2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

On the stability of Lyapunov exponents of discrete linear systems

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***Abstract*— In this paper we present a sufficient condition for continuity, called in the paper stability, of Lyapunov exponents of discrete time-varying linear system. Basing on the result we show that Lyapunov exponents of time-invariant systems depend continuously on the time-varying perturbations.**

Key words: time-varying discrete linear systems, Lyapunov exponents, perturbation theory, characteristic exponents.

I. INTRODUCTION

CONSIDER the linear discrete time-varying system

$$x(n+1) = A(n)x(n), n \geq 0 \quad (1)$$

where $(A(n))_{n\in\mathbb{N}}$ is a bounded sequence of invertible s-by-s real matrices such that sequence $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded. Together with (1) we consider the following perturbed system

$$y(n+1) = (A(n) + \Delta(n))\, y(n), \quad (2)$$

where $(\Delta(n))_{n\in\mathbb{N}}$ is a bounded sequence of s-by-s real matrices. By $\|\cdot\|$ we denote the Euclidean norm and the induced operator norm in $R^{s\times s}$. The transition matrix is defined as

$$\mathcal{A}(m) = A(m-1)...A(0)$$

for $m > 1$ and $\mathcal{A}(0) = I$, where I is the identity matrix. For an initial condition x_0 the solution of (1) is denoted by $x(n, x_0)$ so

$$x(n, x_0) = \mathcal{A}(n)x_0.$$

Let $a = (a(n))_{n\in\mathbb{N}}$ be a sequence of real numbers. The number (or the symbols $\pm\infty$) defined as

$$\lambda(a) = \limsup_{n\to\infty} \frac{1}{n} \ln |a(n)|$$

are called the characteristic exponent of sequence $(a(n))_{n\in\mathbb{N}}$. For $x_0 \in R^s$, $x_0 \neq 0$ the Lyapunov exponent $\lambda(x_0)$ of (1) is defined as the characteristic exponent of $(\|x(n, x_0)\|)_{n\in\mathbb{N}}$ that is

$$\lambda_A(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n, x_0)\|.$$

Discrete linear difference equations (1) often appear in control theory directly as models of real systems or as a linearization of nonlinear models, and finally, as a result of discretization of continuous-time linear models. In the latter case it should be noted that, even if we discretize stationary linear system but the sampling rate is variable, then the resulting discrete model is nonstationary. In practical applications, a typical situation is that the model parameters are known only to some precision (e.g., are the result of the observed model parameter estimation). In this case, the set of Lyapunov exponents for the approximated values of the parameters generally differ from the values corresponding to actual values (see e.g. [9]). It is possible even if the estimation error tends to 0, because as it is well known, that Lyapunov exponents are discontinuous functions of coefficients (see [6]). In control theory, the main purpose for which the Lyapunov exponents are used, is to analyse stability of the system (see, e.g. [1]). Discontinuity of Lyapunov exponents causes that, the analysis of the stability of the system (1) with employs of Lyapunov exponents of the estimated system may not be correct.

In this paper we will propose certain conditions that guarantee continuous dependence of the spectrum with respect to the coefficients of (1).

This problem for continuous-time case is known as the problem of stability of characteristic exponents and it is completely solved. Necessary and sufficient conditions for the stability of Lyapunov exponents were published by Bylov and Izobov (joint papers [4] and [5]) and Milionschikov [14].

From the control theory point of view the following question is very important. Provided that (1) is exponentially stable find the maximal value δ such that if $\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \delta$, then system (2) is exponentially stable. This problem, known as a problem stability radius, has been formulated and solved for time-invariant system by Hinrichen and Pritchard in [11]. If the maximal Lyapunov exponent of (1) does not depend continuously on the coefficients, then the solution to the above formulated problem does not have to exist. The results of our paper provide sufficient conditions for solvability of this problem.

Yet another motivation to study the above-mentioned issue is the problem of stability by the linear approximation. Together with (1) we consider the following nonlinear system

$$y(n+1) = A(n)y(n) + f(n, y(n)), \quad (3)$$

where the $f : \mathbb{N} \times R^s \to R^s$ belongs to the class F_m consisting of all functions $g : \mathbb{N} \times R^s \to R^s$ for which there exists a constant C_g such that

$$\|g(n, x)\| \leq C_g \|x\|^m$$

for all $n \in \mathbb{N}$ and $x \in R^s$. The problem of stability by the linear approximation consists in finding condition on exponentially stable system (1), that imply exponential stability of (3) for all $f \in \cup_{m>1} F_m$ The problem of stability

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by the linear approximation has been intensively investigated for continuous time systems since the fundamental Lyapunov's paper [12]. He proved that if the system of the first approximation is regular and all its Lyapunov exponents are negative, then the solution of the original system is asymptotically Lyapunov stable. In 1930, it was stated by O. Perron that the requirement of regularity of the first approximation is substantial. He constructed an example of the second-order system of the first approximation, which has negative characteristic exponents along a zero solution of the original system but, at the same time, this zero solution of original system is Lyapunov unstable. Furthermore, in a certain neighborhood of this zero solution almost all solutions of original system have positive characteristic exponents. Finally, it is clear that continuity of the greatest Lyapunov exponent of (1) is a necessary condition for the stability by the linear approximation.

Denote the greatest and the smallest exponent of (1) by $\lambda_g(A)$ and $\lambda_s(A)$, respectively. The quantities

$$\Lambda_g(\mathfrak{M}) = \sup\{\lambda_g(A+\Delta) : \Delta \in \mathfrak{M}\}$$

$$\Lambda_s(\mathfrak{M}) = \inf\{\lambda_g(A+\Delta) : \Delta \in \mathfrak{M}\}$$

are referred to us the maximal upper and minimal lower movability boundary of the greatest exponent of (1) with perturbation in the class $\mathfrak{M}$. It is clear that if $\mathfrak{M}$ consists of all sequences tending to zero and the Lyapunov exponents of (1) are continuous then $\Lambda_g(\mathfrak{M}) = \lambda_g(A) = \Lambda_s(\mathfrak{M})$.

In case when the Laypunov exponents of (1) are not continuous the determination of the movability boundaries of the higher exponent under various perturbations is one of the main problems of Lyapunov exponent theory. For certain classes of perturbation this problem is solved in [6] and [7]. The problem of continuity of Lyapunov exponents have been investigated in [8].

II. Main results

It is well known [2] that the set of all Lyapunov exponents of system (1) contains at most s elements, say $-\infty < \lambda_1(A) < \lambda_2(A) < \ldots < \lambda_r(A) < \infty$ and the set $\{\lambda_1(A), \lambda_2(A), \ldots, \lambda_r(A)\}$ is called the spectrum of (1). For each λ_i, $i = 1, \ldots, r$ we consider the following subspace of R^s

$$E_i = \{v \in R^s : \lambda(v) \le \lambda_i\}$$

and we set $E_0 = \{0\}$. The multiplicities n_i of Lyapunov exponent λ_i are defined as $\dim E_i - \dim E_{i-1}$. For a base $V = \{v_1, \ldots, v_s\}$ of R^s we define the sum σ_V of Lyapunov exponents

$$\sigma_V = \sum_{i=1}^{s} \lambda(v_i).$$

It is known (see [10]) that if $v_1, \ldots, v_s$ is a basis or R^s then the following Lyapunov inequality holds:

$$\limsup_{n\to\infty} \frac{1}{n} \ln|\det \mathcal{A}(n)| \le \sum_{l=1}^{s} \lambda(v_l). \tag{4}$$

The basis $v_1, \ldots, v_s$ is called normal if for each $i = 1, \ldots, r$ there exists a basis of E_i composed of some vectors from the set $\{v_1, \ldots, v_s\}$. Formally, we should say that a basis is normal with respect of family E_i, $i = 1, \ldots, r$. It can be shown (see [3], Theorem 1.2.3) that for the normal basis the sum σ_V of Lyapunov exponents is minimal and then, according to Lyapunov inequality (see [10]), equal to

$$\limsup_{n\to\infty} \frac{1}{n} \ln|\det \mathcal{A}(n)|.$$

For a basis $v_1, \ldots, v_s$ of R^s matrix $\mathcal{V}(n)$, $n \in \mathbb{N}$ whose columns are $x(n, v_1)$,..., $x(n, v_s)$ is called fundamental matrix of (1). For a fundamental matrix the kernel $\mathcal{G}(n, m) = \mathcal{V}(n)\mathcal{V}^{-1}(m)$, $n, m \in \mathbb{N}$ is called Green's matrix of (1). If the base is normal, then the fundamental and Green's matrices are called normal.

Consider the values

$$\lambda_1'(A) \le \lambda_2'(A) \le \ldots \le \lambda_s'(A) \tag{5}$$

of the Lyapunov exponents of (1), counted with their multiplicities. Denote by $\lambda_1'(A+\Delta) \le \lambda_2'(A+\Delta) \le \ldots \le \lambda_s'(A+\Delta)$ the Lyapunov exponents of (2) counted with their multiplicities. We have following definition.

Definition 1: The Lyapunov exponents of system (1) are called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \delta \tag{6}$$

implies the inequality

$$|\lambda_i'(A) - \lambda_i'(A+\Delta)| < \varepsilon,\ i = 1, \ldots, s.$$

To formulate our main results for a Green's matrix of (1) denote by $x_i(m, n)$ the i-th column of it and by μ_i the characteristic exponent of the sequence $(\|x_i(m, n)\|)_{m\in\mathbb{N}}$, $i = 1, \ldots, s$. The next theorem constitutes discrete-time version of Malkin's (see, [13]) sufficient condition for continuity of Lyapunov exponents.

Theorem 1: Suppose that for certain Green's matrix $\mathcal{G}(m, n)$ of (1) and any $\gamma > 0$ there exists $d > 0$ such that

$$\|x_i(m, n)\| \le d \exp[(\mu_i + \gamma)(m - n)] \tag{7}$$

$$\text{for } m, n \in \mathbb{N},\ m \ge n,\ i = 1, \ldots, s$$

and

$$\|x_i(m, n)\| \le d \exp[(\mu_i - \gamma)(m - n)] \tag{8}$$

$$\text{for } m, n \in \mathbb{N},\ n \ge m,\ i = 1, \ldots, s,$$

then the Lyapunov exponents of system (1) are stable.

Proof: The proof of the theorem consists of the following three parts:

1. the shift of the characteristic exponents to the right is small,

2. there exists limit $\lim_{n\to\infty} \frac{1}{n} \ln|\det \mathcal{A}(n)| = \sum_{i=1}^{s} \mu_i$,

3. the shift of the characteristic exponents to the left is small.

1. Let $n_0 \in \mathbb{N}$. According to the variation of constant formula ([1], pp.58) any solution $y(n, y_0)$ of (2) satisfying $y(n_0, y_0) = y_0$, satisfies the equation

$$y(n,y_0) = \begin{cases} \mathcal{G}(n,n_0)\, y_0 + \\ \sum_{l=n_0+1}^{n} \mathcal{G}(n,l)\, \Delta(l-1) y(l-1,y_0) \\ \text{for } n \geq n_0 \\ \mathcal{G}(n,n_0)\, y_0 - \\ \sum_{l=n+1}^{n_0} \mathcal{G}(n,l)\, \Delta(l-1) y(l-1,y_0) \\ \text{for } n < n_0 \end{cases} . \quad (9)$$

For system (1) consider a normal basis $v_1, ..., v_s$ and consider s solutions $y_i(n, v_i)$, $i = 1, ..., s$ of (2) satisfying $y(n_0, v_i) = v_i$, $i = 1, ..., s$. Assume that the numeration of the basis is such that

$$\lambda_A(v_i) \leq \lambda_A(v_{i+1}),\ i = 1, ..., s-1.$$

From (9) we have

$$y_i(n,v_i) = \begin{cases} x_i(n,v_i) + \\ \sum_{l=n_0+1}^{n} \mathcal{G}(n,l)\, \Delta(l-1) y_i(l-1,v_i) \\ \text{for } n \geq n_0 \\ x_i(n,v_i) - \\ \sum_{l=n+1}^{n_0} \mathcal{G}(n,l)\, \Delta(l-1) y_i(l-1,v_i) \\ \text{for } n < n_0 \end{cases} . \quad (10)$$

Take an $\varepsilon > 0$ such that

$$\varepsilon < (\lambda_A(v_s) - \lambda_A(v_i))/2 \quad (11)$$

for all $i = 1, ..., s$ such that $\lambda_A(v_s) \neq \lambda_A(v_i)$. For such ε there exists positive constant c such that

$$\|x_i(n,v_i)\| \leq c \exp[(\lambda_A(v_i) + \varepsilon)\, n]$$

for all $i = 1, ..., s$ and $n \in \mathbb{N}$. We will show that

$$\|y_i(n,v_i)\| \leq 2c \exp[(\lambda_A(v_i) + \varepsilon)\, n] \quad (12)$$

for all $i = 1, ..., s$ and $n \in \mathbb{N}$. For $n = 0$ (12) is true. Consider first the case of i such that $\lambda_A(v_s) = \lambda_A(v_i)$. Suppose that (12) holds for $n = 0, ..., p-1$. Let estimate $\|y_i(p, v_i)\|$. According to the first equality in (10) with $n_0 = 0$ we have

$$\|y_i(p,v_i)\| \leq \|x_i(p,v_i)\| + \sum_{l=1}^{p} \|\mathcal{G}(p,l)\| \, \|\Delta(l-1)\| \, \|y_i(l-1,v_i)\| .$$

For $\gamma = \varepsilon/2$ let find d such that (7) and (8) hold. Then

$$\|y_i(p,v_i)\| \leq c \exp[(\lambda_A(v_i) + \varepsilon)\, p] + 2cd\delta \sum_{l=1}^{p} \exp\left[\left(\lambda_A(v_i) + \frac{\varepsilon}{2}\right)(p-l)\right] \cdot \exp[(\lambda_A(v_i) + \varepsilon)(l-1)] \leq$$

$$c \exp[(\lambda_A(v_i) + \varepsilon)\, p] + \frac{2cd\delta e^{-\lambda_A(v_i)}}{e^{\frac{\varepsilon}{2}} - 1} \cdot \exp[(\lambda_A(v_i) + \varepsilon)\, p] .$$

Taking

$$\delta < \frac{e^{\frac{\varepsilon}{2}} - 1}{2dce^{-\lambda_A(v_i)}}$$

we get that (12) holds for $n = p$. Consider now the case of i such that $\lambda_A(v_s) > \lambda_A(v_i)$. Let estimate $\|y_i(p, v_i)\|$ .According to the second equality in (10) with $n_0 = \infty$ we have

$$\|y_i(p,v_i)\| \leq \|x_i(p,v_i)\| + \sum_{l=p+1}^{\infty} \|\mathcal{G}(p,l)\| \, \|\Delta(l-1)\| \, \|y_i(l-1,v_i)\| .$$

By (8) and (11) we have

$$\|y_i(p,v_i)\| \leq c \exp[(\lambda_A(v_i) + \varepsilon)\, p] + 2cd\delta \sum_{l=p+1}^{\infty} \exp\left[\left(\lambda_A(v_s) - \frac{\varepsilon}{2}\right)(p-l)\right] \cdot \exp[(\lambda_A(v_i) + \varepsilon)(l-1)] .$$

and as in previous case we obtain that (12) is true for $n = p$ and sufficiently small δ. It is also clear, from the estimates for

$$\sum_{l=1}^{n} \mathcal{G}(n,l)\, \Delta(l-1) y_i(l-1,v_i),$$

that for sufficiently small δ vectors $y_i(0, v_i)$, $i = 1, ...s$ differ little from the vectors v_i, $i = 1, ..., s$ and therefore are linear independent. Moreover from the estimates (12) we obtain

$$\lambda_{A+\Delta}(v_i) \leq \lambda_A(v_i) + \varepsilon.$$

If the basis $v_1, ..., v_s$ is not normal for (2), then, by passing to a normal basis, the exponents can only diminish; therefore, we have

$$\lambda'_i(A+\Delta) \leq \lambda'_i(A) + \varepsilon \quad (13)$$

for $i = 1, ..., s$.

2. Using Hadamard's inequality and (8) we have

$$|\det \mathcal{G}(0,n)| \leq d^s \prod_{i=1}^{s} \exp[(\mu_i - \gamma)(0-n)] = d^s \exp\left[-n \sum_{i=1}^{s} \mu_i\right] \exp[n\gamma s]$$

and therefore

$$\liminf_{n\to\infty} \frac{1}{n} \ln|\det \mathcal{A}(n)| = -\limsup_{n\to\infty} \frac{1}{n}(-\ln|\det \mathcal{A}(n)|) = -\limsup_{n\to\infty} \frac{1}{n}\left(\ln\left|\det \mathcal{A}^{-1}(n)\right|\right) = -\limsup_{n\to\infty} \frac{1}{n}(\ln|\det \mathcal{G}(0,n)|) \geq$$

$$\geq \gamma s + \sum_{i=1}^{s} \mu_i. \tag{14}$$

By Lyapunov inequality

$$\sum_{i=1}^{s} \mu_i \geq \limsup_{n\to\infty} \frac{1}{n} \left(\ln \left|\det \mathcal{A}(n)\right|\right)$$

combining this with (14) and taking into account that γ is arbitrarily small, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|\det \mathcal{A}(n)\right| = \sum_{i=1}^{s} \mu_i = \sum_{i=1}^{s} \lambda_i'(A). \tag{15}$$

3. Applying the Lyapunov inequality (4) to the perturbed system (2) we have

$$\sum_{i=1}^{s} \lambda_i'(A+\Delta) \geq$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left|\det \left(A(n)+\Delta(n)\right)\right| \geq$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left|\det A(n) \left(I + A^{-1}(n)\Delta(n)\right)\right| =$$

$$\sum_{i=1}^{s} \lambda_i'(A) + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln \left|\det \left(I + A^{-1}(n)\Delta(n)\right)\right|, \tag{16}$$

where I is the identity matrix of size s-by-s. Fix $\varepsilon > 0$. Since the sequence $\left(A^{-1}(n)\right)_{n\in\mathbb{N}}$ is bounded, then there exists $\delta_1 > 0$ such that for all matrices X such that $\|X\| < \delta_1$ the following inequality is true

$$\left|\ln \left|\det \left(I + A^{-1}(n)X\right)\right|\right| \leq \frac{\varepsilon}{s}$$

for all $n \in \mathbb{N}$. For the perturbation satisfying (6) with δ_1 we have from (16) the following inequality

$$\sum_{i=1}^{s} \lambda_i'(A+\Delta) \geq \sum_{i=1}^{s} \lambda_i'(A) - \frac{\varepsilon}{s}. \tag{17}$$

Moreover, according to (13), we can find $\delta_2 > 0$ such that $\lambda_i'(A+\Delta) \leq \lambda_i'(A) + \frac{\varepsilon}{s}$ for $i = 1, ..., s$ and

$$\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \delta_2.$$

Consider now perturbations $(\Delta(n))_{n\in\mathbb{N}}$ satisfying (6) with $\delta = \min(\delta_1, \delta_2)$. Introduce $\gamma_i \geq 0$ such that

$$\lambda_i'(A+\Delta) = \lambda_i'(A) + \frac{\varepsilon}{s} - \gamma_i,\ i = 1, ..., s. \tag{18}$$

Substituting this expression to (17) we obtain

$$\left(1 + \frac{1}{s}\right) \varepsilon \geq \sum_{i=1}^{s} \gamma_i \geq \gamma_i.$$

From this bound and (18) we have

$$\lambda_i'(A+\Delta) \geq \lambda_i'(A) - \varepsilon.$$

■

Using this result it can be easily shown, that the Lyapunov exponents of time-invariant system are stable.

Theorem 2: Lyapunov exponents of time-invariant system

$$x(n+1) = Ax(n), \tag{19}$$

with invertible matrix A are stable.

If we replaced the time-varying perturbation in Definition 1 by time-invariant one, we would obtain, from the last theorem, the well-known fact that the eigenvalues of a matrix are continuous functions of its elements. But it is worth to notice, that Theorem 2 shows a much deeper fact, because we allow the perturbations to be time-varying in Definition 1.

III. Acknowledgment

The research presented here was done by the first author as a part of the project funded by the National Science Centre granted according to decision DEC-2012/05/B/ST7/00065. The research realized by the second and third author were supported by the Ministry of Science and Higher Education funds BK 219/Rau1/2012 t.1 and BKM-233/RAu1/2012 t.1, respectively. .

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