

Analysis and Comparison of Port-Hamiltonian Formulations for Field Theories - demonstrated by means of the Mindlin plate

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Abstract—This paper focuses on the port-Hamiltonian formulation of systems described by partial differential equations. Based on a variational principle we derive the equations of motion as well as the boundary conditions in the well-known Lagrangian framework. Then it is of interest to reformulate the equations of motion in a port-Hamiltonian setting, where we compare the approach based on Stokes-Dirac structures to a Hamiltonian setting that makes use of the involved bundle structure similar to the one on which the variational approach is based. We will use the Mindlin plate, a distributed parameter system with spatial domain of dimension two, as a running example.

I. INTRODUCTION

Distributed parameter systems described by partial differential equations arise in systems theory from a modeling and a control theoretic point of view and are without doubt a challenging research problem, where lot of progress has been achieved in the last years. Also the port-Hamiltonian setting, originally developed in the finite dimensional scenario has been transferred to infinite-dimensional systems, where e.g. the well-known approach based on (Stokes-)Dirac structures (also known from the lumped parameter scenario) is available, see e.g. [1], [2], [3], [4], [5], [6] and references therein.

Also in mathematical physics, systems described by partial differential equations (pdes) are interpreted in a Hamiltonian setting, e.g. in [7], [8], [9] and references therein, but in most cases systems with trivial boundary conditions are considered, which is not the case in many engineering applications. Therefore, the approach based on Stokes-Dirac structures has been setup to overcome the problem of non-zero energy flow through the boundary.

A different port-Hamiltonian approach that is not necessarily relying on an underlying Stokes-Dirac structure but uses the induced jet-bundle structure with respect to dependent and independent variables explicitly has been discussed in [10], [11], [12], [13], [14], [15], [16]. These references are all based on [9] but adapted to control purposes, i.e. modified in a sense, such that non-zero energy flow through the boundary can be considered (such that boundary ports are included) and furthermore control inputs on the domain and/or the boundary can be included.

The main difference of the approach relying on Stokes-Dirac structures and the approach using bundles, is that the

Stokes-Dirac scenario is based on the choice of proper energy variables (flows and efforts) for which the power balance is formulated, whereas the second approach is based on a given Hamiltonian density (the total energy density) and the evaluation of the power balance is performed based on the underlying bundle formalism in order to restructure the pdes such that the energy flows are linked to the physics. This will have the consequence, that the variational derivative is interpreted differently and the choice of state variables is different, in the two mentioned approaches.

The purpose of this paper is, that based on the well-known Lagrangian setting for first order field theories i) the partial differential equations and the boundary conditions derived using a variational principle are reinterpreted using two different port-Hamiltonian settings, which describe the same physical phenomenon but using a completely different port-Hamiltonian representation, ii) by using the example of a Mindlin plate all these concepts are visualized and compared in great detail.

II. NOTATION

We will use differential geometric methods for our considerations and the notation is similar to the one in [7], where the interested reader can find much more details about this geometric machinery. To keep the formulas short and readable we will use tensor notation and especially Einstein's convention on sums.

We use the standard symbol \wedge for the exterior product (wedge product), d is the exterior derivative, \lrcorner the natural contraction between tensor fields. By ∂_α^B are meant the partial derivatives with respect to coordinates with the indices α_B and $[m^{\alpha\beta}]$ corresponds to the matrix representation of the (second-order) tensor m with components $m^{\alpha\beta}$. E.g. taking a second-order tensor m and a co-vector ω , the components of the contraction $m \lrcorner \omega$ read in local coordinates as $m^{\alpha\beta} \omega_\alpha$, where the summation over α is performed (Einstein convention on sums).

Furthermore $C^\infty(\cdot)$ denotes the set of the smooth functions on the corresponding manifold. Moreover we will not indicate the range of the used indices when they are clear from the context. Additionally, pull backs and pull back bundles are only stated when necessary, when they follow from the context they are not indicated to avoid exaggerated notation.

Let us consider the bundle $\mathcal{Y} \rightarrow \mathcal{D}$ with coordinates (X^A, y^α) for \mathcal{Y} and (X^A) for \mathcal{D} . The first jet manifold $\mathcal{J}^1(\mathcal{Y})$ possesses the coordinates $(X^A, y^\alpha, y_A^\alpha)$, where the

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capital Latin indices A, B, \dots are used for the base manifold \mathcal{D} (independent coordinates) and y_A^α denote derivative coordinates of first order (derivatives of the dependent coordinates with respect to the independent ones) as well as $\partial_A = \partial/\partial X^A$, $\partial_\alpha = \partial/\partial y^\alpha$, $\partial_\alpha^A = \partial/\partial y_A^\alpha$. The jet structure also induces the so-called total derivative

$$d_A = \partial_A + y_A^\alpha \partial_\alpha + y_{AB}^\alpha \partial_\alpha^B$$

acting on elements including first order derivatives and y_{AB}^α correspond to derivative coordinates of second order living in $\mathcal{J}^2(\mathcal{Y})$, the second jet manifold. Based on the bundle structure $\mathcal{Y} \rightarrow \mathcal{D}$ let us introduce the vertical tangent bundle $\mathcal{V}(\mathcal{Y})$, as well as $\Lambda_1^d(\mathcal{Y}) = \mathcal{T}^*(\mathcal{Y}) \wedge (\wedge^d \mathcal{T}^*(\mathcal{D}))$, see also [7], with a typical element $\omega = \omega_\alpha dy^\alpha \wedge dV$ for $\Lambda_1^d(\mathcal{Y})$ where dV denotes the volume element on the manifold \mathcal{D} , i.e. $dV = dX^1 \wedge \dots \wedge dX^d$ with $\dim(\mathcal{D}) = d$ and the functions ω_α may depend on derivative coordinates. Furthermore, a typical element for $\mathcal{V}(\mathcal{Y})$ reads as $v = v^\alpha \partial_\alpha$ and when v^α depends on derivative coordinates we call v a generalized vertical vector field, see [9].

III. BACKGROUND MATERIAL

A. Geometric preliminaries

We will consider densities \mathfrak{F} in the sequel (a quantity that can be integrated), $\mathfrak{F} = \mathcal{F}dV$ with $\mathcal{F} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$ (we restrict ourselves to the first-order case). By $F = \int_{\mathcal{D}} \mathfrak{F}$ we denote the integrated quantity, where of course a section of the bundle $\mathcal{Y} \rightarrow \mathcal{D}$, i.e. a map $y = \Phi(X)$ leading to $y_A = \partial_A \Phi(X)$ has to be plugged in to be able to evaluate the integral properly.

Proposition 1: Given the density $\mathfrak{F} = \mathcal{F}dV$ and a generalized vertical vector field $v : \mathcal{Y} \rightarrow \mathcal{V}(\mathcal{Y})$, together with its first jet-prolongation $j^1(v) = v^\alpha \partial_\alpha + d_A(v^\alpha) \partial_\alpha^A$, see [9], [7], we obtain the decomposition

$$\int_{\mathcal{D}} j^1(v)(\mathcal{F}dV) = \int_{\mathcal{D}} v \rfloor \delta \mathfrak{F} + \int_{\partial \mathcal{D}} v \rfloor \delta^\partial \mathfrak{F}. \quad (1)$$

Here the map $\delta \mathfrak{F} = \delta_\alpha \mathcal{F} dy^\alpha \wedge dV$ (corresponding to the Euler Lagrange operator of \mathfrak{F}), see [7], with the coefficients $\delta_\alpha \mathcal{F} = \partial_\alpha \mathcal{F} - d_A \partial_\alpha^A \mathcal{F}$ (called the variational derivatives) is used, as well as the boundary operator $\delta^\partial \mathfrak{F} = \partial_\alpha^A \mathcal{F} dy^\alpha \wedge dV_A$ with $dV_A = \partial_A \rfloor dV$ (the boundary volume form).

Proof: The proof follows by evaluating the Lie-derivative of the geometric object \mathfrak{F} with respect to the vector field $j^1(v)$

$$\begin{aligned} j^1(v)(\mathcal{F}dV) &= (v^\alpha (\partial_\alpha \mathcal{F} - d_A \partial_\alpha^A \mathcal{F}) + d_A(v^\alpha \partial_\alpha^A \mathcal{F})) dV \\ &= (v^\alpha \delta_\alpha \mathcal{F} + d_A(v^\alpha \partial_\alpha^A \mathcal{F})) dV \end{aligned} \quad (2)$$

and applying the Theorem of Stokes [9] to (2). ■

The relation (1) will be of key interest in the forthcoming, since it provides a natural decomposition of the expression $\int_{\mathcal{D}} j^1(v)(\mathcal{F}dV)$ into a term on the domain \mathcal{D} and one on the boundary $\partial \mathcal{D}$. Important is the case when the generalized vector-field v is linked to the solution of a pde system (via its semi-group, that v may generate), then the formal change of $F = \int_{\mathcal{D}} \mathcal{F}dV$ along solutions of a pde system can be

computed as $\int_{\mathcal{D}} j^1(v)(\mathcal{F}dV)$ (provided all operations are admissible), which we denote by $\dot{F} = \int_{\mathcal{D}} j^1(v)(\mathcal{F}dV)$ in this special case.

B. Dirac structures

Based on the space of power variables $\mathcal{F} \times \mathcal{E}$ (flows and efforts) and the symmetric bilinear pairing

$$\ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the dual product of the linear spaces \mathcal{F} and $\mathcal{E} = \mathcal{F}^*$ a Dirac structure is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\mathbb{D} = \mathbb{D}^\perp$ with respect to the pairing (3). For $(f, e) \in \mathbb{D}$ one has $\langle e, f \rangle = 0$ such that the Dirac structure preserves power. This concept can be transferred to the case where \mathcal{F} and \mathcal{E} are spaces of vector-valued functions over a spatial domain \mathcal{D} , then infinite dimensional systems are the focus, and to allow for non-zero energy flow through the boundary the so-called Stokes-Dirac structure is introduced, see [1], [2], [3], [6] and section V-B.

IV. LAGRANGIAN FRAMEWORK

In this section we recapitulate the well-known Lagrangian framework for first-order field theories, and we will derive the partial differential equations as well as the boundary conditions in a geometric fashion. Thus, we consider a bundle $\mathcal{Q} \rightarrow \mathcal{D}_L$ with coordinates (t^0, X^A, q^α) for \mathcal{Q} and (t^0, X^A) for \mathcal{D}_L . We use the shortcut $x^i = (t^0, X^A)$ such that the independent variables are the time t^0 and the spatial ones X^A , $A = 1, \dots, \dim(\mathcal{D}_L) - 1 = d$. A first order Lagrangian takes the form $\mathcal{L} : \mathcal{J}^1(\mathcal{Q}) \rightarrow \wedge^{d+1} \mathcal{T}^*(\mathcal{D}_L)$, $\mathcal{L} = \mathcal{L}\omega$ with $\mathcal{L} \in C^\infty(\mathcal{J}^1(\mathcal{Q}))$ together with the volume element ω that meets $\omega = dt^0 \wedge dX^1 \dots \wedge dX^d$, $\omega_i = \partial_i \rfloor \omega$, $i = 0, \dots, d$. The variational problem for a section $s : \mathcal{D}_L \rightarrow \mathcal{Q}$ is the following¹

$$\left(d_\epsilon \int_{\mathcal{D}_L} j^1(\psi_\epsilon \circ s)^* \mathcal{L} \right) \Big|_{\epsilon=0} = 0, \quad (4)$$

where the flow ψ_ϵ is used to deform sections $s : \mathcal{D}_L \rightarrow \mathcal{Q}$ and whose generator is a vertical vector field $v_L : \mathcal{Q} \rightarrow \mathcal{V}(\mathcal{Q})$.

This is a well-known problem and treated for example in [7], [9] and references therein. It is obvious that (4) is equivalent to

$$\int_{\mathcal{D}_L} (j^2 s)^* (j^1(v_L)(\mathcal{L})) = 0, \quad (5)$$

see for example [7]. Adapting (1) (with respect to the bundle structure, and replacing \mathcal{F} by \mathcal{L}) we obtain the decomposition

$$\int_{\mathcal{D}_L} v_L \rfloor \delta \mathcal{L} + \int_{\partial \mathcal{D}_L} v_L \rfloor \delta^\partial \mathcal{L} = 0. \quad (6)$$

Consequently, the partial differential equations for a first order Lagrangian follow as

$$\delta_\alpha(\mathcal{L}) = 0, \quad \delta_\alpha = \partial_\alpha - d_i \partial_\alpha^i \quad (7)$$

¹At this point the pull back is essential and therefore indicated, i.e. $j^1(\psi_\epsilon \circ s)^* \mathcal{L}$ means, that the first prolongation of $(\psi_\epsilon \circ s)$ has to be plugged in into $\mathcal{L} = \mathcal{L}\omega$ in order to evaluate the integral properly.

and the boundary term is the second term in (6) and reads in local coordinates as

$$\int_{\partial\mathcal{D}_L} v_L^\alpha \partial_\alpha^i \mathcal{L} \omega_i = 0. \quad (8)$$

The boundary conditions can be fulfilled by either allowing for no variations (Dirichlet condition) on (a part of) $\partial\mathcal{D}_L$, i.e. $v_L^\alpha = 0$ or by $\partial_\alpha^i \mathcal{L} = 0$ or by a combinations of both approaches. Possible is also the inclusion of external boundary variables $F_{e,\alpha}^i$ such that

$$\int_{\partial\mathcal{D}_L} v_L^\alpha (\partial_\alpha^i \mathcal{L} - F_{e,\alpha}^i) \omega_i = 0. \quad (9)$$

has to be met.

V. PORT-HAMILTONIAN PICTURE

Now we turn to the Hamiltonian picture, where we discuss two different port-Hamiltonian formulations. We will restrict ourselves to systems without dissipation and without distributed control for simplicity, but these properties can be included in both formalisms in a straightforward manner, see [2], [11].

A. Geometric approach based on underlying bundle structure

We will introduce port-Hamiltonian systems described by pdes based on a power balance relation, such that the power balance relation together with the structure of the equations represent the physical process.

Definition 1: A port-Hamiltonian boundary control system without dissipation on a bundle $\mathcal{X} \rightarrow \mathcal{D}_H$ with coordinates (x^α, X^A) for \mathcal{X} and (X^A) for \mathcal{D}_H takes the form of

$$\dot{x} = \mathcal{J}(\delta\mathfrak{H}), \quad (\dot{x}^\alpha = \mathcal{J}^{\alpha\beta} \delta_\beta \mathcal{H}) \quad (10)$$

with the Hamiltonian $\mathfrak{H} = \mathcal{H}\Omega$, $\mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{X}))$. Here the volume form $\Omega = dX^1 \wedge \dots \wedge dX^d$ is used and furthermore (10) has to be completed by boundary conditions that include boundary inputs (optionally leading to so-called boundary ports). The map \mathcal{J} is of the form $\mathcal{J} : \Lambda_1^d(\mathcal{X}) \rightarrow \mathcal{V}(\mathcal{X})$ where \mathcal{J} is a skew-symmetric map.

In general the map \mathcal{J} can be a differential operator, see our paper [16], but within this contribution we exclude this case (since in many examples, e.g. mechanics this is not required).

Remark 1: An evolutionary PDE of second order (without domain-input) is in general of the form

$$\dot{x}^\alpha = f^\alpha(X^A, x_B^\beta, x_{BC}^\beta) \quad (11)$$

and the boundary conditions (first-order, with boundary input) may take the form $f_\partial(X^A, x_B^\beta, x_{BC}^\beta, u_\partial)|_{\partial\mathcal{D}_H} = 0$. The system (10) is a special case of (11), i.e. it is its Hamiltonian representation.

Now we make use of proposition 1 and replace \mathfrak{F} by \mathfrak{H} in (1). Setting $v = \dot{x}$ we obtain

$$\dot{H} = \int_{\partial\mathcal{D}_H} \dot{x} \rfloor \delta^\partial \mathfrak{H} = \int_{\partial\mathcal{D}_H} \dot{x}^\alpha \partial_\alpha^A \mathcal{H} \Omega_A \quad (12)$$

where $\Omega_A = \partial_A \rfloor \Omega$, which reflects the power balance, since the total change of the functional H along solutions of (10), is affected by a boundary port (if it exists) depending on the boundary conditions. See e.g. [11] for a formal introduction concerning the boundary ports.

Remark 2: If the system can be derived by a variational principle, then from (9) we see that in the case of non Dirichlet boundary conditions with no variation on the temporal boundary $\partial_\alpha^A \mathcal{L} = F_{e,\alpha}^A$, $A = 1, \dots, d$ has to hold, which equivalently leads to the boundary port $\dot{x}^\alpha F_{e,\alpha}^A$ when the Hamiltonian follows from the Lagrangian as $\mathcal{H} = p_\alpha q_0^\alpha - \mathcal{L}$, where $p_\alpha = \partial_\alpha^0 \mathcal{L}$.

B. Approach based on underlying Stokes-Dirac Structure

Following [6] we shortly recapitulate the port-Hamiltonian framework based on Stokes-Dirac structures. For more details we refer to [1], [2].

We consider the space of flows \mathcal{F} and the space of efforts \mathcal{E} which are spaces of vector-valued functions over a spatial domain \mathcal{D} . Given \mathcal{J}_{SD} a skew-adjoint matrix differential operator [6], the space

$$\mathbb{D} = \{(f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \mid f = -\mathcal{J}_{SD}e, w = \mathcal{B}_D(e)\}$$

is a Stokes-Dirac structure, regarding the pairing

$$\begin{aligned} \ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg &= \\ &= \int_{\mathcal{D}} [e_1^T f_2 + e_2^T f_1] dV + \int_{\partial\mathcal{D}} \mathcal{B}_{\mathcal{J}_{SD}}(w_1, w_2) dA \end{aligned}$$

where $\mathcal{B}_{\mathcal{J}_{SD}}$ is a boundary differential operator induced by \mathcal{J}_{SD} and by slight abuse of notation dA corresponds to the boundary volume element. The map \mathcal{B}_D is a boundary operator and the boundary variables are w .

If $(f, e, w) \in \mathbb{D}$ then

$$0 = \int_{\mathcal{D}} e^T f dV + \frac{1}{2} \int_{\partial\mathcal{D}} \mathcal{B}_{\mathcal{J}_{SD}}(w, w) dA \quad (13)$$

holds. Given an energy density $\mathcal{H}dV$ where \mathcal{H} depends on the energy variables, a port-Hamiltonian boundary control system without dissipation can be stated as

$$f = -\mathcal{J}_{SD}e, \quad w = \mathcal{B}_D(e) \quad (14)$$

where the energy variables f , are linked to the state variables χ via $f = -\dot{\chi}$ and the effort variables are $e = \partial_\chi \mathcal{H}$.

Remark 3: Originally, in [6] instead of $e = \partial_\chi \mathcal{H}$ the authors use $e = \delta_\chi H$ where $H = \int_{\mathcal{D}} \mathcal{H} dV$, but since \mathcal{H} depends on energy variables, the variational derivative degenerates to a 'partial' one.

Furthermore, from $H = \int_{\mathcal{D}} \mathcal{H} dV$ and the relations (13) and (14) one has

$$\dot{H} = \frac{1}{2} \int_{\partial\mathcal{D}} \mathcal{B}_{\mathcal{J}_{SD}}(w, w) dA \quad (15)$$

since $\dot{H} = \int_{\mathcal{D}} \partial_\chi \mathcal{H} \dot{\chi} dV = - \int_{\mathcal{D}} e^T f dV$.

Remark 4: To derive this energy balance also proposition 1 can be applied, but since no jet-variables are included it simplifies to $\dot{H} = \int_{\mathcal{D}} j^1(v)(\mathcal{H}dV) = \int_{\mathcal{D}} v^\alpha \partial_\alpha \mathcal{H} dV$ and to derive (15) a further integration by parts must be performed, since v corresponds to $-f$ which involves the differential operator \mathcal{J}_{SD} .

VI. THE MINDLIN PLATE

Let us consider a rectangular plate with lengths l_x, l_y , where h will denote the thickness, which will be modeled based on the hypothesis stated by Mindlin. Therefore, we choose as independent coordinates the vertical deflection w of the mid-plane as well as the rotations of a transverse normal to the X and Y direction termed ψ and ϕ , respectively. The kinetic energy density \mathcal{K} and the potential energy density \mathcal{V} can be stated as $\mathcal{K} = \frac{\rho}{2}(\frac{h^3}{12}(\psi_t^2 + \phi_t^2) + hw_t^2)$ and

$$\begin{aligned} \mathcal{V} &= \frac{1}{2}kGh [(w_X - \psi)^2 + (w_Y - \phi)^2] \\ &+ \frac{1}{2}D\frac{1-\nu}{2}(\psi_Y + \phi_X)^2 \\ &+ \frac{1}{2}(D(\psi_X^2 + \nu\phi_Y\psi_X) + D(\phi_Y^2 + \nu\phi_Y\psi_X)), \end{aligned}$$

where ν is the Poisson ratio, $k = \frac{E}{12}$, and G, D are the plate stiffness and the plate module, respectively, see [6] and references therein.

Remark 5: The subscripts t, X, Y correspond to the derivatives with respect to these independent variables, according to the jet-bundle structure in the Lagrangian framework. The subscripts x, y to be used later, correspond to quantities which are connected to the spatial variables X and Y but they must not be confused with derivative variables. Since we are in a time-invariant setting, we will use later on also the $\dot{}$ notation, for time derivatives, instead of the subscripts t .

To derive the equations of motion we will use the variational principle in a Lagrangian setting. Then given the partial differential equations, we will interpret them in a Hamiltonian setting, either using the approach presented in section V-A and using an approach based on the Stokes-Dirac structure as in section V-B.

A. The Lagrangian picture

In the Lagrangian framework we consider the bundle $\mathcal{Q} \rightarrow \mathcal{D}_L$ with coordinates (w, ψ, ϕ, t, X, Y) for \mathcal{Q} and (t, X, Y) for \mathcal{D}_L together with the Lagrangian density $\mathcal{L} = \mathcal{K} - \mathcal{V}$ with $\omega = dt \wedge dX \wedge dY$. The variational derivatives follow from the chosen bundle structure and follow to

$$\begin{aligned} \delta_w &= \partial_w - d_t \partial_w^t - d_X \partial_w^X - d_Y \partial_w^Y \\ \delta_\psi &= \partial_\psi - d_t \partial_\psi^t - d_X \partial_\psi^X - d_Y \partial_\psi^Y \\ \delta_\phi &= \partial_\phi - d_t \partial_\phi^t - d_X \partial_\phi^X - d_Y \partial_\phi^Y. \end{aligned}$$

From $\delta_w \mathcal{L} = 0$, $\delta_\psi \mathcal{L} = 0$, $\delta_\phi \mathcal{L} = 0$ corresponding to (7) we derive the partial differential equations

$$\begin{aligned} \rho h w_{tt} &= kGh(w_{XX} - \psi_X) + kGh(w_{YY} - \phi_Y) \\ \rho \frac{h^3}{12} \psi_{tt} &= kGh(w_X - \psi) + D(\psi_{XX} + \nu\phi_{XY}) \\ &+ \frac{1}{2}D(1-\nu)(\psi_{YY} + \phi_{XY}) \\ \rho \frac{h^3}{12} \phi_{tt} &= kGh(w_Y - \phi) + D(\phi_{YY} + \nu\psi_{XY}) \\ &+ \frac{1}{2}D(1-\nu)(\psi_{XY} + \phi_{XX}). \end{aligned}$$

If we introduce

$$\begin{aligned} M_x &= D(\psi_X + \nu\phi_Y) = -\partial_\psi^X \mathcal{L} \\ M_y &= D(\phi_Y + \nu\psi_X) = -\partial_\phi^Y \mathcal{L} \\ M_{xy} &= D\frac{1-\nu}{2}(\psi_Y + \phi_X) = -\partial_\psi^Y \mathcal{L} = -\partial_\phi^X \mathcal{L} \quad (16) \\ Q_x &= kGh(w_X - \psi) = -\partial_w^X \mathcal{L} \\ Q_y &= kGh(w_Y - \phi) = -\partial_w^Y \mathcal{L} \end{aligned}$$

then the equations of motion take the familiar form²

$$\begin{aligned} \rho h w_{tt} &= \partial_X Q_x + \partial_Y Q_y \\ \rho \frac{h^3}{12} \psi_{tt} &= Q_x + \partial_X M_x + \partial_Y M_{xy} \quad (17) \\ \rho \frac{h^3}{12} \phi_{tt} &= Q_y + \partial_Y M_y + \partial_X M_{xy}. \end{aligned}$$

The boundary conditions follow from

$$\int_{\partial \mathcal{D}_L} (w_L \partial_w^i \mathcal{L} + \psi_L \partial_\psi^i \mathcal{L} + \phi_L \partial_\phi^i \mathcal{L}) \partial_i] (dt \wedge dX \wedge dY) = 0$$

where the variational vector field v_L takes the form $v_L = w_L \partial_w + \psi_L \partial_\psi + \phi_L \partial_\phi$ and on the time-boundary no variation takes place (i.e. when $i = 0$ then $v_L = 0$) and (16) has to be used. Therefore we have

$$\begin{aligned} \int_{\partial \mathcal{D}_L} (w_L Q_x + \psi_L M_x + \phi_L M_{xy}) dt \wedge dY \\ - \int_{\partial \mathcal{D}_L} (w_L Q_y + \psi_L M_{xy} + \phi_L M_y) dt \wedge dX = 0 \quad (18) \end{aligned}$$

such that, e.g. if at $X = 0$ we have that w_L is arbitrary, then Q_x has to vanish or has to be compensated by an external boundary term as in (9), such that the familiar boundary conditions are recovered.

B. The Hamiltonian picture

Based on the partial differential equations (17) and the boundary conditions (18) we discuss the two presented port-Hamiltonian formulations as well as the power balance relations corresponding to the particular representation.

1) *Geometric approach:* Now we consider the bundle (which is different from the one in the Lagrangian setting) $\mathcal{X} \rightarrow \mathcal{D}_H$ with coordinates $(w, \psi, \phi, p_w, p_\psi, p_\phi, X, Y)$ for \mathcal{X} and (X, Y) for \mathcal{D}_H and the volume form $\Omega = dX \wedge dY$. From the Legendre transform we derive the temporal momenta $p_w = \partial_w^t \mathcal{L}$, $p_\psi = \partial_\psi^t \mathcal{L}$, $p_\phi = \partial_\phi^t \mathcal{L}$ which read as

$$p_w = \rho h w_t, \quad p_\psi = \rho \frac{h^3}{12} \psi_t, \quad p_\phi = \rho \frac{h^3}{12} \phi_t \quad (19)$$

and the Hamiltonian follows as

$$\mathcal{H} = \dot{w} p_w + \dot{\psi} p_\psi + \dot{\phi} p_\phi - \mathcal{L}.$$

In the coordinates $(w, \psi, \phi, p_w, p_\psi, p_\phi)$ together with (19) one has $\mathcal{H} = \mathcal{K} + \mathcal{V}$.

²In order to be comparable with the literature we sometimes use ∂_X and ∂_Y although in a strict mathematical sense it should be d_X and d_Y .

To derive the Hamiltonian formulation as in (10) we set $x = (w, \psi, \phi, p_w, p_\psi, p_\phi)$ and obtain

$$\dot{x} = \mathcal{J}(\delta\mathfrak{H}) \quad (20)$$

which reads as

$$\begin{bmatrix} \dot{w} \\ \dot{\psi} \\ \dot{\phi} \\ \dot{p}_w \\ \dot{p}_\psi \\ \dot{p}_\phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H} \\ \delta_\psi \mathcal{H} \\ \delta_\phi \mathcal{H} \\ \delta_{p_w} \mathcal{H} \\ \delta_{p_\psi} \mathcal{H} \\ \delta_{p_\phi} \mathcal{H} \end{bmatrix}.$$

The variational derivatives in this setting take the form

$$\begin{aligned} \delta_w &= \partial_w - d_X \partial_w^X - d_Y \partial_w^Y \\ \delta_\psi &= \partial_\psi - d_X \partial_\psi^X - d_Y \partial_\psi^Y \\ \delta_\phi &= \partial_\phi - d_X \partial_\phi^X - d_Y \partial_\phi^Y \end{aligned}$$

and $\delta_{p_w} = \partial_{p_w}$, $\delta_{p_\psi} = \partial_{p_\psi}$, $\delta_{p_\phi} = \partial_{p_\phi}$ because of the different bundle structure compared to the Lagrangian approach.

The boundary ports follow from (12) and we obtain

$$\begin{aligned} \dot{H} &= \int_{\partial\mathcal{D}_H} \dot{x}^\alpha \partial_\alpha^A \mathcal{H} \partial_A \rfloor (dX \wedge dY) \\ &= \int_{\partial\mathcal{D}_H} (\dot{x}^\alpha \partial_\alpha^X \mathcal{H} dY - \dot{x}^\alpha \partial_\alpha^Y \mathcal{H} dX). \end{aligned}$$

From the special choice of the Hamiltonian \mathcal{H} we observe that the expressions $\partial_\alpha^A \mathcal{L}$ and $\partial_\alpha^A \mathcal{H}$ correspond (apart from the sign), and therefore we also have

$$\begin{aligned} \partial_w^X \mathcal{H} &= Q_x, & \partial_w^Y \mathcal{H} &= Q_y \\ \partial_\psi^X \mathcal{H} &= M_x, & \partial_\psi^Y \mathcal{H} &= M_{xy} \\ \partial_\phi^X \mathcal{H} &= M_{xy}, & \partial_\phi^Y \mathcal{H} &= M_y \end{aligned}$$

which consequently leads to the power balance relation

$$\begin{aligned} \dot{H} &= \int_{\partial\mathcal{D}_H} (\dot{w}Q_x + \dot{\psi}M_x + \dot{\phi}M_{xy})dY \\ &\quad - \int_{\partial\mathcal{D}_H} (\dot{w}Q_y + \dot{\psi}M_{yx} + \dot{\phi}M_y)dX \quad (21) \end{aligned}$$

which is based on (12).

Remark 6: It should be noted that depending on the boundary conditions, along $\partial\mathcal{D}_H$ a boundary port appears only if in the pairings $\dot{x}^\alpha \partial_\alpha^A \mathcal{H}$ both 'players' are not equal to zero. Furthermore it should be noted that in \dot{x}^α only $(\dot{w}, \dot{\psi}, \dot{\phi})$ remain, since in \mathcal{H} only jet variables with respect to w, ψ, ϕ appear, i.e. there is a w_X present but no $(p_w)_X$ and so on.

2) *The Stokes-Dirac approach:* This approach does not make use on an underlying bundle structure explicitly, which distinguishes dependent and independent variables strictly, but uses so-called energy variables instead. Therefore, let us introduce the strain variables as in [6]

$$\begin{aligned} \Gamma_x &= -\psi_X \\ \Gamma_y &= -\phi_Y \\ \Gamma_{xy} &= -(\psi_Y + \phi_X) \\ \Gamma_{xz} &= w_X - \psi \\ \Gamma_{yz} &= w_Y - \phi. \end{aligned} \quad (22)$$

Then one can introduce the state χ which consists of the momentum variables and the strains

$$\chi = (\rho h \dot{w}, \Gamma_{xz}, \Gamma_{yz}, \rho \frac{h^3}{12} \dot{\psi}, \rho \frac{h^3}{12} \dot{\phi}, \Gamma_x, \Gamma_y, \Gamma_{xy})$$

and the Hamiltonian $\mathcal{H} = \mathcal{K} + \mathcal{V}$ can be rewritten as

$$\mathcal{K} = \frac{\rho}{2} \left(\frac{h^3}{12} (\dot{\psi}^2 + \dot{\phi}^2) + h \dot{w}^2 \right)$$

and

$$\mathcal{V} = \frac{1}{2} (Q_x \Gamma_{xz} + Q_y \Gamma_{yz} - M_{xy} \Gamma_{xy} - M_x \Gamma_x - M_y \Gamma_y)$$

such that $e = \partial_\chi \mathcal{H}$ follows to

$$e = (\dot{w}, Q_x, Q_y, \dot{\psi}, \dot{\phi}, -M_x, -M_y, -M_{xy}).$$

The partial differential equations can be stated as

$$\dot{\chi} = \mathcal{J}_{SD} e = \mathcal{J}_{SD} \partial_\chi \mathcal{H} \quad (23)$$

where \mathcal{J}_{SD} takes the form

$$\begin{bmatrix} 0 & \partial_X & \partial_Y & 0 & 0 & 0 & 0 & 0 \\ \partial_X & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \partial_Y & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\partial_X & 0 & -\partial_Y \\ 0 & 0 & 1 & 0 & 0 & 0 & -\partial_Y & -\partial_X \\ 0 & 0 & 0 & -\partial_X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_Y & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_Y & -\partial_X & 0 & 0 & 0 \end{bmatrix}.$$

The energy balance follows from equation (15) where (13) has to be evaluated by a further integration by parts, since \mathcal{J}_{SD} is a differential operator which contributes to the boundary expression, i.e. $\mathcal{B}_{\mathcal{J}_{SD}}$ has to be constructed, see [6].

VII. DISCUSSION AND COMPARISON

In this section we will discuss the main differences of the two presented port-Hamiltonian scenarios as in sections (V-A) and (V-B) where we will highlight these aspects by focusing on the presented example, the Mindlin plate.

A. State variables

The state variables in the geometric approach consist of the displacements/deflections and the temporal momenta, i.e. $x = (w, \psi, \phi, p_w, p_\psi, p_\phi)$ in our example where in many cases (mechanical systems) the temporal momenta can be derived from a given Lagrangian by means of the Legendre transformation. These temporal momenta are introduced mainly to obtain explicit partial differential equations where the state variables are differentiated with respect to a curve parameter, which is the time.

In contrast to this, in the approach based on the Stokes-Dirac structures, energy variables are used, such that the strains are introduced in mechanical applications and one has $\chi = (\rho h \dot{w}, \Gamma_{xz}, \Gamma_{yz}, \rho \frac{h^3}{12} \dot{\psi}, \rho \frac{h^3}{12} \dot{\phi}, \Gamma_x, \Gamma_y, \Gamma_{xy})$. From a conceptual point of view the use of energy/power variables may be beneficial since they are linked to the power balance relation in a simple manner, but as can be seen already using the Mindlin plate example, the five strain variables have to be

derived from the three independent deflection/displacement variables (w, ψ, ϕ) by differentiation, see (22), such that additionally to the partial differential equations (23) also the compatibility conditions (22) must be listed, such that a constrained Hamiltonian representation is apparent.

Thus, since for mechanical applications the configuration plays a central role, the choice of coordinates including displacements/deflections seems advantageous, this however is different e.g. in electrodynamics where in the Maxwell equations it is appropriate to deal with the fields (electrical/magnetic) instead of the vector potential.

B. Control issues

From a control point of view control methods like damping injection or control by interconnection can be performed equivalently using the two presented Hamiltonian representations, see for example [14], [5] where the Timoshenko beam is analyzed. However as stated also above, the use of energy variables allows for controlling the system for instance to zero strain configuration, but the global position in space cannot be controlled in a straightforward manner since the deflection/displacement coordinates do not enter the formalism, in contrast to the approach as in (V-A).

C. The skew-symmetric operators \mathcal{J} and \mathcal{J}_{SD}

By inspection it becomes apparent that \mathcal{J} and \mathcal{J}_{SD} differ significantly, since \mathcal{J} is no differential operator in contrast to \mathcal{J}_{SD} . This also has severe consequences for the expressions $\delta\mathfrak{H}$ as in (10) and $e = \partial_X \mathcal{H}$ as in (14), where it is vice versa, i.e. δ is a variational derivative and in the Stokes-Dirac approach a partial derivative appears. Let us consider for instance the fourth equation of (23) which reads as

$$\rho \frac{h^3}{12} \ddot{\psi} = 1 \cdot Q_x - \partial_X(-M_x) - \partial_Y(-M_{xy})$$

as well as the fifth equation of (20) which is

$$\dot{p}_\psi = -\delta_\psi \mathcal{H} = -(\partial_\psi - d_X \partial_\psi^X - d_Y \partial_\psi^Y) \mathcal{H}.$$

From $\partial_\psi \mathcal{H} = -Q_x$ as well as from $\partial_\psi^X \mathcal{H} = M_x$, $\partial_\psi^Y \mathcal{H} = M_{xy}$ we easily observe that a part of the variational derivative $\delta_\psi = \partial_\psi - d_X \partial_\psi^X - d_Y \partial_\psi^Y$ is incorporated in \mathcal{J}_{SD} (namely $(1, -\partial_X, -\partial_Y)$) and that the variables Q_x , M_x and M_{xy} are used directly.

D. The energy balances

From the relation (12) the power balance is derived easily once the state x as well as the Hamiltonian density \mathcal{H} is chosen - this is very simple, since \mathcal{J} is no differential operator and does not contribute to the boundary term. This is different in the approach as in section (V-B) since the boundary operator $\mathcal{B}_{\mathcal{J}_{SD}}$ has to be derived from the special choice of \mathcal{J}_{SD} and by an additional integration by parts one ends up again by the same relation as in (21).

VIII. CONCLUSION

We have presented two different port-Hamiltonian representations based on a given set of partial differential equations together with their boundary conditions derived by the Lagrangian formalism using jet-bundles. By means of the running example, the Mindlin plate, we have extensively discussed and compared these two different Hamiltonian scenarios. Further investigations should also include the field theoretic Hamiltonian concepts coming from mathematical physics, like the polysymplectic and/or the multisymplectic approach as in [7], [8] or in the spirit as in [17].

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