

# On reduced-order interval observers for time-delay systems

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**Abstract**—The estimation problem for uncertain time-delay systems is addressed. A design method of reduced-order interval observers is proposed. The observer estimates the set of admissible values (the interval) for the state at each instant of time. The cases of known fixed delays and uncertain time-varying delays are analyzed. The proposed approach can be applied to linear time-delay systems and nonlinear time-delay systems in the output canonical form. The framework efficiency is demonstrated on examples of nonlinear systems.

## I. INTRODUCTION

The problem of observer design for nonlinear delayed systems is rather complex [24], as well as the stability conditions for analysis of functional differential equations are rather complicated [22]. Especially the observer synthesis is problematical for the cases when the model of a nonlinear delayed system contains parametric and signal uncertainties, or when the delay is time-varying or uncertain [3], [4], [7], [10], [23], [8], [25], [27]. An observer solution for these more complex situations are highly demanded in many real-world applications.

In this work we are going to address this problem proposing an interval observer for time-delay systems. In opposite to a conventional observer, that in the absence of measurement noise and uncertainties has to converge to the “exact” value of the state of the estimated system (it gives a *pointwise* estimation of the state), the interval observers evaluate at each time instant a set of admissible values for the state consistent with the measured output (i.e. they provide an *interval* estimation) [11], [14], [20]. Usually the interval observers have an enlarged dimension with respect to the system dimension since the upper and lower estimate of the state interval are generated by an observer (two times bigger than the system, see, for example, the paper [14] where an interval framer/predictor

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has been proposed for time-delay systems). Therefore, for applications, the problem of reduction of an interval observer dimension is of great importance, this is why in this work we will consider the reduced-order observers. The reduced order interval observers for some particular cases have been already used implicitly in the literature [1], [15], in this work a theoretical framework is established for a class of delay systems. Comparing with [14], where a framer is proposed dependent on integral of some auxiliary variables, in this work a more simple computational scheme is presented (see the comparison after Theorem 2), the observer gain derivation scheme is formulated as a linear programming problem and the case of time-varying uncertain delays is additionally studied.

The paper is organized as follows. Some preliminaries are given in Section 2. The reduced-order observer definition is given in Section 3, in the same section the observer design is performed for a class of linear time-delay systems (or a class of nonlinear systems in the output canonical form). Examples of numerical simulation are presented in Section 4.

## II. NOTATIONS AND DEFINITIONS

In the rest of the paper, the following definition will be used:

- $\mathbb{R}$  is the set of real numbers ( $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ ),  $\mathcal{C}_\tau = C([- \tau, 0], \mathbb{R})$  is the set of continuous maps from  $[- \tau, 0]$  into  $\mathbb{R}$ ,  $\mathcal{C}_\tau^n = (\mathcal{C}_\tau)^n$ ;  $\mathcal{C}_{\tau+} = \{y \in \mathcal{C}_\tau : y(s) \in \mathbb{R}_+, s \in [- \tau, 0]\}$ ;
- $x_t$  is an element of  $\mathcal{C}_\tau^n$  associated with a map  $x_t : \mathbb{R} \rightarrow \mathbb{R}^n$  by  $x_t(s) = x(t + s)$ , for all  $s \in [- \tau, 0]$ ;
- $|x|$  denotes the absolute value of  $x \in \mathbb{R}$ ,  $\|x\|$  is the Euclidean norm of a vector  $x \in \mathbb{R}^n$ ,  $\|\varphi\| = \sup_{t \in [- \tau, 0]} \|\varphi(t)\|$  for  $\varphi \in \mathcal{C}_\tau^n$ ;
- for a measurable and locally essentially bounded input  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^p$  the symbol  $\|u\|_{[t_0, t_1]}$  denotes its  $L_\infty$  norm  $\|u\|_{[t_0, t_1]} = \text{ess sup}\{|u(t)|, t \in [t_0, t_1]\}$ , if  $t_1 = +\infty$  then we will simply write  $\|u\|$ , we will denote as  $\mathcal{L}_\infty^p$  the set of all such inputs  $u \in \mathbb{R}^p$  with the property  $\|u\| < \infty$ ;
- for a matrix  $A \in \mathbb{R}^{n \times n}$  the vector of its eigenvalues is denoted as  $\lambda(A)$ ;
- $E_n \in \mathbb{R}^n$  is stated for a vector with unit elements,  $I_n$  and  $0_n$  denotes the identity and zero matrices of dimension  $n \times n$  respectively;

- for two integers  $i \leq j$  the symbol  $\overline{i, j}$  denotes the sequence  $i, i+1, \dots, j-1, j$ ;
- $a \mathcal{R} b$  corresponds to an elementwise relation  $\mathcal{R}$  ( $a$  and  $b$  are vectors or matrices): for example  $a < b$  (vectors) means  $\forall i : a_i < b_i$ ; for  $\phi, \varphi \in \mathcal{C}_\tau^n$  the relation  $\phi \mathcal{R} \varphi$  has to be understood elementwise for all domain of definition of the functions, i.e.  $\phi(s) \mathcal{R} \varphi(s)$  for all  $s \in [-\tau, 0]$ .

#### A. Functional Differential Equation

A large number of processes can be modeled by a *Functional Differential Equation* (FDE):

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x_t, d), & y(t) &= h(t, x(t), x_t, d), \\ x_{t_0} &= \varphi \in \mathcal{C}_\tau^n, \end{aligned} \quad (1)$$

where  $t \in \mathbb{R}$  is the time variable,  $d \in \mathcal{S}_d$  is either a vector or a function representing *disturbances or parameter uncertainties* of the system,  $\mathcal{S}_d \subset \mathcal{L}_\infty^q$  is a set of vectors or functions for which some bounds are usually supposed to be known,  $x(t) \in \mathbb{R}^n$  is a vector of internal variables,  $x_t \in \mathcal{C}_\tau^n$  and  $\tau \in \mathbb{R}_+$  is the maximal delay,  $y(t) \in \mathbb{R}^p$  is the output vector.

It is assumed that the system (1) has solutions (for example  $f$  satisfies Carathéodory conditions, see [13]) defined over a maximal interval denoted by  $\mathcal{I}_{(1)}(t_0, \varphi)$  where  $t_0$  is the initial time and  $\varphi$  is the initial function from  $\mathcal{C}_\tau^n$ .

#### B. Linear cooperative systems with delays

Consider a linear system with constant delays

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^N A_i x(t - \tau_i) + b(t), \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x_t \in \mathcal{C}_\tau^n$  for  $\tau = \max_{1 \leq i \leq N} \tau_i$  where  $\tau_i \in \mathbb{R}_+$  are the delays; a piecewise continuous function  $b \in \mathcal{L}_\infty^n$  is the input; the constant matrices  $A_i$ ,  $i = \overline{0, N}$  have appropriate dimensions. The matrix  $A_0$  is called Metzler if all its off-diagonal elements are nonnegative. The matrices  $A_i$  are called nonnegative if  $A_i \geq 0$  (elementwise). The function  $g(t, x, x_t) = A_0 x(t) + \sum_{i=1}^N A_i x(t - \tau_i) + b(t)$  is mixed quasi-monotone non-decreasing in  $x$ , non-decreasing in  $x_t$  if  $A_0$  is Metzler and  $A_i$ ,  $i = \overline{1, N}$  are nonnegative.

**Definition 1.** The system (2) is called *cooperative* (or nonnegative [12]) if  $A_0$  is Metzler and  $A_i$ ,  $i = \overline{1, N}$  are nonnegative matrices.

The cooperative system (2) admits  $x_t \in \mathcal{C}_{\tau_+}^n$  for all  $t \geq t_0$  provided that  $x_{t_0} \in \mathcal{C}_{\tau_+}^n$  and  $b : \mathbb{R} \rightarrow \mathbb{R}_+$ .

**Lemma 1.** [6], [5], [12] *A cooperative system (2) is asymptotically stable for  $b(t) \equiv 0$  for all  $\tau \in \mathbb{R}_+$  iff there are  $p, q \in \mathbb{R}_+^n$  ( $p > 0$  and  $q > 0$ ) such that*

$$p^T \sum_{i=0}^N A_i + q^T = 0.$$

Under conditions of the above lemma the system has bounded solutions for  $b \in \mathcal{L}_\infty^n$  with  $b(t) \in \mathbb{R}_+$  for all  $t \in \mathbb{R}$  (see also [2] for  $L_1$  and  $L_\infty$  gain conditions).

**Lemma 2.** [20] *Given the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{p \times n}$ . If there is a matrix  $L \in \mathbb{R}^{n \times p}$  such that the matrices  $A - LC$  and  $R$  have the same eigenvalues, then there is a  $P \in \mathbb{R}^{n \times n}$  such that  $R = P(A - LC)P^{-1}$  provided that the pairs  $(A - LC, e_1)$  and  $(R, e_2)$  are observable for some  $e_1 \in \mathbb{R}^{1 \times n}$ ,  $e_2 \in \mathbb{R}^{1 \times n}$ .*

This result was used in [20] to design interval observers for LTI systems with a Metzler matrix  $R$  (in other words, the lemma establishes the conditions when the matrix  $A - LC$  is similar to a Metzler matrix). The main difficulty is to prove the existence of a *real* matrix  $P$ , and to provide a constructive approach of its calculation. In [20] the matrix  $P = O_R O_{A-LC}^{-1}$ , where  $O_{A-LC}$  and  $O_R$  are the observability matrices of the pairs  $(A - LC, e_1)$  and  $(R, e_2)$  respectively. Another (more strict) condition is that the Sylvester equation  $PA - RP = QC$ ,  $Q = PL$  has a unique solution  $P$  provided that the pair  $(A, C)$  is observable (in this case there exists a matrix  $L$  such that  $\lambda(A) \neq \lambda(A - LC) = \lambda(R)$ , that is equivalent to existence of a unique  $P$ ). Note that if the matrix  $A - LC$  has only real positive eigenvalues, then  $R$  can be chosen as diagonal or Jordan representation of  $A - LC$ .

#### C. Interval analysis

Given a matrix  $A \in \mathbb{R}^{m \times n}$  define  $A^+ = \max\{0, A\}$ ,  $A^- = A^+ - A$  and  $|A| = A^+ + A^-$ . Let  $x \in \mathbb{R}^n$  be a vector variable,  $\underline{x} \leq x \leq \bar{x}$  for some  $\underline{x}, \bar{x} \in \mathbb{R}^n$ , and  $A \in \mathbb{R}^{m \times n}$  be a constant matrix, then

$$A^+ \underline{x} - A^- \bar{x} \leq Ax \leq A^+ \bar{x} - A^- \underline{x}. \quad (3)$$

This claim follows from the equation  $Ax = (A^+ - A^-)x$ , that for  $\underline{x} \leq x \leq \bar{x}$  gives the required estimates.

### III. MAIN RESULT

In this section an interval observer will be designed for a linear time-delay system. The possibility of the interval observer application in the case of an uncertain or time-varying delay is discussed thereafter.

#### A. Linear cooperative time-delay system

Consider the system (2) equipped with an output  $y \in \mathbb{R}^p$  available for measurements with a noise  $v \in \mathcal{L}_\infty^p$ :

$$y = Cx, \quad \psi = y + v(t), \quad (4)$$

where  $C \in \mathbb{R}^{p \times n}$ .

**Assumption 1.** *Let*

- $x \in \mathcal{L}_\infty^n$  with  $\underline{x}_0 \leq x_{t_0} \leq \bar{x}_0$  for some  $\underline{x}_0, \bar{x}_0 \in \mathcal{C}_\tau^n$ ;
- $\|v\| \leq V$  for a given  $V > 0$ ;
- $\tau_i \in \mathbb{R}_+$  are known and

•  $\underline{b}(t) \leq b(t) \leq \bar{b}(t)$  for all  $t \geq t_0$  for some known  $\underline{b}, \bar{b} \in \mathcal{L}_{\infty}^n$ .

In this assumption we suppose that the state of the system (2) is bounded with an unknown upper bound, but with a specified admissible set for initial conditions  $[x_0, \bar{x}_0]$ . The upper bound on the measurement noise amplitude  $V$  as well as the constant delays  $\tau_i$  are assumed to be given. All uncertainty of the system is collected in the external input  $b$  with known bounds on the incertitude  $\underline{b}, \bar{b}$ .

*Remark 1.* Note that under such formulation we also can take into account nonlinear systems which are diffeomorphic to the following output canonical form:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^N A_i x(t - \tau_i) + g(y_t, u) + \rho(t),$$

where the nonlinear term  $g$  and the external input  $\rho$  can be represented as  $b(t) = g(y_t, u) + \rho(t)$  with the known interval bounds for  $y_t \in [\psi_t - V, \psi_t + V]$  and the control  $u$ , that allows us to calculate the functions  $\underline{b}, \bar{b}$  taking into account the interval of  $\rho$ .

For the system (2), (4) there exists a nonsingular matrix  $S \in \mathbb{R}^{n \times n}$  such that  $x = S[y^T z^T]^T$  for an auxiliary variable  $z \in \mathbb{R}^{n-p}$  (define  $S^{-1} = [C^T Z^T]^T$  for a matrix  $Z \in \mathbb{R}^{(n-p) \times n}$ ), then

$$\begin{aligned} \dot{y}(t) &= R_1 y(t) + R_2 z(t) + \sum_{i=1}^N [D_{1i} y(t - \tau_i) \\ &\quad + D_{2i} z(t - \tau_i)] + C b(t), \\ \dot{z}(t) &= R_3 y(t) + R_4 z(t) + \sum_{i=1}^N [D_{3i} y(t - \tau_i) \\ &\quad + D_{4i} z(t - \tau_i)] + Z b(t), \end{aligned}$$

for some matrices  $R_k, D_{ki}, k = \overline{1, 4}, i = \overline{1, N}$  of appropriate dimensions. Introducing a new variable  $w = z - Ky = Ux$  for a matrix  $K \in \mathbb{R}^{(n-p) \times p}$  with  $U = Z - KC$  we obtain

$$\begin{aligned} \dot{w}(t) &= G_0 \psi(t) + M_0 w(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) \\ &\quad + M_i w(t - \tau_i)] + \beta(t), \\ \beta(t) &= U b(t) - G_0 v(t) - \sum_{i=1}^N G_i v(t - \tau_i), \end{aligned} \quad (5)$$

where  $\psi(t)$  is defined in (4),  $G_0 = R_3 - KR_1 + (R_4 - KR_2)K$ ,  $M_0 = R_4 - KR_2$ , and  $G_i = D_{3i} - KD_{1i} + \{D_{4i} - KD_{2i}\}K$ ,  $M_i = D_{4i} - KD_{2i}$  for  $i = \overline{1, N}$ . Under Assumption 1 using the relations (3) we get

$$\begin{aligned} \underline{\beta}(t) &\leq \beta(t) \leq \bar{\beta}(t), \\ \underline{\beta}(t) &= U^+ \underline{b}(t) - U^- \bar{b}(t) - \sum_{i=0}^N |G_i| E_p V, \\ \bar{\beta}(t) &= U^+ \bar{b}(t) - U^- \underline{b}(t) + \sum_{i=0}^N |G_i| E_p V. \end{aligned}$$

Then the following interval reduced-order observer can be proposed for (2):

$$\begin{aligned} \dot{\underline{w}}(t) &= G_0 \psi(t) + M_0 \underline{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) \\ &\quad + M_i \underline{w}(t - \tau_i)] + \underline{\beta}(t), \\ \dot{\bar{w}}(t) &= G_0 \psi(t) + M_0 \bar{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) \\ &\quad + M_i \bar{w}(t - \tau_i)] + \bar{\beta}(t). \end{aligned} \quad (6)$$

The applicability conditions for (6) are given below.

**Theorem 1.** *Let Assumption 1 be satisfied and the matrices  $M_0, M_i, i = \overline{1, N}$  form an asymptotically stable cooperative system (see Definition 1 and Lemma 1). Then  $\underline{x}, \bar{x} \in \mathcal{L}_{\infty}^n$  and*

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

for all  $t \geq t_0 = 0$ , where

$$\begin{aligned} \underline{x}(t) &= S^+ [y(t)^T z(t)^T]^T - S^- [\bar{y}(t)^T \bar{z}(t)^T]^T, \\ \bar{x}(t) &= S^+ [\bar{y}(t)^T \bar{z}(t)^T]^T - S^- [y(t)^T z(t)^T]^T, \end{aligned} \quad (7)$$

$$y(t) = \psi(t) - V, \bar{y}(t) = \psi(t) + V, \quad (8)$$

$$z(t) = \underline{w}(t) + K^+ y - K^- \bar{y}, \bar{z}(t) = \bar{w}(t) + K^+ \bar{y} - K^- \underline{y},$$

provided that  $\underline{w}_0 = U^+ \underline{x}_0 - U^- \bar{x}_0$ ,  $\bar{w}_0 = U^+ \bar{x}_0 - U^- \underline{x}_0$ .

All proofs are skipped due to space limitations.

The main condition of Theorem 1 is rather straightforward: the matrices  $M_0, M_i, i = \overline{1, N}$  have to form a stable cooperative system. It is a standard LMI problem to find a matrix  $K$  such that the system composed by  $M_0, M_i, i = \overline{1, N}$  is stable, but to find a matrix  $K$  making the system stable and cooperative simultaneously could be more complicated. However, the advantage of Theorem 1 is that its main condition can be reformulated as a *linear programming problem* following the idea of [21].

**Proposition 1.** *Let there exist  $\varsigma \in \mathbb{R}_+$ ,  $p \in \mathbb{R}_+^{n-p}$ ,  $q \in \mathbb{R}_+^{n-p}$  and  $B \in \mathbb{R}^{(n-p) \times p}$  such that the following linear programming problem is satisfied:*

$$\begin{aligned} p^T \Pi_0 - E_{n-p}^T B \Pi_1 + q^T &\leq 0, \quad p > 0, \quad q > 0, \\ \text{diag}[p] R_4 - B R_2 + \varsigma I_{n-p} &\geq 0, \quad \varsigma > 0, \\ \text{diag}[p] D_{4i} - B D_{2i} &\geq 0, \quad i = \overline{1, N}, \\ \Pi_0 = R_4 + \sum_{i=1}^N D_{4i}, \quad \Pi_1 = R_2 + \sum_{i=1}^N D_{2i}, \end{aligned}$$

then  $K = \text{diag}[p]^{-1} B$  and the matrices  $M_0 = R_4 - KR_2$ ,  $M_i = D_{4i} - KD_{2i}$ ,  $i = \overline{1, N}$  represent a stable cooperative system in (6).

If this linear programming problem is not satisfied, then the assumption that the matrix  $M_0$  is Metzler and the matrices  $M_i, i = \overline{1, N}$  are nonnegative can be relaxed using Lemma 2.

### B. Relaxed conditions of interval observer existence

According to Lemma 2 there exists a coordinate transformation  $\omega = Pw$  that maps  $M_0$  to a Metzler matrix  $PM_0P^{-1}$ , but Lemma 1 also requires the transformed matrices  $PM_iP^{-1}$  to be nonnegative, that is hard to satisfy. Fortunately, as we are going to show, the non-negativity of  $PM_iP^{-1}$  is not necessary.

Let us start with assumption confirming the conditions of Lemma 1.

**Assumption 2.** *There is a matrix  $K \in \mathbb{R}^{(n-p) \times p}$  such that the matrix  $M_0 = R_4 - KR_2$  and a Metzler matrix  $Y_0$  have the same eigenvalues and the pairs  $(M_0, e_1)$  and  $(Y_0, e_2)$  are observable for some  $e_1 \in \mathbb{R}^{1 \times n}$ ,  $e_2 \in \mathbb{R}^{1 \times n}$ .*

Under Assumption 2 there is a matrix  $P \in \mathbb{R}^{(n-p) \times (n-p)}$  such that  $Y_0 = PM_0P^{-1}$ . Define the set of new coordinates  $\omega = Pw$  and  $Y_i = PM_iP^{-1}$ ,  $T_i = PG_i$  for  $i = \overline{0, N}$ , then (5) yields:

$$\dot{\omega}(t) = T_0\psi(t) + Y_0\omega(t) + \sum_{i=1}^N [T_i\psi(t - \tau_i) + Y_i\omega(t - \tau_i)] + \gamma(t), \quad (9)$$

where  $\gamma(t) = P\beta(t)$  and  $\underline{\gamma}(t) = P^+\beta(t) - P^-\bar{\beta}(t)$ ,  $\bar{\gamma}(t) = P^+\bar{\beta}(t) - P^-\beta(t)$ . The matrices  $Y_i$  may be sign indefinite, thus the following modification of the interval reduced-order observer (6) is proposed:

$$\begin{aligned} \dot{\underline{\omega}}(t) &= T_0\psi(t) + Y_0\underline{\omega}(t) + \sum_{i=1}^N [T_i\psi(t - \tau_i) + Y_i^+\underline{\omega}(t - \tau_i) \\ &\quad - Y_i^-\bar{\omega}(t - \tau_i)] + \underline{\gamma}(t), \\ \dot{\bar{\omega}}(t) &= T_0\psi(t) + Y_0\bar{\omega}(t) + \sum_{i=1}^N [T_i\psi(t - \tau_i) + Y_i^+\bar{\omega}(t - \tau_i) \\ &\quad - Y_i^-\underline{\omega}(t - \tau_i)] + \bar{\gamma}(t). \end{aligned} \quad (10)$$

Comparing with (6), the observer (10) contains coupling terms between dynamics of  $\bar{\omega}$  and  $\underline{\omega}$ .

**Theorem 2.** *Let assumptions 1, 2 be satisfied, and there exist some  $p, q \in \mathbb{R}_+^{2(n-p)}$  ( $p > 0$  and  $q > 0$ ) such that*

$$p^T \sum_{i=0}^N \Psi_i + q^T = 0,$$

where

$$\Psi_0 = \begin{bmatrix} Y_0 & 0_{n-p} \\ 0_{n-p} & Y_0 \end{bmatrix}, \quad \Psi_i = \begin{bmatrix} Y_i^+ & Y_i^- \\ Y_i^- & Y_i^+ \end{bmatrix} \quad (11)$$

for all  $i = \overline{1, N}$ . Then  $\underline{x}, \bar{x} \in \mathcal{L}_+^n$  and

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

for all  $t \geq 0$ , where  $\underline{x}(t)$ ,  $\bar{x}(t)$  are defined by (7), (8), (10) and

$$\underline{\omega}(t) = [P^{-1}]^+\underline{\omega} - [P^{-1}]^-\bar{\omega}, \quad \bar{\omega}(t) = [P^{-1}]^+\bar{\omega} - [P^{-1}]^-\underline{\omega}, \quad (12)$$

where  $\underline{\omega}_0, \bar{\omega}_0$  are chosen as  $\underline{\omega}_0 = O^+\underline{x}_0 - O^-\bar{x}_0$ ,  $\bar{\omega}_0 = O^+\bar{x}_0 - O^-\underline{x}_0$  for  $O = PU$ .

Theorem 2 relax the applicability conditions of Theorem 1 skipping the requirement that the matrices  $M_i$ ,  $i = \overline{1, N}$  have to be nonnegative.

*Remark 2.* In the paper [14] a similar estimation problem is studied, the observer proposed there (see equation (4.14) in [14]) has more terms and it additionally depends on integrals of some auxiliary variables (i.e.  $\nu$  and  $W$ ), whose calculation increases the computational complexity of the scheme. Despite that, both observers ((10) in this work and in [14]) have similar applicability conditions (it is also required that the matrix  $\sum_{i=0}^N \Psi_i$  is Hurwitz in [14]). The problem of application of the coordinate transformation  $P$  and the uncertain delay treatment (considered below) are not analyzed in [14].

### C. Estimation for an uncertain delay

Assume that in the system (2) the delays  $\tau_i : \mathbb{R} \rightarrow [-\tau, 0]$  are time-varying:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + \sum_{i=1}^N A_ix(t - \tau_i(t)) + b(t), \\ \underline{\tau}_i &\leq \tau_i(t) \leq \bar{\tau}_i \quad t \geq 0, i = \overline{1, N}, \end{aligned}$$

with  $\tau = \max_{1 \leq i \leq N} \bar{\tau}_i$  for some given  $\underline{\tau}_i, \bar{\tau}_i \in \mathbb{R}_+$ , then applying the same transformations of coordinates we obtain a system similar to (9):

$$\begin{aligned} \dot{\omega}(t) &= T_0\psi(t) + Y_0\omega(t) + \sum_{i=1}^N [T_i\psi\{t - \tau_i(t)\} \\ &\quad + Y_i\omega\{t - \tau_i(t)\}] + \gamma(t). \end{aligned}$$

Next, the idea is to replace in the interval reduced-order observer (10) the delayed term  $\omega\{t - \tau_i\}$  with its minimum and maximum over the interval  $[\underline{\tau}_i, \bar{\tau}_i]$ :

$$\underline{m}_i[\omega(t)] = \min_{s \in [\underline{\tau}_i, \bar{\tau}_i]} \omega(t - s), \quad \bar{m}_i[\omega(t)] = \max_{s \in [\underline{\tau}_i, \bar{\tau}_i]} \omega(t - s),$$

that does not influence on the possibility of interval estimation. Thus the observer equations can be rewritten as follows:

$$\begin{aligned} \dot{\underline{\omega}}(t) &= T_0\psi(t) + Y_0\underline{\omega}(t) + \sum_{i=1}^N \{T_i^+ \underline{m}_i[\psi(t)] \\ &\quad - T_i^- \bar{m}_i[\psi(t)] + Y_i^+ \underline{m}_i[\underline{\omega}(t)] \\ &\quad - Y_i^- \bar{m}_i[\bar{\omega}(t)]\} + \underline{\gamma}(t), \\ \dot{\bar{\omega}}(t) &= T_0\psi(t) + Y_0\bar{\omega}(t) + \sum_{i=1}^N \{T_i^+ \bar{m}_i[\psi(t)] \\ &\quad - T_i^- \underline{m}_i[\psi(t)] + Y_i^+ \bar{m}_i[\bar{\omega}(t)] \\ &\quad - Y_i^- \underline{m}_i[\underline{\omega}(t)]\} + \bar{\gamma}(t). \end{aligned} \quad (13)$$

It is worth to stress that the observer (13) is nonlinear.

**Theorem 3.** *Let assumptions 1, 2 be satisfied. Then*

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

for all  $t \geq 0$ , where  $\underline{x}(t)$ ,  $\bar{x}(t)$  are defined by (7), (8) and (12) provided that  $\underline{\omega}_0 = O^+ \underline{x}_0 - O^- \bar{x}_0$ ,  $\bar{\omega}_0 = O^+ \bar{x}_0 - O^- \underline{x}_0$  for  $O = PU$ .

In Theorem 3 we did not prove that the variables  $\underline{x}, \bar{x}$  are bounded, that is rest for a future work, the idea is that

$$\underline{m}_i[\underline{\omega}(t)] = \underline{\omega}[t - \underline{\theta}_i(t)], \quad \bar{m}_i[\bar{\omega}(t)] = \bar{\omega}[t - \bar{\theta}_i(t)]$$

for some *known* functions  $\underline{\theta}_i : \mathbb{R}_+ \rightarrow [\underline{\tau}_i, \bar{\tau}_i]$ ,  $\bar{\theta}_i : \mathbb{R}_+ \rightarrow [\underline{\tau}_i, \bar{\tau}_i]$ ,  $i = \overline{1, N}$ , next the results of [9], [18] can be directly applied to prove boundedness of  $\underline{x}, \bar{x}$ . Now, the objective of the last theorem is to show that the interval observers are natural in the case of an uncertain delay function.

*Remark 3.* As in Remark 1, in the same way the uncertain delays can be treated in the nonlinear terms.

Let us show the performance of the proposed interval reduced-order observers (6), (10), (13) on examples of numerical simulation.

#### IV. APPLICATIONS

##### A. HIV/AIDS dynamics

Consider a variant of four-dimensional model of HIV/AIDS [16], [19], [26]:

$$\begin{aligned} \dot{T}(t) &= s(t) - dT(t) - \gamma v(t - \tau(t))T(t), \\ \dot{T}_1(t) &= q_1(t)\gamma v(t - \tau(t))T(t) - \mu_1 T_1(t), \\ \dot{T}_2(t) &= q_2(t)\gamma v(t - \tau(t))T(t) + k_1 T_1(t - \tau_1) \\ &\quad - \mu_2 T_2(t), \\ \dot{v}(t) &= k_2 T_2(t) - cv(t), \end{aligned}$$

where  $T \in \mathbb{R}_+$  is the size of population of uninfected/healthy cells;  $T_1 \in \mathbb{R}_+$  is the population of cells which are infected by the virus, but which do not yet produce new virus particles;  $T_2 \in \mathbb{R}_+$  denotes the population of infected cells which do produce virus particles; and  $v \in \mathbb{R}_+$  is the population size of free virus particles. The signal  $s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an uncertain time-varying rate of new cells production from internal sources in the body;  $d > 0$ ,  $c > 0$ ,  $\mu_1 > 0$  and  $\mu_2 > 0$  are known death rates of the corresponding populations;  $\gamma$  is the rate of influence of the viral population on other cells,  $q_i : \mathbb{R}_+ \rightarrow [\underline{q}_i, \bar{q}_i]$  characterizes an uncertain influence of  $v$  on production of  $T_i$ , the constants  $0 < \underline{q}_i < \bar{q}_i \leq 1$  are given,  $i = 1, 2$ ;  $0 < k_1 < \mu_1$  and  $k_2 > 0$  are known rates of transition from populations  $T_1$  and  $T_2$  to  $T_2$  and  $v$  respectively;  $\tau : \mathbb{R}_+ \rightarrow [\underline{\tau}, \bar{\tau}]$  is an uncertain time-varying delay representing the time shift in the influence of virus cells on other populations,  $0 \leq \underline{\tau} < \bar{\tau} < +\infty$  are some known constants;  $\tau_1 > 0$  is a given small delay in the transition from the population  $T_1$  to  $T_2$ . In [16], [19], [26] all parameters and delays are constant. As in [26] assume that

$$y_1(t) = T(t), \quad y_2(t) = v(t),$$

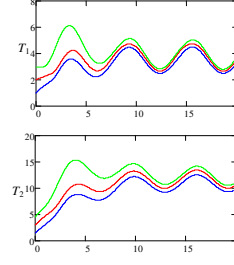


Figure 1. Results of interval estimation for HIV/AIDS model

note that from the consideration below the measurement  $y_2 = v(t - \tau_v)$  with  $0 \leq \tau_v < \bar{\tau}$  is also acceptable. Thus it is necessary to design the interval observer for the variables  $w = [T_1, T_2]^T$ :

$$\begin{aligned} \dot{w} &= M_0 w + M_1 w(t - \tau_1) + \beta(t), \\ M_0 &= \begin{bmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 \\ k_1 & 0 \end{bmatrix}, \\ \underline{\beta}(t) \leq \beta(t) &= \gamma y_1(t) y_2(t - \tau(t)) \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \leq \bar{\beta}(t), \\ \underline{\beta}(t) &= \gamma y_1(t) \min_{s \in [t - \bar{\tau}, t - \underline{\tau}]} y_2(s) \begin{bmatrix} \underline{q}_1 \\ \underline{q}_2 \end{bmatrix}, \\ \bar{\beta}(t) &= \gamma y_1(t) \max_{s \in [t - \bar{\tau}, t - \underline{\tau}]} y_2(s) \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} \end{aligned}$$

and the matrix  $M_0$  is Metzler and Hurwitz,  $M_1$  is nonnegative. It is obvious that all conditions of Theorem 2 are satisfied, the results of the system simulation for

$$\begin{aligned} s(t) &= 3 + 2 \sin(t), \quad d = \beta = \mu_1 = \mu_2 = 0.5, \quad c = 1, \\ k_1 &= 1, \quad k_2 = 0.7, \quad \tau_1 = 0.2, \quad \tau(t) = 1 + 0.5 \sin(t), \\ q_1(t) &= 0.7 + 0.02 \sin(2t), \quad q_2(t) = 0.8 + 0.02 \sin(3t) \end{aligned}$$

are shown in Fig. 1.

##### B. Glucose-insulin model

Following a recent paper [17], consider a modification of the second order time-delay glucose-insulin model

$$\begin{aligned} \dot{G}(t) &= -k_{GI} G(t) I(t) + r_1(t), \\ \dot{I}(t) &= -k_I I(t) + r_2(t) f[G(t - \tau_G(t))] + u(t), \\ f(G) &= \frac{\left(\frac{G}{G^*}\right)^\gamma}{1 + \left(\frac{G}{G^*}\right)^\gamma}, \end{aligned}$$

where  $G \in \mathbb{R}_+$  and  $I \in \mathbb{R}_+$  denote the plasma glycemia and insulinemia, respectively;  $k_{GI} > 0$  and  $k_I > 0$  are known decay rates;  $r_i : \mathbb{R}_+ \rightarrow [\underline{r}_i, \bar{r}_i]$  is an uncertain signal representing a production rate, the constants  $0 < \underline{r}_i \leq \bar{r}_i < +\infty$  are given,  $i = 1, 2$ ;  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is the control;  $\tau_G : \mathbb{R}_+ \rightarrow [\underline{\tau}_G, \bar{\tau}_G]$  is an uncertain time-varying delay,  $0 \leq \underline{\tau}_G < \bar{\tau}_G < +\infty$  are some known constants. In [17] all parameters were selected to be constant (here the

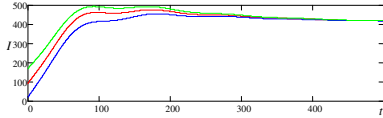


Figure 2. Results of interval estimation for glucose-insulin model

uncertain function  $r_2$  represents also a possible incertitude of the function  $f$  and its parameters) and

$$y(t) = G(t).$$

Then  $w(t) = I(t)$  and the design of interval observer is trivial taking into account the uncertain delay treatment proposed in the present paper. The results of interval estimation of  $I(t)$  for

$$\begin{aligned} k_{GI} &= 3.11e - 5, \quad k_I = 1.211e - 2, \\ r_1(t) &= 0.016 + 0.01 \sin(0.01t), \\ u(t) &= 5 + 5 \sin^2(0.03t)e^{-0.01t}, \\ r_2(t) &= 0.393(1 + 0.25 \sin(0.02t)), \\ \tau_G(t) &= 24 + 8 \sin(0.04t), \quad G^* = 9, \quad \gamma = 3.205 \end{aligned}$$

are shown in Fig. 2.

Thus the main advantage of the proposed approach is that it can easily take into account the presence of an additional uncertainty with respect to conventional methods [17], [26].

## V. CONCLUSION

The concept of interval reduced-order observers for nonlinear systems is introduced. Several observer solutions for linear and nonlinear time-delay systems are proposed. It is shown that if under a suitable coordinate transformation the undelayed subsystem is cooperative, then the delayed estimation error dynamics inherits this property. An approach for interval estimation of systems with uncertain and time-varying delays is presented. Examples of numerical simulation for two nonlinear systems confirm the efficiency of the proposed method.

## REFERENCES

- [1] O. BERNARD, J.-L. GOUZÉ, "Closed loop observers bundle for uncertain biotechnological models", *J. Process Control*, 14, 2004, pp. 765–774.
- [2] C. BRIAT, "Robust stability and stabilization of uncertain Linear Positive Systems via Integral Linear Constraints:  $L_1$ - and  $L_\infty$ -gains characterizations", *Int. J. Robust and Nonlinear Control*, 2013, DOI: 10.1002/rnc.2859.
- [3] C. BRIAT, O. SENAME, J.-F. LAFAY, "Design of LPV observers for LPV time-delay systems: an algebraic approach", *Int. J. Control*, 84(9), 2011, pp. 1533–1542.
- [4] C. CALIFANO, L.A. MARQUEZ-MARTINEZ, C.H. MOOG, "On the observer canonical form for Nonlinear Time-Delay Systems", *Proc. 18th IFAC World Congress*, 2011, Milano.
- [5] M. DAMBRINE, "Contribution à l'étude des systèmes à retards", *Ph.D. Thesis*, University of Sciences and Technology of Lille, France, 1994.
- [6] M. DAMBRINE, J.-P. RICHARD, "Stability Analysis of Time-Delay Systems", *Dynamic Systems and Applications*, 2, 1993, pp. 405–414.
- [7] M. DAROUACH, "Linear functional observers for systems with delays in state variables," *IEEE Transactions on Automatic Control*, 46(3), 2001, pp. 491–496.
- [8] A. FATTOUH, O. SENAME, J.M. DION, "Robust observer design for time-delay systems: a Riccati equation approach", *Kybernetika*, 35(6), 1999, pp. 753–764.
- [9] E. FRIDMAN, "Descriptor Discretized Lyapunov Functional Method: Analysis and Design", *IEEE Transactions on Automatic Control*, 51(5), 2006, pp. 890–897.
- [10] A. GERMANI, C. MANES, P. PEPE, "A state observer for nonlinear delay systems", *Proc. the 37th IEEE CDC*, 1998, Tampa, FL, pp. 355–360.
- [11] J.-L. GOUZÉ, A. RAPAPORT, Z. HADJ-SADOK, "Interval observers for uncertain biological systems", *Ecological Model.*, 133, 2000, pp. 45–56.
- [12] W.M. HADDAD, V. CHELLABOINA, "Stability theory for nonnegative and compartmental dynamical systems with time delay", *Syst. Control Lett.*, 51, 2004, pp. 355–361.
- [13] J.K. HALE, "Theory of Functional Differential Equations", Springer-Verlag, New York, 1977.
- [14] F. MAZENC, S. NICULESCU, O. BERNARD, "Exponentially stable interval observers for linear systems with delay", *SIAM J. Control Optim.*, 50(1), 2012, pp. 286–305.
- [15] M. MOISAN, O. BERNARD, J.-L. GOUZÉ, "Near optimal interval observers bundle for uncertain bioreactors", *Automatica*, Volume 45(1), 2009, pp. 291–295.
- [16] P.W. NELSON and A.S. PERELSON, "Mathematical analysis of delay differential equation models of HIV-1 infection", *Mathematical Biosciences*, 179, 2002, pp. 73–94.
- [17] P. PALUMBO, P. PEPE, S. PANUNZI, and A. D. GAETANO, "Time-delay model-based control of the glucose-insulin system, by means of a state observer", *European Journal of Control*, 18(6), 2012, pp. 591–606.
- [18] A. PAPACHRISTODOULOU, M.M. PEET, S. NICULESCU, "Stability Analysis of Linear Systems with Time-Varying Delays: Delay Uncertainty and Quenching", *Proc. IEEE CDC 2007*, New Orleans, pp. 2117–2122.
- [19] A.S. PERELSON and P.W. NELSON, "Mathematical analysis of HIV-1 dynamics in vivo", *SIAM Rev.*, 41, 1999, pp. 3–44.
- [20] T. RAÏSSI, D. EFIMOV, A. ZOLGHADRI, "Interval state estimation for a class of nonlinear systems", *IEEE Trans. Automatic Control*, 57, 2012, pp. 260–265.
- [21] RAMI M.A., HELMKE U., TADEO F., "Positive observation problem for linear time-delay positive systems," *Proc. Mediterranean Conf. Control & Automation (MED '07)*, 2007, pp.1–6.
- [22] J.-P. RICHARD, "Time delay systems: an overview of some recent advances and open problems", *Automatica*, 39(10), 2003, pp. 1667–1694.
- [23] O. SENAME, C. BRIAT, "Observer-based  $H_\infty$  control for time-delay systems: A new LMI solution", *Proc. 6th IFAC Workshop on Time Delay Systems*, 2006, L'Aquila, Italy.
- [24] R. SIPAHI, S.-I. NICULESCU, C. ABDALLAH, W. MICHIELS, AND K. GU, "Stability and stabilization of systems with time delay limitations and opportunities," *IEEE Control Systems Magazine*, 31(1), 2011, pp. 38–65.
- [25] A. SEURET, T. FLOQUET, J.-P. RICHARD, S.K. SPURGEON, "Observer design for systems with nonsmall and unknown time-varying delay", *Proc. IFAC Workshop on Time Delay Systems*, 2007, Nantes.
- [26] X. XIA and C.H. MOOG, "Identifiability of Nonlinear Systems with Application to HIV/AIDS Models", *IEEE Trans. Automatic Control*, 48(2), 2003, pp. 330–335.
- [27] G. ZHENG, J.-P. BARBOT, D. BOUTAT, T. FLOQUET, J.-P. RICHARD, "On observation of time-delay systems with unknown inputs", *IEEE Trans. Automatic Control*, 56(8), 2011, pp. 1973–1978.