, we conclude

$$((\Gamma_i^\perp)^T [\mathbf{I} \quad -H_i])^T = \tilde{\mathbf{P}} \Phi \quad (62)$$

which is (41) in Theorem 1. This completes the proof. \square

For the convenience of computation, it is recommended to take $\Xi = \mathbf{I}$ and correspondingly, it can be obtained that

$$\begin{bmatrix} \Gamma_i^\perp \\ -H_i^T \Gamma_i^\perp \end{bmatrix} = \tilde{\mathbf{P}} = \begin{bmatrix} \tilde{\mathbf{P}}_y \\ \tilde{\mathbf{P}}_u \end{bmatrix} \quad (63)$$

where $\tilde{\mathbf{P}}_y \in \mathfrak{R}^{n_y \times (n_y - n_x)}$ and $\tilde{\mathbf{P}}_u \in \mathfrak{R}^{n_u \times (n_y - n_x)}$.

Correspondingly, there are

$$\Gamma_i = \tilde{\mathbf{P}}_y^\perp \quad (64)$$

$$-\tilde{\mathbf{P}}_y^T H_i = \tilde{\mathbf{P}}_u^T \quad (65)$$

In the sequel, the state matrices (A, B, C, D) can be retrieved from an analytical shift-invariant approach as developed in [15] and [16], which is briefly summarized as below.

$$C = \Gamma_i(1:n_y, :) \quad (66)$$

The other matrices can be derived by the standard LS algorithm from

$$\Gamma_i(1:n_y, (i-1):i) A = \Gamma_i(n_y+1:n_y, i):i) \quad (67)$$

$$\begin{bmatrix} D \\ B \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n_y \times n_y} & \mathbf{0}_{n_y \times n_x} \\ \mathbf{0}_{n_y(i-1) \times n_y} & \Gamma_i(1:n_y(i-1), :) \end{bmatrix}^\dagger H_{i-1} \quad (68)$$

where $H_{i-1} = [D^T, B^T C^T, B^T A^T C^T, \dots, B^T (A^{i-2})^T C^T]^T$.

Note that the above algorithm gives a shift-invariant estimation of the state matrices (A, B, C, D) up to a similarity transformation. Model validation should be performed by verifying the eigenvalues of A that equal the plant transfer function poles of the transfer matrix consisting of a combination of (A, B, C, D) .

IV. ILLUSTRATION

Consider an illustrated example studied in the references [4, 10], where the controller plant is described by

$$A = \begin{bmatrix} 4.4 & 1 & 0 & 0 & 0 \\ -8.9 & 0 & 1 & 0 & 0 \\ 7.83 & 0 & 0 & 1 & 0 \\ -4.0 & 0 & 0 & 0 & 1 \\ 0.86 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.00098 \\ 0.01299 \\ 0.01859 \\ 0.0033 \\ -0.0002 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad K = \begin{bmatrix} 2.3 \\ -6.64 \\ 7.515 \\ -4.0146 \\ 0.86336 \end{bmatrix}$$

and the output feedback controller is

$$A_c = \begin{bmatrix} 2.65 & -3.11 & 1.75 & -0.39 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C_c^T = \begin{bmatrix} -0.4135 \\ 0.8629 \\ -0.7625 \\ 0.2521 \end{bmatrix}$$

with $D_c = 0.61$.

For illustration, the simulation tests adopted in [4] are performed here: The noise innovation sequence is a zero-mean Gaussian white noise with variance $1/9$, and the setpoint input excitation is taken as another zero-mean Gaussian white noise sequence with the unity variance. 100 Monte-Carlo tests are performed with a data length of $N = 1000$ taken in each test for computation. By taking $p = f = 10$, the proposed identification algorithm based on Theorem 1 gives the plant transfer function poles (i.e., the eigenvalues of A) as listed in Table 1, where the result of each pole is shown by the mean value along with the standard deviation in parentheses.

It is seen that good accuracy is obtained by the proposed identification algorithm. Note that similar results can also be obtained by [4] based on the knowledge of the rank condition for making the corresponding SVD for computation. Otherwise, additional eigenvalues may be taken in the computation, which will lead to erratic identification results.

V. CONCLUSION

For the development of OPA based closed-loop SIMs, some consistency analysis has been presented in this paper. To eliminate the noise influence for consistent estimation, the rank condition for performing the PCA based SVD to derive the extended observability matrix and the lower triangular

block-Toeplitz matrix of the plant state-space model has been disclosed by a strict proof, together with the requirement on the setpoint input excitation. It is a notable merit that no knowledge of the closed-loop controller is required in the proposed consistency analysis. An illustrative example has well demonstrated the effectiveness of the proposed rank condition for application of the OPA based closed-loop SIMs.

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TABLE 1 COMPARISON OF IDENTIFIED PLANT POLES (EIGENVALUES OF A) WITH THE TRUE VALUES

True poles	Identified poles
$0.7319 \pm 0.6007i$	$0.7339 (\pm 0.0153) \pm 0.6014 (\pm 0.0108)i$
$0.9681 \pm 0.1486i$	$0.9674 (\pm 0.0134) \pm 0.1477 (\pm 0.0112)i$
1.0	$0.9999 (\pm 0.0006)$