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Krylov Subspace Methods for Block Patterned Linear Systems

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Abstract— **This article considers the moment matching using Krylov subspace methods for block patterned linear systems that comprise symmetrically interconnected as well as block circulant systems. Thereby, the block patterned system structure is preserved so that subsequent analysis and synthesis approaches can exploit the special structure of the reduced order system. It is shown that a specific number of moments at arbitrary points can be matched. For the order reduction numerical efficient algorithms can be applied to determine the corresponding Krylov subspaces. The proposed approximation technique is illustrated by means of a simple symmetrically interconnected system.**

I. INTRODUCTION

One of the most successful techniques for the order reduction of large-scale systems is *moment matching*. This means that some of the coefficients of the Taylor series expansion of the transfer matrices of the large-scale system and its approximation coincide for given expansion points. Consequently, an approximation of the transfer behaviour of the large-scale system in prescribed frequency ranges is possible. This can be achieved by using a projection of the large-scale dynamics onto special Krylov subspaces of lower dimensions. For the computation of the corresponding Krylov subspaces numerically reliable algorithms of low complexity exist enabling the application of this *Krylov subspace method* to systems of very high order (see e. g. [1], [2]).

An important property of an order reduction technique is to preserve structural properties of the original system. Therefore, the formulation of structure preserving Krylov subspace methods attracted the attention of researchers in the last decade. In [3] the preservation of passivity was considered, which also implies a stability preserving order reduction. Krylov subspace methods for second order systems that retain the system structure were presented in [4]. Since a lot of large-scale systems can be modelled as port-Hamiltonian systems, also the extension of Krylov subspace methods to preserve this system structure were proposed in [5], [6].

In this article the structure preserving order reduction of *block patterned linear systems* using Krylov subspace methods is considered. This system class was introduced in [7] to model complex systems that arise from the interconnection of a large number of identical subsystems. Members of this system class are *symmetrically interconnected systems* (see [8]) as well as *block circulant systems* (see [9]). An order reduction of these systems is of special interest if the subsystems itself are of high order. Examples are the modelling of flexible structures with a special spatial symmetry (see e. g. [10]) or the distributed control of Micro-Electro-Mechanical Systems (MEMS) (see [11]). Thereby, the order reduction should preserve the special system structure so that a simplified analysis and synthesis using the special structure of the resulting reduced order model is still possible.

In order to achieve a structure preserving order reduction of block patterned linear systems a transformation into block diagonal form is introduced. This leads to a small set of decoupled subsystems of high order which are the starting point of the order reduction. An advantage of this approach is that the classical Krylov subspace method can be directly applied to these subsystems without taking the system structure into account. Then, on the basis of this result a structure preserving projection achieving moment matching for the block patterned system can be derived. This *decomposition approach* was already used in the literature for the distributed control (see [12]) and the identification (see [10]) of large-scale systems and is applied in this article for the structure preserving order reduction.

The next section introduces the class of block patterned systems. Thereby, the definition of patterned systems in [7] is extended to the larger class of block patterned systems. Then, the transformation of the system into decoupled subsystems is presented. Section III considers the structure preserving *Petrov-Galerkin method* that forms the basis for the moment matching contained in Section IV. A simple numerical example of a symmetrically interconnected system is used to demonstrate the proposed structure preserving order reduction.

II. BLOCK PATTERNED SYSTEMS

A linear time-invariant *block patterned linear system* is given by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx, \tag{2}$$

where $A \in \mathbb{R}^{Nl\times Nl}$, $B \in \mathbb{R}^{Nl\times pl}$, $p \leq N$, and $C \in \mathbb{R}^{ml\times Nl}$, $m \leq N$, are block Γ-patterned matrices. This class of matrices is characterized by the following definition.

Definition 1 (block Γ-patterned matrix): The set of *block Γ-patterned matrices* with respect to the *base matrix* $\Gamma \in$

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$\mathbb{R}^{l\times l}$ is

$$\mathscr{BF}_{nm}(\Gamma) = \{M \in \mathbb{C}^{nl\times ml} \mid \exists\, M_0,\dots,M_{r-1} \in \mathbb{C}^{n\times m} \text{ such that } M = \sum_{i=0}^{r-1} \Gamma^i \otimes M_i\}, \quad (3)$$

in which $r \leq l$ is the number of different eigenvalues of Γ and $\otimes$ denotes the *Kronecker product* (see e. g. [13]).

Remark 1: An alternative definition of a block patterned matrix M is to assume that it is a block matrix where the blocks are Γ-*patterned matrices*, i. e. they are an element of $\mathscr{F}(\Gamma) = \{M \in \mathbb{C}^{l\times l} \mid \exists\, m_0,\dots,m_{r-1} \in \mathbb{C}$ such that $M = \sum_{i=0}^{r-1} m_i\Gamma^i\}$, $r \leq l$, with the base matrix $\Gamma \in \mathbb{R}^{l\times l}$ (see [7]). Since both forms of block patterned matrices can be transformed to each other by a simple transformation the results of the article are also directly applicable to this alternative definition.

Thus, the system matrices can be characterized by

$$A = \sum_{i=0}^{r-1} \Gamma^i \otimes A_i,\ B = \sum_{i=0}^{r-1} \Gamma^i \otimes B_i \text{ and } C = \sum_{i=0}^{r-1} \Gamma^i \otimes C_i, \quad (4)$$

where $A_i \in \mathbb{C}^{N\times N}$, $B_i \in \mathbb{C}^{N\times p}$ and $C_i \in \mathbb{C}^{m\times N}$ such that $A \in \mathscr{BF}_{NN}(\Gamma)$, $B \in \mathscr{BF}_{Np}(\Gamma)$ and $C \in \mathscr{BF}_{mN}(\Gamma)$. In the following it is assumed that the corresponding base matrix Γ is diagonalizable.

Remark 2: If $r = 2$ in (3) and Γ is a symmetric *Laplacian* of the graph with all possible interconnections (see e. g. [14]), then (1) is a *symmetrically interconnected system* (see e. g. [8]). For $r = l$ and Γ being the *shift operator* (see [15]) the system (1) is a *block circulant system* (see [9]).

An interesting property of block patterned systems is, that different systems with the same dimensions and the same base matrix can be block diagonalized by a common transformation. This significantly simplifies the analysis of this system class. In order to block diagonalize the system matrix A in (4) consider the transformation

$$T = R \otimes I_N, \quad (5)$$

where R is the matrix of eigenvectors with respect to the eigenvalues μ_i, $i = 1,2,\dots,l$, of the base matrix Γ, i. e. $\Gamma R = R\Lambda$ and

$$\Lambda = \operatorname{diag}(\mu_1,\dots,\mu_l). \quad (6)$$

Then,

$$S = R^{-1} \quad (7)$$

is the matrix of the corresponding left eigenvectors of Γ. When taking

$$(M_1 \otimes M_2)^{-1} = M_1^{-1} \otimes M_2^{-1} \quad (8)$$

for any invertible matrices M_1 and M_2 into account (see e. g. [13]) then $T^{-1} = S \otimes I_N$ in view of (7). Now, consider the similarity transformation

$$T^{-1}AT = \sum_{i=0}^{r-1}(S\otimes I_N)(\Gamma^i\otimes A_i)(R\otimes I_N) = \sum_{i=0}^{r-1}(S\Gamma^i R\otimes A_i), \quad (9)$$

in which the property

$$(M_1 \otimes M_2)(M_3 \otimes M_4) = M_1M_3 \otimes M_2M_4 \quad (10)$$

for suitable matrices was used (see e. g. [13]). Since $S\Gamma^i R = \Lambda^i$ the equation (9) becomes with (6)

$$T^{-1}AT = \sum_{i=0}^{r-1}(\Lambda^i \otimes A_i) = \begin{bmatrix} A(\mu_1) & & \\ & \ddots & \\ & & A(\mu_l)\end{bmatrix} = \operatorname{bdiag}_{i\in I}(A(\mu_i)), \quad (11)$$

when defining the *representation polynomial matrix*

$$A(s) = \sum_{i=0}^{r-1} A_i s^i \in \mathbb{C}^{N\times N}[s] \quad (12)$$

of A and

$$\operatorname{bdiag}_{i\in I}(A(\mu_i)) = \operatorname{bdiag}(A(\mu_1),\dots,A(\mu_l)) \quad (13)$$

with the *index set* $I = \{1,2,\dots,l\}$. Since the base matrix Γ has the distinct eigenvalues μ_i, $i = 1,2,\dots,r$, so that the *spectrum* of Γ satisfies $\sigma(\Gamma) = \{\mu_1,\dots,\mu_r,\mu_{12},\dots,\mu_{rn_r}\}$ with $\mu_{ij} = \mu_i$, $i = 1,2,\dots,r$, $j = 2,3,\dots,n_i$, where $\sum_{i=1}^{r} n_i = l$ the alternative notation

$$\operatorname{bdiag}^{\Gamma}(A_1,\dots,A_r) = \operatorname{bdiag}(A_1,\dots,A_r,A_{12},\dots,A_{1n_1},\dots,A_{rn_r}) \quad (14)$$

is useful in the sequel where for the block diagonal matrix (11) $A_i = A(\mu_i)$, $i = 1,2,\dots,r$, and $A_{ij} = A(\mu_{ij})$, $j = 2,3,\dots,n_i$, is satisfied.

The next lemma shows that also a converse result holds, that relates a block diagonal matrix with the set $\mathscr{BF}_{mn}(\Gamma)$.

Lemma 1 (block diagonal matrices and $\mathscr{BF}_{mn}(\Gamma)$): Consider the block diagonal matrix

$$M = \operatorname{bdiag}(M_1,\dots,M_r,M_{12},\dots,M_{1n_1},\dots,M_{rn_r}), \quad (15)$$

where $M_i, M_{ij} \in \mathbb{C}^{m\times n}$, $i = 1,2,\dots,r$, $j = 2,3,\dots,n_i$. If $M = \operatorname{bdiag}^{\Gamma}(M_1,\dots,M_r)$, then

$$(R\otimes I_m)\operatorname{bdiag}^{\Gamma}(M_1,\dots,M_r)(S\otimes I_n) \in \mathscr{BF}_{mn}(\Gamma), \quad (16)$$

where $\Gamma \in \mathbb{R}^{l\times l}$ is any diagonalizable matrix with r distinct eigenvalues, R is the matrix of its eigenvectors and $S = R^{-1}$.

Proof: In order to prove $M \in \mathscr{BF}_{mn}(\Gamma)$ consider $(S\otimes I_m)\sum_{i=0}^{r-1}(\Gamma^i\otimes N_i)(R\otimes I_n) = \operatorname{bdiag}_{i\in I}(\sum_{j=0}^{r-1} N_j\mu_i^j)$ with $N_i \in \mathbb{C}^{m\times n}$ to be determined (see (11)). By comparing this with (15) one obtains r distinct interpolation points $\sum_{j=0}^{r-1} N_j\mu_i^j = M_i$, $\quad i = 1,2,\dots,r$, for each polynomial element of degree $r-1$ on the left hand side. Consequently, by the fundamental theorem on polynomial interpolation the matrices N_i, $i = 0,1,\dots,r-1$, are uniquely specified showing that $M(s) = \sum_{j=0}^{r-1} N_j s^j$ is the representation polynomial matrix of M. Thus, (16) holds, i. e. $(R\otimes I_m)M(S\otimes I_n) \in \mathscr{BF}_{mn}(\Gamma)$. ■

The proof of Lemma 1 clarifies that it is reasonable to choose $r-1$ as highest degree in $M(s)$ (see also (3)) since this assures that the coefficient matrices N_i are uniquely determined by the corresponding block diagonal matrix (15).

Similarly, the input matrix B can be block diagonalized by using (5) and the *input transformation* $u = T_u u^*$ where $T_u = R \otimes I_p$. Application of these transformations to B gives

$$T^{-1} B T_u = \text{bdiag}_{i \in I}(B(\mu_i)) \tag{17}$$

with representation polynomial matrix

$$B(s) = \sum_{i=0}^{r-1} B_i s^i \in \mathbb{C}^{N \times p}[s] \tag{18}$$

of B. Finally, the *output transformation* $y^* = T_y^{-1} y$ with $T_y = R \otimes I_m$ and $T_y^{-1} = S \otimes I_m$ yields with (5)

$$T_y^{-1} C T = \text{bdiag}_{i \in I}(C(\mu_i)), \tag{19}$$

in which

$$C(s) = \sum_{i=0}^{r-1} C_i s^i \in \mathbb{C}^{m \times N}[s] \tag{20}$$

is the representation polynomial matrix of C. Consequently, by using the *state transformation* $z = \text{col}(z_1, \ldots, z_l) = Tx$, the input transformation $u = T_u u^* = T_u \,\text{col}(u_1^*, \ldots, u_l^*)$ and the output transformation $y^* = \text{col}(y_1^*, \ldots, y_l^*) = T_y^{-1} y$ the block patterned system (1)–(2) can be equivalently represented by a set of r *decoupled subsystems*

$$\dot{z}_i = A(\mu_i) z_i + B(\mu_i) u_i^*, \quad i = 1, 2, \ldots, r \tag{21}$$

$$y_i^* = C(\mu_i) z_i, \tag{22}$$

where $z_i(t) \in \mathbb{C}^N$, $u_i^*(t) \in \mathbb{C}^p$ and $y_i^*(t) \in \mathbb{C}^m$. In (21)–(22) μ_i, $i = 1, 2, \ldots, r$, are the different eigenvalues of Γ after a suitable ordering of the eigenvalues μ_i in Λ (see (6)). This shows that, in general, only $r \leq l$ subsystems have to be considered since the number of different subsystems is given by the number of different eigenvalues of Γ. Consequently, the analysis and synthesis of block patterned systems can be simplified since lesser computational effort is needed to consider the systems (21)–(22) when compared to (1)–(2).

Remark 3: If the representation polynomial matrices $A(s)$, $B(s)$ and $C(s)$ are constant, then only one decoupled subsystem has to be analysed independent of the eigenvalues of Γ.

III. STRUCTURE PRESERVING PETROV-GALERKIN METHOD

Typically, block patterned systems belong to the class of large scale systems so that a model order reduction is of interest. Thereby, the resulting reduced order model should also be a block patterned system, i. e. a *structure preserving order reduction* has to be applied. In this section a model reduction approach with this property on the basis of the *Petrov-Galerkin method* (see e. g. [16]) is derived. To this end, assume that $n < N$ and introduce the matrices

$$V = \begin{bmatrix} v_1 & \ldots & v_{nl} \end{bmatrix} \in \mathbb{C}^{Nl \times nl} \tag{23}$$

$$W = \begin{bmatrix} w_1 & \ldots & w_{nl} \end{bmatrix}^H \in \mathbb{C}^{nl \times Nl} \tag{24}$$

with $v_i, w_i \in \mathbb{C}^{Nl}$, $i = 1, 2, \ldots, nl$, satisfying

$$\det WV \neq 0 \tag{25}$$

and $\dim V = \dim W = nl$ with the corresponding subspaces $\mathcal{V} = \operatorname{ran} V$ and $\mathcal{W} = \operatorname{ran} W^H$. Now, consider the decomposition $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}^\perp$ of the *state space* $\mathcal{X} = \mathbb{C}^{Nl}$ with

$$\mathcal{V} \cap \mathcal{W}^\perp = \{0\}, \tag{26}$$

where $\mathcal{W}^\perp$ is the *orthogonal complement* of $\mathcal{W}$. It is straightforward to show that (26) is implied by (25). In order to determine a reduced order model the *projection*

$$P = V(WV)^{-1} W \in \mathbb{C}^{Nl \times Nl} \tag{27}$$

of $\mathcal{X}$ onto $\mathcal{V}$ along $\mathcal{W}^\perp$ is used which exists in view of (25) and induces a unique decomposition $x = Px + (I_{Nl} - P)x$ with $Px \in \mathcal{V}$ and $(I_{Nl} - P)x \in \mathcal{W}^\perp$. Note, that (27) is in general a *non-orthogonal projection*. When compared to an orthogonal projection this introduces additional degrees of freedom in the order reduction that can be used to influence further properties of the approximant. A reduced order model is then obtained by projecting the state $x \in \mathcal{X}$ onto $\mathcal{V}$ giving $x_n = Px \in \mathcal{V}$. The corresponding time derivative reads

$$\dot{x}_n = PAx + PBu = PAx_n + PBu + PA(I_{Nl} - P)x \tag{28}$$

in view of (1). By neglecting $PA(I_{Nl} - P)x$ a state equation

$$\dot{\hat{x}}_n = PA\hat{x}_n + PBu \tag{29}$$

of the order nl for $\hat{x}_n$ is obtained that approximates x_n. Since $\hat{x}_n \in \mathcal{V} \subset \mathbb{C}^{Nl}$ it has the unique representation $\hat{x}_n = V\xi$, $\xi \in \mathbb{C}^{nl}$, yielding

$$V\dot{\xi} = PAV\xi + PBu. \tag{30}$$

Thus, the reduced order system with the state space $\mathcal{X}_n = \mathbb{C}^{nl}$ takes the form

$$\dot{\xi} = A_n \xi + B_n u \tag{31}$$

with

$$A_n = (WV)^{-1} WAV \in \mathbb{C}^{nl \times nl} \tag{32}$$

$$B_n = (WV)^{-1} WB \in \mathbb{C}^{nl \times pl}. \tag{33}$$

The output equation for (29) is $y_n = C\hat{x}_n$ so that

$$y_n = C_n \xi, \tag{34}$$

where

$$C_n = CV \in \mathbb{C}^{ml \times nl}. \tag{35}$$

In general, A_n, B_n and C_n are not block Γ-patterned matrices so that this order reduction does not yield a block patterned system. However, if the Petrov-Galerkin approach is applied to the decoupled subsystems (21)–(22) a structure preserving order reduction can be achieved. This means that the special structure of block patterned system is exploited to solve this model order reduction problem. The next theorem presents this result.

Theorem 1 (Structure preserving approximation): Consider the block patterned system (1)–(2) and the two matrices

$$V = (R \otimes I_N)\,\text{bdiag}^\Gamma(V_1, \ldots, V_r)(S \otimes I_n) \tag{36}$$

$$W = (R \otimes I_n)\,\text{bdiag}^\Gamma(W_1, \ldots, W_r)(S \otimes I_N) \tag{37}$$

with $V_i \in \mathbb{C}^{N\times n}$, $i = 1, 2, \ldots, r$, and $W_i \in \mathbb{C}^{n\times N}$ satisfying $\det W_i V_i \neq 0$. Then, the projection

$$P = V(WV)^{-1}W \in \mathscr{BF}_{NN}(\Gamma) \quad (38)$$

is *structure preserving*, i. e. the reduced order model (31) and (34) of the order nl characterized by the matrices

$$\begin{aligned} A_n &= (WV)^{-1}WAV \in \mathscr{BF}_{nn}(\Gamma) && (39)\\ B_n &= (WV)^{-1}WB \in \mathscr{BF}_{pn}(\Gamma) && (40)\\ C_n &= CV \in \mathscr{BF}_{mn}(\Gamma) && (41)\end{aligned}$$

is a block patterned system.

Proof: Consider the Petrov-Galerkin approximations of the decoupled subsystems (21)–(22) on the basis of the matrices V_i and W_i that lead to

$$\begin{aligned} A_{ni} &= (W_iV_i)^{-1}W_iA(\mu_i)V_i \in \mathbb{C}^{n\times n} && (42)\\ B_{ni} &= (W_iV_i)^{-1}W_iB(\mu_i) \in \mathbb{C}^{n\times p} && (43)\end{aligned}$$

as well as

$$C_{ni} = C(\mu_i)V_i \in \mathbb{C}^{m\times n} \quad (44)$$

for $i = 1, 2, \ldots, r$ with $\det W_iV_i \neq 0$ (see (32)–(33) and (35)). A relation between these approximations and the approximation (39)–(41) of (1)–(2) can be obtained by introducing the transformation $T_{red} = R\otimes I_n$ and $T_{red}^{-1} = S \otimes I_n$ as well as the index set $J = \{1, 2, \ldots, r\}$ yielding

$$\begin{aligned} &T_{red}\,\mathrm{bdiag}^{\Gamma}_{i\in J}((W_iV_i)^{-1}W_iA(\mu_i)V_i)T_{red}^{-1}\\ &= (R\otimes I_n)\,\mathrm{bdiag}^{\Gamma}_{i\in J}((W_iV_i)^{-1}W_iA(\mu_i)V_i)(S\otimes I_n)\\ &= (R\otimes I_n)(\mathrm{bdiag}^{\Gamma}_{i\in J}(W_i)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(V_i))^{-1}\\ &\quad\cdot \mathrm{bdiag}^{\Gamma}_{i\in J}(W_i)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(A(\mu_i))\\ &\quad\cdot \mathrm{bdiag}^{\Gamma}_{i\in J}(V_i)(S\otimes I_n)\\ &= ((R\otimes I_n)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(W_i)(S\otimes I_N)(R\otimes I_N)\\ &\quad \mathrm{bdiag}^{\Gamma}_{i\in J}(V_i)(S\otimes I_n))^{-1}\\ &\quad\cdot(R\otimes I_n)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(W_i)(S\otimes I_N)(R\otimes I_N)\\ &\quad\cdot \mathrm{bdiag}^{\Gamma}_{i\in J}(A(\mu_i))(S\otimes I_N)\\ &\quad\cdot(R\otimes I_N)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(V_i)(S\otimes I_n)\\ &= (WV)^{-1}WAV = A_n \in \mathbb{C}^{nl\times nl} \quad (45)\end{aligned}$$

in view of (11), (32)–(33) and (36)–(37). Note, that $\det WV \neq 0$ is implied by $\det W_iV_i \neq 0$. Using the same reasoning one can readily show that also $(R \otimes I_n)\,\mathrm{bdiag}^{\Gamma}_{i\in J}((W_iV_i)^{-1}W_iB(\mu_i))(S \otimes I_p) = (WV)^{-1}WB = B_n \in \mathbb{C}^{nl\times pl}$ and $(R \otimes I_m)\,\mathrm{bdiag}^{\Gamma}_{i\in J}(C(\mu_i)V_i)(S \otimes I_n) = CV = C_n \in \mathbb{C}^{ml\times nl}$. Thus, in view of (32)–(33) and (35) the matrices (39)–(41) characterize a Petrov-Galerkin approximation of the block patterned system (1)–(2). It is easy to verify using the block diagonal matrix representation of a block Γ-patterned matrix that if $M_1 \in \mathscr{BF}_{nm}(\Gamma)$ and $M_2 \in \mathscr{BF}_{mn}(\Gamma)$ then $M_1M_2 \in \mathscr{BF}_{nn}(\Gamma)$ and if $M \in \mathscr{BF}_{nn}(\Gamma)$ then $M^{-1} \in \mathscr{BF}_{nn}(\Gamma)$ provided M^{-1} exists. Consequently, since Lemma 1 implies that (36)–(37) are block Γ-patterned matrices the same is true for (39)–(41) so that the corresponding approximation is a block patterned system. ■

Remark 4: Note, that for $N = 1$ no structure preserving order reduction is possible since the order n of the reduced system has to satisfy $n < N$, though the system (1)–(2) may have a high order l. However, this system can be described by a set of first order systems (see (21)–(22)), so that a reduction of the system order is not necessary.

In view of Lemma 1 the matrices (36)–(37) are block Γ-patterned matrices. Thus, Theorem 1 reveals that a structure preserving Petrov-Galerkin approximation of a block patterned system is possible if the matrices V and W are block Γ-patterned matrices. This reduces the degrees of freedom in the choice of the matrices V and W. However, it is shown in the next section, that a moment matching using Krylov subspaces is still achievable.

IV. STRUCTURE PRESERVING MOMENT MATCHING

The degrees of freedom contained in choosing the matrices V_i and W_i in Theorem 1 can be used to assure desirable properties of the reduced order model (39)–(41). A well-known approach is to match the moments with the original system using *Krylov subspaces* (see e. g. [17], [1], [2], [16]) which means that the transfer behaviour of (1)–(2) is approximated. In order to shortly review the idea of this approach consider the *transfer matrix*

$$F(s) = C(sI_{Nl} - A)^{-1}B, \quad s \in \rho(A) \quad (46)$$

of (1)–(2) with the *resolvent set* $\rho(A) = \mathbb{C} \setminus \sigma(A)$. This transfer matrix has the convergent *Taylor series expansion*

$$F(s) = \sum_{i=0}^{\infty} -M_i^{s_0}(s - s_0)^i \quad (47)$$

about $s = s_0 \in \rho(A)$ in $|s-s_0| < R$, $R = \min_i(|s_0-s_{\infty,i}|)$, where $s_{\infty,i}$ are the poles of $F(s)$. The coefficient matrices in (47) are

$$M_i^{s_0} = C(A - s_0I_{Nl})^{-i-1}B \in \mathbb{C}^{ml\times pl}, \quad i \geq 0 \quad (48)$$

and coincide with the *moments at* s_0 of $F(s)$. Since (47) is a Taylor series its coefficient matrices can be computed from

$$M_i^{s_0} = -\frac{1}{i!}\frac{d^iF}{ds^i}(s_0), \quad i \geq 0, \quad (49)$$

which shows that if the moments of the original system (1)–(2) and the moments $\hat{M}_i^{s_0}$ of its structure preserving Petrov-Galerkin approximation with the matrices (39)–(41) match, i. e.

$$\hat{M}_i^{s_0} = C_n(A_n - s_0I_{nl})^{-i-1}B_n = M_i^{s_0} \quad (50)$$

for some i then the corresponding transfer matrices nearly coincide in a neighborhood about $s = s_0$ where $s_0 \in \rho(A_n)$ is assumed. In order to obtain a transfer matrix for the reduced order system that has real coefficients the expansion point s_0 should be real if only one point is considered. Especially, if $s_0 = 0$, then the steady state transfer behaviour of the system (1)–(2) coincides with its Petrov-Galerkin approximation. If special *Krylov subspaces* are a subset of the spaces $\mathcal{V}$ and $\mathcal{W}$ defining the Petrov-Galerkin projection, then this *moment matching* is assured. These Krylov subspaces are characterized in the next definition.

Definition 2 (Krylov subspaces): Consider a system (C, A, B) where $C \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times p}$ as well as suppose that the spaces $K_i^{s_0,q}((A - s_0 I_n)^{-1}, B) = \operatorname{span}\{(A - s_0 I_n)^{-k} b_l, k = 1,\ldots,q/p, l = 1,\ldots,p\}$ and $K_o^{s_0,f}((A^H - \overline{s_0} I_n)^{-1}, C^H) = \operatorname{span}\{(A^H - \overline{s_0} I_n)^{-k} c_l, k = 1,\ldots,f/m, l = 1,\ldots,m\}$ with $\frac{q}{p}, \frac{f}{m} \in \mathbb{N}$ satisfy $\dim K_i^{s_0,q}((A - s_0 I_n)^{-1}, B) = q$ and $\dim K_o^{s_0,f}((A^H - \overline{s_0} I_n)^{-1}, C^H) = f$ where b_l are the columns of the matrix B and c_l are the columns of the matrix C^H. Then, $K_i^{s_0,q}((A - s_0 I_n)^{-1}, B)$ is called *input Krylov subspace* and $K_o^{s_0,f}((A^H - \overline{s_0} I_n)^{-1}, C^H)$ is called *output Krylov subspace*.

In the following, conditions for the matrices V_i and W_i defining the Petrov-Galerkin approximation of the decoupled systems (21)–(22) are derived such that the moments of the approximation (42)–(44) match with the corresponding moments of (21)–(22). Then, the resulting projection (38) yields a structure preserving moment matching for the Petrov-Galerkin approximation (39)–(41) of (1)–(2). This is the statement of the next theorem.

Theorem 2 (Structure preserving moment matching): Suppose $s_0 \in \rho(A)$ and $s_0 \in \rho(A_n)$ as well as $q \leq n$ and $f \leq n$. Then,

1) If $K_i^{s_0,q}((A(\mu_j) - s_0 I_N)^{-1}, B(\mu_j)) \subseteq \operatorname{ran} V_j$, $j = 1, 2, \ldots, r$, then the block patterned approximation (39)–(41) for (1)–(2) has the property

$$\hat{M}_i^{s_0} = M_i^{s_0} \quad \text{for} \quad i = 0, 1, \ldots, \tfrac{q}{p} - 1. \tag{51}$$

2) If $K_o^{s_0,f}((A^H(\mu_j) - \overline{s_0} I_N)^{-1}, C^H(\mu_j)) \subseteq \operatorname{ran} W_j^H$, $j = 1, 2, \ldots, r$, then the block patterned approximation (39)–(41) for (1)–(2) has the property

$$\hat{M}_i^{s_0} = M_i^{s_0} \quad \text{for} \quad i = 0, 1, \ldots, \tfrac{f}{m} - 1. \tag{52}$$

3) If $K_i^{s_0,q}((A(\mu_j) - s_0 I_N)^{-1}, B(\mu_j)) \subseteq \operatorname{ran} V_j$ and $K_o^{s_0,f}((A^H(\mu_j) - \overline{s_0} I_N)^{-1}, C^H(\mu_j)) \subseteq \operatorname{ran} W_j^H$, $j = 1, 2, \ldots, r$, then the block patterned approximation (39)–(41) for (1)–(2) has the property

$$\hat{M}_i^{s_0} = M_i^{s_0} \quad \text{for} \quad i = 0, 1, \ldots, \tfrac{q}{p} + \tfrac{f}{m} - 1. \tag{53}$$

Proof: If Item 1) is satisfied then the moments of the decoupled subsystem (21)–(22) and its Petrov-Galerkin approximation (42)–(44) match (see e. g. [17]), i. e. $\hat{M}_{ij}^{s_0} = C_{nj}(A_{nj} - s_0 I_n)^{-i-1} B_{nj} = M_{ij}^{s_0} = C(\mu_j)(A(\mu_j) - s_0 I_N)^{-i-1} B(\mu_j)$, $j = 1, 2, \ldots, r$. Due to the multiplicity of the eigenvalues μ_j of Γ (see Section II) the same is true for the remaining moments of the decoupled subsystems and their Petrov-Galerkin approximations. Now, consider the transformation

$$\begin{aligned} &T_y \operatorname{bdiag}_{j\in J}^{\Gamma}(\hat{M}_{ij}^{s_0}) T_u^{-1} = R_m \operatorname{bdiag}_{j\in J}^{\Gamma}(\hat{M}_{ij}^{s_0}) S_p \\ &= R_m \operatorname{bdiag}_{j\in J}^{\Gamma}(C_{nj}) S_n (R_n \\ &\quad \cdot \operatorname{bdiag}_{j\in J}^{\Gamma}(A_{nj} - s_0 I_n) S_n \cdot \ldots \cdot R_n \\ &\quad \cdot \operatorname{bdiag}_{j\in J}^{\Gamma}(A_{nj} - s_0 I_n) S_n)^{-1} R_n B_{nj} S_p \\ &= C_n (A_n - s_0 I_n)^{-i-1} B_n = \hat{M}_i^{s_0}, \end{aligned} \tag{54}$$

where $R_m = R \otimes I_m$, $R_n = R \otimes I_n$, $S_p = S \otimes I_p$, $S_n = S \otimes I_n$ and $\hat{M}_i^{s_0}$ is the moment of (39)–(41). Similarly, $T_y \operatorname{bdiag}_{i\in J}^{\Gamma}(M_{ij}^{s_0}) T_u^{-1} = M_i^{s_0}$ holds for the moments of (1)–(2), so that the moments $M_i^{s_0}$ of (1)–(2) and the moments $\hat{M}_i^{s_0}$ of its block patterned Petrov-Galerkin approximation (39)–(41) match in view of $\hat{M}_{ij}^{s_0} = M_{ij}^{s_0}$. Using the same reasoning as well as the corresponding results on moment matching in [17] for the remaining Items 2) and 3) the corresponding statements can readily be proved for the application of the output Krylov subspaces and of the combination of the input and output Krylov subspaces. ■

Remark 5: The direct application of the Definition 2 to determine input and output Krylov subspaces is not numerically reliable for large scale systems. Therefore, the numerically stable *Arnoldi algorithm* or *Lanczos algorithm* are usually applied to find an orthonormal basis of these subspaces (see [18], [19], [17]).

Remark 6: If $q < n$ and/or $f < n$ then the remaining $n - q$ vectors v_{ij}, $i = 1, 2, \ldots, r$, $j = q + 1, \ldots, n$, and/or $n - f$ vectors w_{ij}, $j = f + 1, \ldots, n$, of V_i and W_i can be determined such that moment matching is achieved for additional expansion points s_i, $i > 0$, of the Krylov-based model order reduction. In order to obtain an approximation with a transfer matrix having real coefficients the resulting set of expansion points has to be self-conjugate. Furthermore, if the expansion points lie on the imaginary axis, then an approximation of the frequency response of (1)–(2) in prescribed frequency ranges is possible. This approach, which is known as *rational interpolation* (see [17], [2]), can directly be formulated in a structure preserving form using the results of this section (see also the example in the next section).

V. EXAMPLE

Consider an asymptotically stable system (1)–(2) with the matrices

$$\begin{aligned} A &= I_3 \otimes A_0 + \Gamma \otimes A_1 && (55) \\ B &= I_3 \otimes B_0 + \Gamma \otimes B_1 \\ C &= I_3 \otimes C_0 + \Gamma \otimes C_1, && (56) \end{aligned}$$

in which $A_i \in \mathbb{R}^{10\times 10}$, $B_i \in \mathbb{R}^{10\times 1}$, $C_i \in \mathbb{R}^{1\times 10}$ for $i = 0, 1$, hold and the base matrix reads

$$\Gamma = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{57}$$

so that a symmetrically interconnected system results (see Remark 2). The matrix Γ has the eigenvalue $\mu_1 = 2$ and the multiple eigenvalue $\mu_2 = \mu_{22} = -1$ so that only two decoupled subsystems (21)–(22) have to be considered, i. e. $r = 2$. The first subsystem has the spectrum $\sigma(A(2)) = \{-0.0071 + j, -0.0071 - j, -2, -3, -4, -5, -6, -7, -8, -9\}$ and is represented in controller normal form with $C(2) = e_1^T$. The spectrum of the second subsystem is $\sigma(A(-1)) = \{-10, -11, -12, -0.0919 + j12.9997, -0.0919 - j12.9997, -14, -15, -18, -19, -20\}$ and the corresponding state space model is also in controller normal form with $C(-1) = e_1^T$. In order to obtain a second order approximation with moment matching the expansion points $s_0 = 0$ and $s_{1,2} = \pm j0.9999$ are chosen. The first expansion point assures that the reduced order system has the same steady state transfer behaviour as the original system. The other expansion points assure that the resonant peak with the lowest frequency in the Bode plot is approximated by the order reduction. This leads to the moment matching $\hat{M}_0^{s_0} = M_0^{s_0}$, $\hat{M}_1^{s_0} = M_1^{s_0}$ and $\hat{M}_0^{s_1} = M_0^{s_1}$, $\hat{M}_0^{s_2} = M_0^{s_2}$ where the first two conditions imply that the value and the slope of the frequency response at $s = 0$ of the original system and its approximation are the same. In order to obtain a structure preserving moment matching the corresponding moments of the decoupled subsystems have to coincide. This is achieved by choosing the matrices $V_j = [v_{j1} \quad v_{j2}]$ and $W_j = [w_{j1} \quad w_{j2}]^H$, $j = 1, 2$, such that $\operatorname{ran} V_j = K_i^{s_0,2}((A(\mu_j))^{-1}, B(\mu_j))$, $\operatorname{ran} w_{1j} = K_o^{s_1,1}((A(\mu_j) - s_1 I)^{-1}, C^H(\mu_j))$, and $\operatorname{ran} w_{2j} = K_o^{s_2,1}((A(\mu_j) - s_2 I)^{-1}, C^H(\mu_j))$ hold (see item 3) of Theorem 2). Since $\dim \operatorname{ran} V_j = 2$, $j = 1, 2$, this leads to decoupled subsystems of the reduced order $n = 2$, so that the approximation of (55) has the order $nl = 6$. The corresponding structure preserving Petrov-Galerkin projection (38) which achieves the moment matching for the original system (55) is specified by $V = (R \otimes I_{10}) \operatorname{bdiag}(V_1, V_2, V_2)(S \otimes I_2)$ and $W = (R \otimes I_2) \operatorname{bdiag}(W_1, W_2, W_2)(S \otimes I_{10})$ (see Theorem 1). The matrices (39)–(41) of the corresponding reduced order system, which is asymptotically stable, can be represented by

$$\begin{aligned}
A_{red} &= I_3 \otimes \begin{bmatrix} 3.49 & 11 \\ -3.98 & -8.25 \end{bmatrix} + \Gamma \otimes \begin{bmatrix} -1.75 & -2.98 \\ 1.89 & 4.12 \end{bmatrix} \\
B_{red} &= I_3 \otimes \begin{bmatrix} 0.18 \cdot 10^{-5} \\ -0.02 \cdot 10^{-5} \end{bmatrix} + \Gamma \otimes \begin{bmatrix} 0.18 \cdot 10^{-5} \\ -0.02 \cdot 10^{-5} \end{bmatrix} \\
C_{red} &= I_3 \otimes \begin{bmatrix} 0.23 & 0.27 \end{bmatrix} + \Gamma \otimes \begin{bmatrix} -0.57 & -0.33 \end{bmatrix}
\end{aligned}$$

meaning that the reduced order system is also symmetrical. This shows that a structure preserving reduction has been achieved. The corresponding moment matching can be verified by the Bode plots in Figure 1. Thereby, only the Bode plots of the unequal elements of the symmetric transfer matrices $F(s) = C(sI_{30} - A)^{-1}B$ of the original system and $F_{red}(s) = C_{red}(sI_6 - A_{red})^{-1}B_{red}$ of its approximation are shown.

VI. CONCLUDING REMARKS

A topic for further research is to investigate whether an application of other structure preserving Krylov subspace methods such as passivity preservation to the decoupled subsystems also lead to a corresponding structure preserving order reduction of the block patterned system. It is interesting to note, that a structure preserving balanced truncation of block patterned systems is, in general, not possible, since the balancing transformation destroys the system structure.

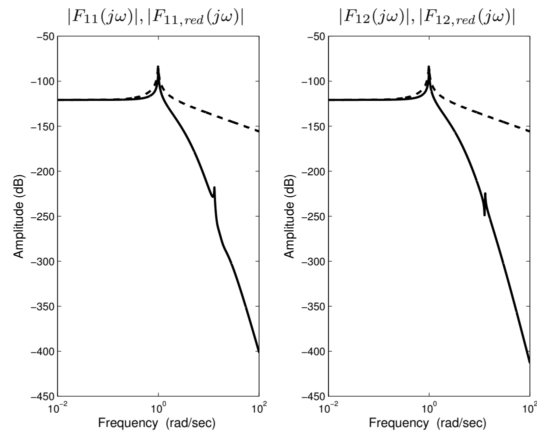

Fig. 1. Bode plots $|F(j\omega)|_{dB}$ of the high order system (solid line) and $|F_{red}(j\omega)|_{dB}$ of its approximation based on structure preserving moment matching (dashed line).

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