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Input-output Linearization by Dynamic Output Feedback*

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Abstract— **The problem of input-output linearization by dynamic output feedback of discrete-time multi-input multi-output nonlinear control systems is considered in this paper. The system is described by the set of higher order input-output difference equations. First, an algorithm is given to find a set of functions, which are then used to provide sufficient solvability conditions and the equations of the dynamic output feedback. The results are specified for the special subclass of systems, called ANARX systems and comparisons with the single-input single-output case is made.**

I. INTRODUCTION

It is natural to use the output feedback, instead of the state feedback, to solve various control problems, because not always all the states are available for measurement. The alternative would be to construct the state observers which is not a simple task for nonlinear state equations [25]. Moreover, not all nonlinear systems can be described via state equations. However, there are not many results, unlike the linear case, that address the output feedback solutions for various problems in the nonlinear case. Besides input-output (i/o) linearization addressed in this paper, output feedback has been used in stabilization of nonlinear control systems [7], [5], [4], in output regulation [6], in the disturbance decoupling [8], [9], [16] and in the i/o decoupling problems [3], [20]. Some results on output feedback are also available for systems described by higher order i/o difference equations [1], [11], [13], [18].

For continuous-time systems, the solutions for the i/o linearization problem by output feedback are given in [3], [19], [23]. In [19] and [23] necessary and sufficient solvability conditions by static output feedback for multi-input multi-output (MIMO) systems are given, whereas the solution by dynamic output feedback for single-input single-output (SISO) systems is given in [3].

The input-output linearization problem by output feedback for discrete-time nonlinear systems has not received much attention. For SISO systems, [21] gives sufficient solvability conditions both by static and dynamic output feedback. In [9] the results of [21] were extended to weaken the solvability conditions (by dynamic output feedback) of [21] within the solution of the disturbance decoupling problem.

In this paper our goal is to extend the results of [9] for the MIMO case. We assume the nonlinear discrete-time systems to be given by the set of input-output difference equations. In [9], an algorithm was given, that finds a certain set of functions, which play the key role in the construction of the output feedback. Now, we extend this algorithm for the MIMO systems. Then, as in the SISO case, based on this algorithm, we give sufficient conditions for linearizability of i/o equations by regular dynamic output feedback. Obtained results are then specified for a subclass of additive NARX (ANARX) systems. The application of the algorithm and construction of the dynamic feedback is then demonstrated on two examples. Finally, note also that the MIMO case has not been directly studied for continuous-time systems, but it has been studied within the i/o decoupling problem for systems described by the state equations, see [22].

II. PRELIMINARIES

Throughout the paper we assume that $i, \tau = 1, \ldots, p$ and $j = 1, \ldots, m$. Also, we denote y for $y(t)$ and $y^{[k]}$ for $y(t+k)$, $k \geq 1$. Similar notations are used for the other variables. Consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the difference equations

$$y_i^{[k_i]} = \Phi_i(y_\tau, \ldots, y_\tau^{[k_{i\tau}]}, u_j, \ldots, u_j^{[k_i-1]}) \tag{1}$$

where Φ_i are supposed to be analytic functions of their arguments and the indices in (1) satisfy the relations

$$\begin{aligned} &k_1 \leq k_2 \leq \cdots \leq k_p, \quad k_{i\tau} < k_\tau \\ &k_{i\tau} < k_i, \quad \tau \leq i \\ &k_{i\tau} \leq k_i, \quad \tau > i. \end{aligned} \tag{2}$$

The conditions (2) mean that the equations (1) are assumed to be in the so-called doubly (row- and column) reduced form. Note that whenever the system is well-defined, one can always transform an arbitrary set of i/o equations, at least locally, under mild rank conditions, into the form (1), see [14]. If the system is in the form (1), then the forward-shift operator, defined below, is explicitly defined.

Also, we assume, like in the majority of papers addressing discrete-time nonlinear systems, that system (1) is submersive, i.e. the map $\Phi = (\Phi_1, \ldots, \Phi_p)^T$ satisfies generically the condition

$$\text{rank}\Big[\frac{\partial \Phi}{\partial (y, u)}\Big] = p,$$

where $y = (y_1, \ldots, y_p)$ and $u = (u_1, \ldots, u_m)$.

Let $\mathcal{K}$ be the field of meromorphic functions in variables y, u and a finite number of their independent forward shifts, i.e. $\{y_i, \ldots, y_i^{[k_i-1]}, u_j^{[k]}; k \geq 0\}$. Next, we define the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ associated with

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system (1) as follows. If function ϕ depends on variables $\{y_i,\ldots,y_i^{[k_i-1]},u_j,\ldots,u_j^{[k]}\}$, then

$$\delta\Big[\phi(y_i,\ldots,y_i^{[k_i-1]},u_j,\ldots,u_j^{[k]})\Big] := \phi(y_i^{[1]},\ldots,\Phi_i(\cdot),u_j^{[1]},\ldots,u_j^{[k+1]}).$$

Under the assumption that system (1) is submersive, operator δ is injective endomorphism of the field $\mathcal{K}$. Then the pair $(\mathcal{K},\delta)$ is an algebraic object called difference field, which we denote simply by $\mathcal{K}$. In general, $\mathcal{K}$ is not inversive, but there always exists an overfield $\mathcal{K}^*$ of field $\mathcal{K}$, called inversive closure of $\mathcal{K}$. Because $\mathcal{K}^*$ is inversive, there exists operator δ^{-1}, which will be interpreted as the backward-shift operator. By δ^k and δ^{-k} we denote the k-fold application of operators δ and δ^{-1}, respectively. The detailed explanation for the construction of $\mathcal{K}^*$ (the rule for computation of δ^{-1}) is given in [15] for the special case of SISO systems. The MIMO case, though technically more involved, can be handled in a similar manner. The crucial point is the choice of the new independent variables of the field extension. From now on, we assume that the field $\mathcal{K}$ is inversive.

Define the vector spaces $\mathcal{U} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_j^{[k]}; k \geq 0\}$, $\mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[s]}; s = 1,\ldots,k_i-1\}$ and the vector space of one-forms $\mathcal{E} = \mathcal{Y}+\mathcal{U}$. So, the set of one-forms $\mathcal{E}$ is a vector space that is formed by taking the linear combinations over the field $\mathcal{K}$ of standard basis elements $\{\mathrm{d}y_i,\ldots,\mathrm{d}y_i^{[k_i-1]},\mathrm{d}u_j^{[k]}; k\geq 0\}$. Also, define $\mathcal{E}^k := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i,\ldots,\mathrm{d}y_i^{[k-1]},\mathrm{d}u_j,\ldots,\mathrm{d}u_j^{[k-1]}\}$ for any $k\in\mathbb{N}$. The forward-shift of a one-form $\omega = \sum_s a_s\mathrm{d}\phi_s$ is defined by

$$\delta\Big(\sum_s a_s\mathrm{d}\phi_s\Big) := \sum_s \delta a_s\mathrm{d}(\delta\phi_s),$$

where $a_s\in\mathcal{K}$ and $\phi_s\in\mathcal{K}$.

Definition 1: The relative degree r_i of the ith output component y_i of the system (1) is defined as the delay between y_i and the control input u.

Note that this definition agrees with the definition in [12].

In general, a one-form $\omega\in\mathcal{E}$ is a linear combination over $\mathcal{K}$ of certain number of standard basis elements of $\mathcal{E}$. However, it is often possible to find a linearly independent set of exact one-forms with less elements than those basis elements of $\mathcal{E}$ in terms of which ω can be expressed.

Definition 2: [2] We say that γ is the rank of a one-form ω, if γ is minimal number of linearly independent exact one-forms necessary to express a one-form ω.

If the rank γ of a one-form ω is one, then $\omega = \xi\mathrm{d}\alpha$ ($\xi,\alpha\in\mathcal{K}$) is integrable and thus $\omega\wedge\mathrm{d}\omega = 0$. In the general case, if γ is the rank of the one-form ω, then $\omega\wedge(\mathrm{d}\omega)^{(\gamma)} = 0$, where $(\mathrm{d}\omega)^{(\gamma)} := \mathrm{d}\omega\wedge\ldots\wedge\mathrm{d}\omega$ is γ-fold wedge product.

For example, a one-form $\omega = u^{[1]}\mathrm{d}y + y\mathrm{d}u^{[1]} + y^{[1]}\mathrm{d}u^{[2]}$ is a linear combination of three standard basis elements $\{\mathrm{d}y,\mathrm{d}u^{[1]},\mathrm{d}u^{[2]}\}$. However, one can express ω as a linear combination of two exact one-forms $\mathrm{d}(yu^{[1]})$ and $\mathrm{d}u^{[2]}$, i.e. $\omega = \mathrm{d}(yu^{[1]}) + y^{[1]}\mathrm{d}u^{[2]}$. Thus, one says that the rank of ω is 2.

III. I/O LINEARIZATION

Problem statement. Given a discrete-time MIMO nonlinear control system of the form (1), we are searching for a regular dynamic output feedback of the form

$$\begin{aligned}\eta^{[1]} &= F(\eta,y,v)\\ u &= H(\eta,y,v),\end{aligned}\qquad(3)$$

where $\eta\in\Delta\subset\mathbb{R}^{\rho}$ and $v\in V\subset\mathbb{R}^m$ are the state and the input of the compensator, respectively, such that the differentials of input-output equations of the closed-loop system satisfy the relation

$$\mathrm{d}y_i^{[k_i]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[k_{i\tau}]},\ldots,\mathrm{d}y_\tau,\mathrm{d}v\}.\qquad(4)$$

This means that, for the closed-loop system, $\mathrm{d}y_i^{[k_i]}$ is equal to a linear combination of the elements from $\{\mathrm{d}y_\tau^{[k_{i\tau}]},\ldots,\mathrm{d}y_\tau,\mathrm{d}v_j\}$ over $\mathbb{R}$. Then, after integrating, which is always possible, one gets that $y_i^{[k_i]}$ is a linear function of variables $\{y_\tau^{[k_{i\tau}]},\ldots,y_\tau,v_j\}$.

If there exists such dynamic output feedback, then we say that system (1) is input-output linearizable. Note that we call the compensator (3) regular, if it generically defines the (y,η)-dependent one-to-one correspondence between the variables v and u. In other word, the compensator (3) is right-invertible, for more information, see [10].

Problem solution. First, we recall the result for the SISO systems. This helps to understand the MIMO case and additionally, allows us later to make comparisons with the SISO case.

Theorem 1: [9] The SISO system with finite relative degree r of output y is input-output linearizable by dynamic output feedback if the differential of its i/o equation can be rewritten as

$$\begin{aligned}\mathrm{d}y^{[n]} &= \lambda_1\mathrm{d}y^{[n-1]}+\cdots+\lambda_{r-1}\mathrm{d}y^{[n-r+1]}\\ &+ \mathrm{d}\phi_{n-r+1}(\cdot,\delta\phi_{n-r-1},\ldots,\delta\phi_1,y,u)\circ\\ &\quad\delta\phi_{n-r}(\cdot,\delta\phi_{n-r-2},\ldots,\delta\phi_1,y,u)\circ\ldots\\ &\ldots\circ\delta\phi_3(\cdot,\delta\phi_1,y,u)\circ\delta\phi_2(\cdot,y,u)\circ\delta\phi_1(y,u)\end{aligned}\qquad(5)$$

for $\lambda_s\in\mathbb{R}$ ($s=1,\ldots,r-1$) and some functions $\phi_1,\ldots,\phi_{n-r+1}$ which are invertible with respect to their first argument.

An algorithm is given in [9] to find the functions $\phi_1,\ldots,\phi_{n-r+1}$ if they exist. In the MIMO case, we are searching for another set of functions $\phi_l^{i,j}$, $l=1,\ldots,k_i-r_i+1$, such that

$$\begin{aligned}\mathrm{d}y_i^{[k_i]} &= \lambda_{i,1,\nu}\mathrm{d}y_\nu^{[k_\nu-1]}+\cdots+\lambda_{i,k_\nu,\nu}\mathrm{d}y_\nu\\ &+ \mathrm{d}\phi^{i,1}_{k_i-r_i+1}(\cdot,\delta\phi^{\tau,j}_\sigma,y,u)\circ\\ &\quad\delta\phi^{i,1}_{k_i-r_i}(\cdot,\delta\phi^{\tau,j}_\kappa,y,u)\circ\cdots\circ\delta\phi^{i,1}_1(y,u),\end{aligned}\qquad(6)$$

where $\nu=1,\ldots,p$, $\sigma=1,\ldots,k_i-r_i$ and $\kappa=1,\ldots,k_i-r_i-1$. When specified for SISO systems, (6) becomes (5). Really, since $i=j=1$, then we can write functions $\phi_l^{i,j}$ just as ϕ_l, where $l=1,\ldots,n-r+1$.

Next, we give an algorithm to find such functions if they exist. Denote by $r_1,\ldots,r_p$ the relative degrees of the outputs $y_1,\ldots,y_p$ respectively.

Algorithm 1. Denote $\omega_i := \mathrm{d}y_i^{[k_i]} \mod \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[k_{i\tau}]}, \ldots, \mathrm{d}y_\tau\}$. Also, assume that the relative degrees r_i are given and define $k_{max} := \max_i\{k_i - r_i + 1\}$.

Step 0. Check whether

$$\omega_i \in \mathcal{E}^{k_i - r_i + 1}. \tag{7}$$

If it is not true, then stop, there do not exist functions $\phi_l^{i,j}$ that satisfy (6). Otherwise, go to step 1.

Step 1. For every i let

$$\bar{\omega}_{i,1} = \sum_{\tau=1}^{p} \alpha_{i,1,\tau}\mathrm{d}y_\tau^{[k_i - r_i]} + \sum_{j=1}^{m} \beta_{i,1,j}\mathrm{d}u_j^{[k_i - r_i]}$$

($\alpha_{i,1,\tau}, \beta_{i,1,j} \in \mathcal{K}^*$) be such that $\omega_i - \bar{\omega}_{i,1} \in \mathcal{E}^{k_i - r_i}$. Check whether $\gamma_{i,1} := \mathrm{rank}\, \bar{\omega}_{i,1} \leq m$. If not, then stop[1]. Otherwise, let $\phi_1^{i,1}, \ldots, \phi_1^{i,\gamma_{i,1}}$ be such that $\{\mathrm{d}\delta^{k_i - r_i}\phi_1^{i,1}, \ldots, \mathrm{d}\delta^{k_i - r_i}\phi_1^{i,\gamma_{i,1}}\}$ is a basis of $\bar{\omega}_{i,1}$ and $\phi_1^{i,\gamma_{i,1}+1}, \ldots, \phi_1^{i,m} = 0$.

Step q. ($q = 2, \ldots, k_{max}$) Rewrite ω_i in terms of forward shifts of functions $\phi_l^{i,j}$, found previously, i.e. such that

$$\begin{aligned} \omega_i \quad \in \quad & \mathcal{E}^{k_i - q - r_i + 2} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\delta^\sigma \phi_\pi^{\tau,\mu} \\ & \pi = 1, \ldots, q-1; \mu = 1, \ldots, \gamma_{\tau,\pi}; \\ & \sigma = 0, \ldots, k_i - r_i - q + 2\}. \end{aligned}$$

Let

$$\begin{aligned} \bar{\omega}_{i,q} \quad = \quad & \sum_{\tau=1}^{p} \alpha_{i,q,\tau}\mathrm{d}y_\tau^{[k_i - r_i - q + 1]} + \sum_{j=1}^{m} \beta_{i,q,j}\mathrm{d}u_j^{[k_i - r_i - q + 1]} \\ + \quad & \sum_{\tau=1}^{p}\sum_{\pi=1}^{q-1}\sum_{\mu=1}^{\gamma_{\tau,\pi}} \theta_{i,q,\tau}^{\pi,\mu}\mathrm{d}\delta^{k_i - r_i - q + 2}\phi_\pi^{\tau,\mu} \end{aligned}$$

($\alpha_{i,q,\tau}, \beta_{i,q,j}, \theta_{i,q,\tau}^{\pi,\mu} \in \mathcal{K}^*$) be such that

$$\begin{aligned} \omega_i - \bar{\omega}_{i,q} \quad \in \quad & \mathcal{E}^{k_i - q - r_i + 1} \\ + \quad & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\delta^\sigma \phi_\pi^{\tau,\mu} \\ & \pi = 1, \ldots, q-1; \mu = 1, \ldots, \gamma_{\tau,\pi}; \\ & \sigma = 0, \ldots, k_i - r_i - q + 1\}. \end{aligned}$$

Check whether $\gamma_{i,q} := \mathrm{rank}\, \bar{\omega}_{i,q} \leq m$. If not, then stop[1]. Otherwise, let $\phi_q^{i,1}, \ldots, \phi_q^{i,\gamma_{i,q}}$ be such that $\{\mathrm{d}\delta^{k_i - r_i - q + 1}\phi_q^{i,1}, \ldots, \mathrm{d}\delta^{k_i - r_i - q + 1}\phi_q^{i,\gamma_{i,q}}\}$ is a basis of $\bar{\omega}_{i,q}$ and $\phi_q^{i,\gamma_{i,q}+1}, \ldots, \phi_q^{i,m} = 0$. End of algorithm.

Remark 1: In the $(k_i - r_i + 1)$th step one has $\bar{\omega}_{i,k_i - r_i + 1} = \omega_i$. Since ω_i is integrable by definition, $\gamma_{i,k_i - r_i + 1} = \mathrm{rank}\, \bar{\omega}_{i,k_i - r_i + 1} = 1$. Then let the function $\phi_{k_i - r_i + 1}^{i,1}$ be such that $\bar{\omega}_{i,k_i - r_i + 1} = \omega_i = \mathrm{d}\phi_{k_i - r_i + 1}^{i,1}$. Since $\gamma_{i,k_i - r_i + 1}$ is always 1, the functions $\phi_{k_i + r_i + 1}^{i,s}$, for $s = 2, \ldots, m$, can be always taken equal to 0.

Now we give the main result of this paper.

Theorem 2: Suppose the relative degree r_i of the output y_i is finite, for $i = 1, \ldots, p$. Then the system (1) is input-output linearizable by the dynamic output feedback of the form (3) if

[1]If $\gamma_{i,1} \not\leq m$ ($\gamma_{i,l} \not\leq m$) then condition (ii) of Theorem 2 given below is not satisfied, so there is no reason to continue.

(i) there exist functions $\phi_l^{i,j}$, where $l = 1, \ldots, k_i - r_i + 1$, that satisfy (6);

(ii)

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\phi_1^{i,1}, \ldots, \phi_1^{i,\gamma_{i,1}})^T}{\partial u} = \sum_{i=1}^{p} \gamma_{i,1}$$

and

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\phi_l^{i,1}, \ldots, \phi_l^{i,\gamma_{i,l}})^T}{\partial(\delta\phi_{l-1}^{i,1}, \ldots, \delta\phi_{l-1}^{i,\gamma_{i,l-1}})} = \sum_{i=1}^{p} \gamma_{i,l},$$

where $l = 2, \ldots, k_i - r_i + 1$ and $\gamma_{i,l}$ are defined by Algorithm 1.

Proof: Denote

$$\begin{aligned} \eta_{i,l,s} \quad &= \quad \phi_l^{i,s}(\cdot) \qquad (8) \\ v_i \quad &= \quad \phi_{k_i - r_i + 1}^{i,1}(\cdot), \qquad (9) \end{aligned}$$

where $l = 1, \ldots, k_i - r_i$ and $s = 1, \ldots, \gamma_{i,l}$. Under condition (ii), one can solve system of equations (8), (9) in variables $\{u, \eta_{i,l,s}^{[1]}; l = 1, \ldots, k_i - r_i; s = 1, \ldots, \gamma_{i,l}\}$ to get a dynamic output feedback. In case some variables remain free, they can be taken equal to new input component v_μ $\mu = p+1, \ldots, m$. Because of (9) and (6) one gets

$$\mathrm{d}y_i^{[k_i]} \quad \in \quad \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[k_{i\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v_i\}.$$

for the closed-loop system. Thus, the differentials of i/o equations (1) of the closed-loop system satisfy (4) and therefore the closed-loop system is input-output linearized. ■

Note that in the SISO case, condition (i) of Theorem 2 corresponds to condition (5) and condition (ii) is always satisfied, since by the definition of the relative degree, ranks of the matrices in (ii) are always equal to 1.

Consider a special subclass of MIMO systems, called ANARX systems, which are described by the equations of the form

$$y_i^{[k_i]} = \sum_{s=1}^{k_i} \varphi_{i,s}\big(y_\tau^{[k_i - s]}, u_j^{[k_i - s]}; k_i - s < k_\tau\big). \tag{10}$$

In ANARX model the restrictions are imposed on the structure (1), not allowing coupling of shifts of different orders in the same term (function $\varphi_{i,s}$). The choice of an appropriate (restricted) structure is a typical approach in control to guarantee that the restricted system structure will satisfy certain properties important for feedback construction.

Next, we specify the solution in Theorem 2 for the i/o linearization problem for systems of the form (10).

Corollary 1: Under the assumption that the relative degrees r_i of outputs y_i are all equal to 1, the system (10) is i/o linearizable by dynamic output feedback of the form (3) if

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\delta^{1-k_1}\varphi_{1,1}, \ldots, \delta^{1-k_p}\varphi_{p,1})^T}{\partial u} = p. \tag{11}$$

Proof: Because all the relative degrees $r_i = 1$, the condition (7) is satisfied and one can always find from Algorithm 1 the functions $\phi_l^{i,1}$ as follows

$$\begin{aligned} \phi_1^{i,1} \quad &= \quad \delta^{-k_i+1}\varphi_{i,1}(\cdot) \qquad (12) \\ \phi_l^{i,1} \quad &= \quad \delta\phi_{l-1}^{i,1} + \delta^{-k_i+l}\varphi_{i,l}(\cdot), \qquad (13) \end{aligned}$$

where $l = 2, \ldots, k_i$, since[2] $\gamma_{i,l} = 1 \leq m$. Then $\mathrm{d}y_i^{[k_i]} = \mathrm{d}\phi_{k_i}^{i,1}$ and thus the condition (i) of Theorem 2 is satisfied. Condition (ii) is also satisfied, because by assumption (11) and (12),

$$\begin{aligned}&\mathrm{rank}_{\mathcal{K}}\frac{\partial(\phi_1^{1,1}, \ldots, \phi_1^{p,1})^T}{\partial u}\\ &= \mathrm{rank}_{\mathcal{K}}\frac{\partial(\delta^{-k_1+1}\varphi_{1,1}, \ldots, \delta^{-k_p+1}\varphi_{p,1})^T}{\partial u} = p\end{aligned}$$

and by (13)

$$\mathrm{rank}_{\mathcal{K}}\frac{\partial(\phi_l^{1,1}, \ldots, \phi_l^{p,1})^T}{\partial(\delta\phi_{l-1}^{p,1}, \ldots, \delta\phi_{l-1}^{p,1})} = \mathrm{rank}_{\mathcal{K}} I_p = p,$$

where $l = 2, \ldots, k_i$ and I_p is $p \times p$ identity matrix. Thus all the conditions of Theorem 2 are satisfied and the system (10) is i/o linearizable. ■

Remark 2: The assumption $r_i = 1$, in Corollary 1 is actually not restrictive, its goal was to make the proof transparent. In the case of arbitrary (finite) relative degrees, the condition (11) of Corollary 1 takes the form

$$\mathrm{rank}_{\mathcal{K}}\frac{\partial(\delta^{r_1-k_1}\varphi_{1,r_1}, \ldots, \delta^{r_p-k_p}\varphi_{p,r_p})^T}{\partial u} = p \qquad (14)$$

and additionally, one has to assume that (7) is satisfied, because otherwise there do not exist functions $\phi_l^{i,j}$ that satisfy (6), see Step 0 of Algorithm 1.

Then, by renumbering the inputs u_j, if necessary, the dynamic output feedback that linearizes the system (10) is given by

$$\begin{aligned}\eta_{i,l}^{[1]} &= \eta_{i,l+1} - \delta^{l+r_i-k_i}\varphi_{i,l+r_i}(\cdot)\\ \eta_{i,k_i-r_i}^{[1]} &= v_i - \varphi_{i,k_i}(\cdot)\\ u_i &= \delta^{r_i-k_i}\varphi_{i,r_i}^{-1}(\cdot)\\ u_s &= v_s,\end{aligned}$$

where $l = 1, \ldots, k_i - r_i - 1$, $s = p+1, \ldots, m$ and inverse of φ_{i,r_i} is taken with respect to the argument $u_i^{[k_i-r_i]}$.

Remark 3: In case of the SISO ANARX systems, condition (14) is always satisfied, since by the definition of relative degree, $\delta^{r_i-k_i}\varphi_{1,r_i}$ depends on control variable u. Also condition (7) is always satisfied and thus the SISO system of the form (10) is always i/o linearizable, which agrees with the results in [15].

IV. Examples

We present below two examples, the first of them is academic and the other describes a 27 tray binary distillation column model operating in a high-purity regime.

Example 1: Consider a system given by the set of i/o equations

$$\begin{aligned}y_1^{[3]} &= y_2u_2u_1^{[2]} + y_1^{[2]}u_2^{[2]}\\ y_2^{[1]} &= y_2u_3.\end{aligned} \qquad (15)$$

[2]This is because in case of ANARX models all different time shifts are decoupled

Note that $r_1 = r_2 = 1$. We check the conditions of Theorem 2 for system (15). To check whether there exist functions that satisfy (6) and in the case of affirmative answer, to find these functions, we use Algorithm 1.

Compute

$$\begin{aligned}\omega_1 &= y_2u_2\mathrm{d}u_1^{[2]} + y_1^{[2]}\mathrm{d}u_2^{[2]} + u_2^{[2]}\mathrm{d}y_1^{[2]}\\ &+ y_2u_1^{[2]}\mathrm{d}u_2 + u_2u_1^{[2]}\mathrm{d}y_2\\ \omega_2 &= y_2\mathrm{d}u_3 + u_3\mathrm{d}y_2.\end{aligned}$$

Step 0. It is obvious, that $\omega_1 \in \mathcal{E}^3$ and $\omega_2 \in \mathcal{E}^1$.

Step 1. For $i = 1$, define a one-form $\bar{\omega}_{1,1}$ as

$$\bar{\omega}_{1,1} = y_2u_2\mathrm{d}u_1^{[2]} + u_2^{[2]}\mathrm{d}y_1^{[2]} + y_1^{[2]}\mathrm{d}u_2^{[2]},$$

because then $\omega_1 - \bar{\omega}_{1,1} \in \mathcal{E}^2$. Note that $\gamma_{1,1} := \mathrm{rank}\ \bar{\omega}_{1,1} = 2 < 3 = m$, since $\bar{\omega}_{1,1}$ may be rewritten as

$$\bar{\omega}_{1,1} = y_2u_2\mathrm{d}u_1^{[2]} + \mathrm{d}(u_2^{[2]}y_1^{[2]}).$$

Thus one may choose $\phi_1^{1,1} = u_1$ and $\phi_1^{1,2} = u_2y_1$, since $\{\mathrm{d}\delta^2u_1, \mathrm{d}\delta^2(u_2y_1)\}$ is a basis for one-form $\bar{\omega}_{1,1}$.

For $i = 2$, Step 1 is the last one, because[3] $k_2 - r_2 + 1 = 1$. The one-form $\omega_2 = \mathrm{d}(y_2u_3)$ and thus $\phi_1^{2,1} = y_2u_3$.

Step 2. Now, the one-form ω_1 can be rewritten as

$$\begin{aligned}\omega_1 &= y_2u_2\mathrm{d}(\delta^2\phi_1^{1,1}) + y_2\delta^2\phi_1^{1,1}\mathrm{d}u_2\\ &\quad + u_2\delta^2\phi_1^{1,1}\mathrm{d}y_2 + \mathrm{d}(\delta^2\phi_1^{1,2}).\end{aligned}$$

Define $\bar{\omega}_{1,2} = y_2u_2\mathrm{d}(\delta^2\phi_1^{1,1}) + \mathrm{d}(\delta^2\phi_1^{1,2})$, since then $\omega_1 - \bar{\omega}_{1,2} \in \mathcal{E}^1 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_1^{i,j}, \mathrm{d}\delta\phi_1^{i,j}; i,j = 1,2\}$. The rank of the one-form $\bar{\omega}_{1,2}$ is $\gamma_{1,2} = 2 < m = 3$ and the basis of $\bar{\omega}_{1,2}$ is $\{\mathrm{d}(\delta^2\phi_1^{1,1}), \mathrm{d}(\delta^2\phi_1^{1,2})\}$. Therefore, we define $\phi_2^{1,1} = \delta\phi_1^{1,1}$ and $\phi_2^{1,2} = \delta\phi_1^{1,2}$.

Step 3. In the last step of the algorithm, we get $\omega_1 = \mathrm{d}(y_2u_2\delta\phi_2^{1,1} + \delta\phi_2^{1,2})$ and so, we choose $\phi_3^{1,1} = y_2u_2\delta\phi_2^{1,1} + \delta\phi_2^{1,2}$.

Note that the conditions (i) and (ii) of Theorem 2 are satisfied. Really, $\mathrm{d}y_1^{[3]} = \mathrm{d}\phi_3^{1,1}$ and $\mathrm{d}y_2^{[1]} = \mathrm{d}\phi_1^{2,1}$, and

$$\begin{aligned}\mathrm{rank}_{\mathcal{K}}\frac{\partial(\phi_1^{1,1}, \phi_1^{1,2}, \phi_1^{2,1})^T}{\partial u} &= \mathrm{rank}_{\mathcal{K}}\begin{pmatrix}1 & 0 & 0\\ 0 & y_1 & 0\\ 0 & 0 & y_2\end{pmatrix}\\ &= 3 = \gamma_{1,1} + \gamma_{2,1}\end{aligned}$$

everywhere except when $y_1 = 0$ or $y_2 = 0$. Also,

$$\begin{aligned}\mathrm{rank}_{\mathcal{K}}\frac{\partial(\phi_2^{1,1}, \phi_2^{1,2})^T}{\partial(\delta\phi_1^{1,1}, \delta\phi_1^{1,2}, \delta\phi_1^{2,1})} &= \mathrm{rank}_{\mathcal{K}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\\ &= 2 = \gamma_{1,2}.\end{aligned}$$

To find a dynamic output feedback compensator, one has to, according to (8), (9), solve the system of equations

$$\begin{aligned}\eta_{1,1,1} &= u_1\\ \eta_{1,1,2} &= y_1u_2\\ \eta_{1,2,1} &= \eta_{1,1,1}^{[1]}\\ \eta_{1,2,2} &= \eta_{1,1,2}^{[1]}\\ v_1 &= y_2u_2\eta_{1,2,1}^{[1]} + \eta_{1,2,2}^{[1]}\\ v_2 &= y_2u_3\end{aligned}$$

[3]See Remark 1.

for variables $\{u_1, u_2, u_3, \eta^{[1]}_{1,1,1}, \eta^{[1]}_{1,1,2}, \eta^{[1]}_{1,2,1}, \eta^{[1]}_{1,2,2}\}$:

$$\begin{aligned}
\eta^{[1]}_{1,1,1} &= \eta_{1,2,1} \\
\eta^{[1]}_{1,1,2} &= \eta_{1,2,2} \\
\eta^{[1]}_{1,2,1} &= v_3 \\
\eta^{[1]}_{1,2,2} &= v_1 - \frac{y_2 v_3 \eta_{1,1,2}}{y_1} \qquad (16) \\
u_1 &= \eta_{1,1,1} \\
u_2 &= \frac{\eta_{1,1,2}}{y_1} \\
u_3 &= \frac{v_2}{y_2}.
\end{aligned}$$

Since we have 6 equations and 7 unknowns, one variable, $\eta^{[1]}_{1,2,1}$, is free, so that one can take $\eta^{[1]}_{1,2,1} = v_3$. Compensator (16) is regular, since one can write the new input $v = (v_1, v_2, v_3)$ as a function of variables $\{y, u\}$ and a finite number of their independent forward-shifts

$$\begin{aligned}
v_1 &= y_2 u_2 u_1^{[2]} + y_1^{[2]} u_2^{[2]} \\
v_2 &= y_2 u_3 \\
v_3 &= u_1^{[2]}
\end{aligned}$$

and this function is invertible with respect to variables $\{u_1^{[2]}, u_2^{[2]}, u_3\}$. Now, the closed-loop system reads $y_1^{[3]} = v_1$ and $y_2^{[1]} = v_2$.

Example 2: Consider a 27 tray binary distillation column model operating in a high-purity regime, described in [24]:

$$\begin{aligned}
y_1^{[3]} &= 0,0012 + 0,98 y_1^{[2]} - 0,18 u_1^{[2]} + 1,1 y_1 u_2^{[2]} \\
&\quad -1,8 y_2^{[2]} u_1^{[2]} \qquad (17) \\
y_2^{[1]} &= 0,0018 + 0,92 y_2 - 0,22 u_1 + 30,4 y_2^2 u_2 - 1,7 u_2^2.
\end{aligned}$$

After transforming (17) into the form (1), (2), one gets

$$\begin{aligned}
y_1^{[3]} &= 0,0012 + 0,98 y_1^{[2]} + 1,1 y_1 u_2^{[2]} \\
&\quad +(-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \qquad (18) \\
&\quad -54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) u_1^{[2]} \\
y_2^{[1]} &= 0,0018 + 0,92 y_2 - 0,22 u_1 + 30,4 y_2^2 u_2 - 1,7 u_2^2.
\end{aligned}$$

Next, we find the functions that satisfy (6), by Algorithm 1. Note that the relative degrees r_1 and r_2 of outputs y_1 and y_2 respectively, are both 1. Compute

$$\begin{aligned}
\omega_1 &= \mathrm{d}\big(1,1 y_1 u_2^{[2]} + (-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \\
&\quad -54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) u_1^{[2]}\big) \\
\omega_2 &= \mathrm{d}\big(-0,22 u_1 + 30,4 y_2^2 u_2 - 1,7 u_2^2\big).
\end{aligned}$$

Step 0. One can easily see that $\omega_1 \in \mathcal{E}^3$ and $\omega_2 \in \mathcal{E}^1$.

Step 1. For $i = 1$, define the one-form

$$\begin{aligned}
\bar{\omega}_{1,1} &= 1,1 y_1 \mathrm{d} u_2^{[2]} + (-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \\
&\quad -54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) \mathrm{d} u_1^{[2]}.
\end{aligned}$$

The rank of the one-form $\bar{\omega}_{1,1}$ is $\gamma_{1,1} = 2 = m$. The basis of $\bar{\omega}_{1,1}$ is $\{\mathrm{d}\delta^2 u_2, \mathrm{d}\delta^2 u_1\}$. Thus $\phi_1^{1,1} = u_2$ and $\phi_1^{1,2} = u_1$.

For $i = 2$, let $\bar{\omega}_{2,1} = \omega_2$ and thus

$$\phi_1^{2,1} = -0,22 u_1 + 30,4 y_2^2 u_2 - 1,7 u_2^2.$$

Step 2. Rewrite ω_1 as

$$\begin{aligned}
\omega_1 &= \mathrm{d}\big(1,1 y_1 \delta^2 \phi_1^{1,1} + (-0,18324 + 0,396 u_1^{[1]} \\
&\quad - 1,656 y_2^{[1]} - 54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) \delta^2 \phi_1^{1,2}\big).
\end{aligned}$$

and define a one-form

$$\begin{aligned}
\bar{\omega}_{1,2} &= 1,1 y_1 \mathrm{d}\big(\delta^2 \phi_1^{1,1}\big) \\
&\quad + \mathrm{d}\big[(-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \\
&\quad -54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) \delta^2 \phi_1^{1,2}\big].
\end{aligned}$$

The rank of this one-form is clearly $\gamma_{1,2} = 2 = m$ and the basis of this one-form is

$$\begin{aligned}
&\{\mathrm{d}\delta^2 \phi_1^{1,1}, \mathrm{d}\big[(-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \\
&-54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) \delta^2 \phi_1^{1,2}\big]\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\phi_2^{1,1} &= \delta \phi_1^{1,1} \\
\phi_2^{1,2} &= (-0,18324 + 0,396 u_1^{[1]} - 1,656 y_2^{[1]} \\
&\quad -54,72 (y_2^{[1]})^2 u_2^{[1]} + 3,06 (u_2^{[1]})^2) \delta^2 \phi_1^{1,2}.
\end{aligned}$$

Step 3. This is the last step and thus $\bar{\omega}_{1,3} = \omega_1 = \mathrm{d}\big(1,1 y_1 \delta \phi_2^{1,1} + \delta \phi_2^{1,2}\big)$. This means that $\phi_3^{1,1} = 1,1 y_1 \delta \phi_2^{1,1} + \delta \phi_2^{1,2}$.

Now, $\mathrm{d} y_1^{[3]} = 0,98 \mathrm{d} y_1^{[2]} + \mathrm{d}\phi_3^{1,1}$ and $\mathrm{d} y_2^{[1]} = \mathrm{d}\phi_2^{1,1}$. Condition (i) of Theorem 2 is satisfied, since we have found functions $\phi_l^{i,j}$ such that (6) is satisfied. But condition (ii) does not hold since

$$\begin{aligned}
&\mathrm{rank}_{\mathcal{K}} \frac{\partial(\phi_1^{1,1}, \phi_1^{1,2}, \phi_1^{2,1})^T}{\partial u} = \\
&= \mathrm{rank}_{\mathcal{K}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -0,22 & 30,4 y_2^2 - 3,4 u_2 \end{pmatrix} = 2 \neq 3.
\end{aligned}$$

Thus, one can not linearize the system (18), at least not using the method described in this paper.

V. Conclusions

The i/o linearization problem for discrete-time nonlinear control systems by regular dynamic output feedback was addressed in this paper. The nonlinear MIMO system is described by the set of i/o equations. An algorithm was given to find a set of functions, which were used to formulate the sufficient conditions for solvability of the problem. Although the results obtained in this paper can be also applied for systems defined by state and output equations (by eliminating the states), the class of systems, addressed in this paper is larger than those described by the state equations. This is because not all systems, given by the set of i/o equations, are realizable in the classical state space form [17].

References

[1] G. Bastin, F. Jarachi, and I.M.Y. Mareels. Dead beat control of recursive nonlinear systems. In *Proc. of the 32th Conference on Decision and Control*, pages 2965–2971. San Antonio, 1993.

[2] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick. *Analysis, Manifolds and Physics*. Elsevier, 1996.

[3] G. Conte, C.H. Moog, and A.M. Perdon. *Algebraic Methods for Nonlinear Control Systems. Theory and Applications.* Springer, 2007.

[4] F. Esfandiari and H.K. Khalil. Output feedback stabilization of fully linearizable systems. *International Journal of Control*, 56(5):1007–1037, 1992.

[5] R. Findeisen, L. Imsland, F. Allgöwer, and B.A. Foss. Output feedback stabilization of constrained systems with nonlinear predictive control. *Int. J. Robust Nonlinear Control*, 13:211–227, 2003.

[6] J. Huang. *Nonlinear Output Regulation: Theory and Applications*. SIAM, 2004.

[7] A. Isidori. *Nonlinear control systems*. Springer, London, 1995.

[8] A. Kaldmäe and Ü. Kotta. Disturbance decoupling of multi-input multi-output discrete-time nonlinear systems by static measurement feedback. *Proc. of the Estonian Academy of Sciences*, 61(2):77–88, 2012.

[9] A. Kaldmäe and Ü. Kotta. Dynamic measurement feedback in discrete-time nonlinear control systems. In *Proc. of the 2012 American Control Conference: Fairmont The Queen Elizabeth, Montreal, Canada, June 27-29, 2012*, pages 214–219. Montreal, 2012.

[10] Ü. Kotta. *Inversion Method in the Discrete-time Nonlinear Control Systems Synthesis Problems*. Springer, Berlin, 1995.

[11] Ü. Kotta. Model matching problem for nonlinear recursive systems. *Proc. of the Estonian Academy of Sciences. Phys. Math.*, 46:251–261, 1997.

[12] Ü. Kotta. Inversion algorithm for recursive nonlinear systems. *Proc. of the Estonian Academy of Sciences. Phys. Math.*, 47(1):3–18, 1998.

[13] Ü. Kotta. Input-output decoupling of nonlinear recursive systems. *Kybernetika*, 36(1):43–51, 2000.

[14] Ü. Kotta, Z. Bartosiewicz, S. Nõmm, and E. Pawluszewicz. Linear input-output equivalence and row reducedness of discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 56(6):1421–1426, 2011.

[15] Ü. Kotta, F.N. Chowdhury, and S. Nõmm. On realizability of neural networks-based input-output models in the classical state-space form. *Automatica*, 42(7):1211–1216, 2006.

[16] Ü. Kotta, A.Ye. Shumsky, and A. Zhirabok. Output feedback disturbance decoupling in discrete-time nonlinear systems. In *Proc. of the 18th IFAC World Congress*, pages 239–244. Milano, Italy, 2011.

[17] Ü. Kotta and M. Tõnso. Realization of discrete-time nonlinear input-output equations: polynomial approach. *Automatica*, 48(2):255–262, 2012.

[18] A.U. Levin and K.S. Narenda. Control of nonlinear dynamical systems using neural networks. part 2: Observability, identification and control. *IEEE Trans. Neural Networks*, 7(1):30–42, 1996.

[19] R. Marino and P. Tomei. Dynamic output feedback linearization and global stabilization. *Systems and Control Letters*, 17:115–121, 1991.

[20] C.H. Moog, R. Pothin, Ü. Kotta, and S. Nõmm. Input-output decoupling of nonlinear discrete-time systems by static output feedback. In *Proc. of the 15th IFAC World Congress*, pages 293–298. Barcelona, 2002.

[21] R. Pothin, Ü. Kotta, and C. H. Moog. Output feedback linearization of nonlinear discrete time systems. *IFAC Control Systems Design*, pages 181–186, 2000.

[22] R. Pothin and C.H. Moog. Input-output decoupling of nonlinear systems by dynamic measurement feedback. In *Proc. of the 14th International Symposium on Mathematical Theory of Networks and Systems*. Perpignan, France, 2000.

[23] R. Pothin, C.H. Moog, and X. Xia. Input-output linearization of nonlinear mimo systems by static output feedback. In *Proc. of the 38th Conference on Decision and Control*, pages 732–733. Phoenix, 1999.

[24] G.R. Sriniwas, Y. Arkun, I.-L. Chien, and B. A. Ogunnaike. Nonlinear identification and control of a high-purity distillation column: a case study. *Journal of Process Control*, 45:149–162, 1995.

[25] M. Xiao, N. Kazantzis, C. Kravaris, and A. Krener. Nonlinear discrete-time observer design with linearizable error dynamics. *IEEE Trans. on Automatic Control*, 48(4):622–626, 2003.