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Robust Control of Singular Systems with Time Delay. Part I: Continuous Time Case

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***Abstract*— In this paper for the class of continuous time nonlinear uncertain singular time delay dynamical systems we present robust stability analyze results. Next, we consider a control problem for nonlinear continuous time uncertain singular time delay dynamical systems involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic cost criterion over a prescribed uncertainty set. Further we specialize result to affine uncertain systems to obtain controllers predicated on an *inverse optimal control problem*. In particular, to avoid the complexity in solving the steady-state Hamilton-Jacobi-Bellman equation we parameterize a family of stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the continuous time nonlinear singular time delay system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. By varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing controllers that can meet the closed-loop system response constraints. Obtained results for nonlinear case are further specialized to continuous time linear singular time delay dynamical systems.**

I. INTRODUCTION

For the class of continuous time nonlinear uncertain singular time delay dynamical systems, in this paper we give robust stability analyze results. For that purpose, we specialize robust stability results developed in Haddad, Chellaboina, and Kablar (2001b). At first, for the class of continuous time nonlinear uncertain singular time delay dynamical systems, given a performance functional, we develop sufficient conditions for robust stability. Next, for the dynamics of the system written in form of the nominal dynamics plus perturbation, we derive robust stability results. Then, we specialize results to the continuous time linear uncertain singular time delay case.

Finally, in this paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and let $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x, and $\partial\mathcal{S}, \dot{\mathcal{S}}, \bar{\mathcal{S}}$ denote the boundary, the interior, and a closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Finally, let C^0 denote the set of continuous functions and C^{r} denote the set of functions with r continuous derivatives.

II. ROBUST STABILITY ANALYSIS OF CONTINUOUS TIME NONLINEAR UNCERTAIN SINGULAR TIME DELAY DYNAMICAL SYSTEMS

In this section we present sufficient conditions for robust stability for a class of continuous time nonlinear uncertain singular time delay dynamical systems. We consider the problem of evaluating a performance bound for a nonlinear-nonquadratic cost functional depending upon a class of continuous time nonlinear uncertain singular time delay dynamical systems. It turns out that the cost bound can be evaluated in closed-form as long as the cost functional is related in a specific way to an underlying Lyapunov function that guarantees robust stability over a prescribed uncertainty set. Here, we restrict our attention to continuous time nonlinear state-dependent singular time delay dynamical system Kablar (2003a) $\mathcal{G}$ given by

$$E_{\mathrm{c}}\dot{x}(t) = f_{\mathrm{c}}(x(t),\tau), \quad x(0) = x_0, \tag{1}$$

where $t \geq 0$, $\tau \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, t_k denotes the k^{th} instant of time at which $x(t)$ intersects $\mathcal{Z}$, $f_{\mathrm{c}} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0,0) = 0$. Matrix E_{c}, may be singular matrix. From this we can explore different models:

In case $E_{\mathrm{c}} = I$, (1) represent standard dynamical systems with time delay. Therefore, theory of the singular time delay dynamical systems can be viewed as a generalization of the classical dynamical system with time delay theory. In case that time delay is absent, it specialized to classical dynamical system theory.

We refer to the differential equation (1) as the *continuous-time singular time delay dynamics*. Furthermore, $\mathcal{F}_{\mathrm{c}}$ denote the class of continuous time nonlinear uncertain singular time delay dynamical systems with $f_{\mathrm{c}0}(\cdot) \in \mathcal{F}_{\mathrm{c}}$ defining the nominal continuous time nonlinear singular time delay dynamical system for the continuous-time and the resetting dynamics, respectively. Note that since the resetting set $\mathcal{Z}$ is a subset of the state space $\mathcal{D}$ and is independent of time, state-dependent continuous time singular time delay dynamical systems are time-invariant. Here we assume that existence

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and uniqueness properties of a given state-dependent continuous time singular time delay dynamical systems are satisfied in forward time. For details see Lakshmikantham, Bainov, and Simeonov (1989).

For the following result let $L_{\rm c}\mathcal{D} \times \mathbb{R} \to \mathbb{R}$, and let $\mathcal{F}_{\rm c} \subset \{f_{\rm c} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^n : f_{\rm c}(0,0) = 0\}$ denote the class of continuous time nonlinear uncertain singular time delay dynamical systems with $f_{\rm c0}(\cdot,\cdot) \in \mathcal{F}_{\rm c}$ defining the nominal continuous time nonlinear singular time delay dynamical system. For the following result and a remainder of the chapter we denote the resetting times $\tau_k(x_0)$ by t_k, and for simplicity of exposition, we also define $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}_{\rm c} = \mathcal{F}$. Within the context of robustness analysis, it is assumed that the zero solution $x(t) \equiv 0$ to the nominal continuous time nonlinear singular time delay dynamical system (1) is asymptotically stable. Furthermore, we assume that an infinite number of resetting occurs.

Theorem 2.1: Consider the continuous time nonlinear uncertain singular time delay dynamical system $\mathcal{G}$ given by (1) where $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}$, with performance functional

$$J(E_{\rm c}x_0,\tau) = \int_0^\infty L_{\rm c}(E_{\rm c}x(t),\tau){\rm d}t. \tag{2}$$

Furthermore, assume there exist functions $\Gamma_{\rm c} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ and $V : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, where $V(\cdot,\cdot)$ is C^1 function, such that

$$V(0,0) = 0, \tag{3}$$

$$V(E_{\rm c}x,\tau) \geq 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{4}$$

$$V'(E_{\rm c}x,\tau)f_{\rm c}(x,\tau) \leq V'(E_{\rm c}x,\tau)f_{\rm c0}(x,\tau) + \Gamma_{\rm c}(x,\tau), \quad f_{\rm c}(\cdot,\cdot) \in \mathcal{F}_{\rm c}, \tag{5}$$

$$V'(E_{\rm c}x,\tau)f_{\rm c0}(x,\tau) + \Gamma_{\rm c}(x,\tau) < 0, \quad x \neq 0, \tag{6}$$

$$L_{\rm c}(E_{\rm c}x,\tau) + V'(E_{\rm c}x,\tau)f_{\rm c0}(x,\tau) + \Gamma_{\rm c}(x,\tau) = 0, \tag{7}$$

where $f_{\rm c0}(\cdot,\cdot) \in \mathcal{F}$ defines the nominal continuous time nonlinear singular time delay dynamical system. Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (1) is locally asymptotically stable for all $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}$, and

$$\sup_{f_{\rm c}(\cdot,\cdot)\in\mathcal{F}} J(E_{\rm c}x_0,\tau) \leq \mathcal{J}(E_{\rm c}x_0,\tau) = V(E_{\rm c}x_0,\tau), \tag{8}$$

where

$$\mathcal{J}(E_{\rm c}x_0,\tau) = \textstyle\int_0^\infty [L_{\rm c}(E_{\rm c}x(t),\tau) + \Gamma_{\rm c}(x(t),\tau)]{\rm d}t \tag{9}$$

and where $x(t)$, $t \geq 0$, is a solution to (1) with $f_{\rm c}(x(t),\tau) = f_{\rm c0}(x(t),\tau)$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and

$$V(E_{\rm c}x,\tau) \to \infty \quad \text{as} \quad \|x\| \to \infty, \tag{10}$$

then the zero solution $x(t) \equiv 0$ to (1), is globally asymptotically stable for all $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}$, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: Let $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}$ and $x(t)$, satisfies (1). Then,

$$\dot{V}(E_{\rm c}x(t),\tau) = \tfrac{\rm d}{{\rm d}t}V(E_{\rm c}x(t),\tau) = V'(E_{\rm c}x(t),\tau)f_{\rm c}(x(t),\tau), \quad t_k < t \leq t_{k+1}. \tag{11}$$

Hence, it follows from (5) and (6) that

$$\dot{V}(E_{\rm c}x(t),\tau) < 0,\ x(t) \neq 0,\ t_k < t \leq t_{k+1}. \tag{12}$$

Thus, from (3), (12), it follows from Chapter 2, Theorem 2.3.2 (and Theorem 3.2 of Kablar (2003b) [4] that $V(\cdot,\cdot)$ is Lyapunov function for (1) which proves local asymptotic stability of the zero solution $x(t) \equiv 0$ for all $f_{\rm c}(\cdot,\cdot) \in \mathcal{F}$. Consequently, $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x_0 \in \mathcal{D}_0$ for some neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin. Now, (11) imply that

$$0 = -\dot{V}(E_{\rm c}x(t),\tau) + V'(E_{\rm c}x(t),\tau)f_{\rm c}(x(t),\tau), \quad t_k < t \leq t_{k+1} \tag{13}$$

From (13), using (5) and (7),

$$\begin{aligned}
&L_{\rm c}(x(t),\tau) = \\
&\quad -\dot{V}(E_{\rm c}x(t),\tau) + L_{\rm c}(E_{\rm c}x(t),\tau) \\
&+V'(E_{\rm c}x(t),\tau)f_{\rm c}(x(t),\tau) \\
&\quad \leq -\dot{V}(E_{\rm c}x(t),\tau) + L_{\rm c}(E_{\rm c}x(t),\tau) \\
&\quad +V'(E_{\rm c}x(t),\tau)f_{\rm c0}(x(t),\tau) + \Gamma_{\rm c}(x(t),\tau) \\
&\quad = -\dot{V}(E_{\rm c}x(t),\tau), \quad t_k < t \leq t_{k+1}.
\end{aligned} \tag{14}$$

Now, integrating over the interval $[0,t)$ with $\mathcal{N}_{[0,t)} = \{1,2,\ldots,i,\ldots,j\}$, (14) yield

$$\begin{aligned}
&\int_0^t L_{\rm c}(E_{\rm c}x(s),\tau){\rm d}s = \\
&\quad \int_0^{t_i} L_{\rm c}(E_{\rm c}x(s),\tau){\rm d}s \\
&\quad + \int_{t_i^+}^{t_{i+1}} L_{\rm c}(E_{\rm c}x(s),\tau){\rm d}s \\
&\quad + \ldots + \int_{t_{j-1}^+}^{t_j} L_{\rm c}(E_{\rm c}x(s),\tau){\rm d}s + \int_{t_{j+}}^{t} L_{\rm c}(E_{\rm c}x(s),\tau){\rm d}s \\
&\quad \leq -V(E_{\rm c}x(t_i),\tau) + V(E_{\rm c}x_0,\tau) \\
&\quad +V(E_{\rm c}x(t_i),\tau) \\
&\quad +V(E_{\rm c}x(t_{i+1}),\tau) + \ldots - V(E_{\rm c}x(t_j),\tau) \\
&\quad -V(E_{\rm c}x(t),\tau) \\
&\quad \leq -V(E_{\rm c}x(t_i),\tau) + V(E_{\rm c}x_0,\tau) + V(E_{\rm c}x(t_i),\tau) \\
&\quad -V(E_{\rm c}x(t_{i+1}),\tau) + V(E_{\rm c}x(t_{i+1}),\tau) \\
&\quad + \ldots - V(E_{\rm c}x(t_j),\tau) \\
&\quad +V(E_{\rm c}x(t_j),\tau) - V(E_{\rm c}x(t),\tau) \\
&\quad \leq -V(E_{\rm c}x(t),\tau) + V(E_{\rm c}x_0,\tau).
\end{aligned} \tag{15}$$

Letting $t \to \infty$ and noting that $V(E_{\rm c}x(t),\tau) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $J_{f_{\rm c}((x_0),\tau)} \leq V(E_{\rm c}x_0,\tau)$. Next, let $x(t)$, $t \geq 0$, satisfy (1), with $f_{\rm c}(x(t),\tau) = f_{\rm c0}(x(t),\tau)$. Now, from (13), using (7),

$$\begin{aligned}
L_{\rm c}(E_{\rm c}x(t),\tau) + \Gamma_{\rm c}(x(t),\tau) \;&=\; -\dot{V}(E_{\rm c}x(t),\tau) \\
&+L_{\rm c}(E_{\rm c}x(t),\tau) + V'(E_{\rm c}x(t),\tau)f_{\rm c0}(x(t),\tau) + \Gamma_{\rm c}(x(t),\tau) \\
&= -\dot{V}(E_{\rm c}x(t),\tau), \quad t_k < t \leq t_{k+1}.
\end{aligned} \tag{16}$$

Now, integrating over the interval $[0,t)$ (16) yield

$$\begin{aligned}
&\textstyle\int_0^t [L_{\rm c}(E_{\rm c}x(t),\tau) + \Gamma_{\rm c}(x(t),\tau)]{\rm d}t] \\
&= -V(E_{\rm c}x(t),\tau) + V(E_{\rm c}x_0,\tau).
\end{aligned} \tag{17}$$

Letting $t \to \infty$ and noting that $V(E_{\mathrm{c}}x(t),\tau) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields $\mathcal{J}(E_{\mathrm{c}}x_0,\tau) = V(E_{\mathrm{c}}x_0,\tau)$. Finally, for $\mathcal{D} = \mathbb{R}^n$ and for all $f_{\mathrm{c}}(\cdot,\cdot)$, global asymptotic stability of the zero solution $x(t) \equiv 0$ to (1) is a direct consequence of Chapter 2, Theorem 2.3.2 (Theorem 3.2 of Kablar (2003b)) using radially unbounded condition (10) on $V(E_{\mathrm{c}}x,\tau)$, $x \in \mathbb{R}^n$. ■

Remark 2.1: Theorem 2.1 provides sufficient conditions for robust stability of a class of continuous time nonlinear uncertain singular time delay dynamical systems $f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$. Specifically, (3) assume that $V(x,\tau)$ is a Lyapunov function candidate for the continuous time nonlinear uncertain singular time delay dynamical system (1). Conditions (5), (6) and (**??**) imply $\dot{V}(E_{\mathrm{c}}x(t),\tau) < 0$, $t > 0$, for $x(\cdot)$ satisfying (1) for all $f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$, and hence $V(\cdot)$ is a Lyapunov function guaranteeing robust stability of the continuous time nonlinear uncertain singular time delay dynamical system (1). It is important to note that Conditions (6) are *verifiable* conditions since they are independent of the uncertain system parameters $f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$. To apply Theorem 2.1 we specify the bounding functions $\Gamma_{\mathrm{c}}(\cdot,\cdot)$ for the uncertain set $\mathcal{F}_{\mathrm{c}}$ such that $\Gamma_{\mathrm{c}}(\cdot,\cdot)$ abound $\mathcal{F}_{\mathrm{c}}$. If $\mathcal{F}$ consists only of the nominal continuous time nonlinear singular time delay dynamical system $f_{\mathrm{c}0}(\cdot,\cdot)$, then $\Gamma_{\mathrm{c}}(x,\tau) = 0$ for all $x \in \mathcal{D}$ satisfy (5), respectively, and hence $J_{(f_{\mathrm{c}0},\tau)} = \mathcal{J}(E_{\mathrm{c}}x_0,\tau)$. Finally, a worst-case upper bound to the nonlinear-nonquadratic performance functional is given in terms of a Lyapunov function which can be interpreted in terms of an auxiliary cost defined for the nominal continuous time singular time delay dynamical system.

Next, we specialize Theorem 2.1 to continuous time nonlinear uncertain singular time delay dynamical systems of the form

$$E_{\mathrm{c}}\dot{x}(t) \;=\; f_{\mathrm{c}0}(x(t),\tau) + \Delta f_{\mathrm{c}}(x(t),\tau),\; x(0) = x_0, \quad (18)$$

where $t \geq 0$, $f_{\mathrm{c}0} : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f_{\mathrm{c}0}(0,0) = 0$ and $\Delta f_{\mathrm{c}} \in \mathcal{F}_{\mathrm{c}} = \mathcal{F}$, where

$$\mathcal{F}_{\mathrm{c}} \subset \{\Delta f_{\mathrm{c}} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^n : \Delta f_{\mathrm{c}}(0,0) = 0\}, \qquad (19)$$

Corollary 2.1: Consider the continuous time nonlinear uncertain singular time delay dynamical system (18) with performance functional (2). Furthermore, assume there exists functions $\Gamma_{\mathrm{c}} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, and $V : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, where $V(\cdot,\cdot)$ is a C^1 function, such that (3), hold and

$$V'(E_{\mathrm{c}}x,\tau)\Delta f_{\mathrm{c}}(x,\tau) \leq \Gamma_{\mathrm{c}}(x,\tau),\; \Delta f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}_{\mathrm{c}}, \qquad (20)$$

$$V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) < 0,\; x \neq 0, \qquad (21)$$

$$L_{\mathrm{c}}(E_{\mathrm{c}}x,\tau) + V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) = 0, \qquad (22)$$

Then there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ of the origin such that if $x_0 \in \mathcal{D}_0$, then the zero solution $x(t) \equiv 0$ to (18) is locally asymptotically stable for all $\Delta f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$, and the performance functional (2) satisfies

$$\sup_{\Delta f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}} J(E_{\mathrm{c}}x_0,\tau) \leq \mathcal{J}(E_{\mathrm{c}}x_0,\tau) = V(E_{\mathrm{c}}x_0,\tau), \quad (23)$$

where

$$\mathcal{J}(E_{\mathrm{c}}x_0,\tau) \;=\; \int_0^\infty [L_{\mathrm{c}}(E_{\mathrm{c}}x(t),\tau) + \Gamma_{\mathrm{c}}(x(t),\tau)]\mathrm{d} \qquad (24)$$

and where $x(t)$, $t \geq 0$, is a solution to (18), with $\Delta f_{\mathrm{c}}(x(t),\tau) = 0$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(E_{\mathrm{c}}x)$, $x \in \mathbb{R}^n$, satisfies (10), then the zero solution $x(t) \equiv 0$ to (18) is globally asymptotically stable for all $\Delta f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: The result is direct consequence of Theorem 2.1 with $f_{\mathrm{c}}(x,\tau) = f_{\mathrm{c}0}(x,\tau) + \Delta f_{\mathrm{c}}(x,\tau)$. Specifically, in this case it follows from (20) and (21) that $V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}}(x,\tau) \leq V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) < 0$ for all $x \neq 0$, and $\Delta f_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}_{\mathrm{c}}$. Hence, all the conditions of Theorem 2.1 are satisfied. ■

The following corollary specializes Theorem 2.1 to a class of continuous time linear uncertain singular time delay dynamical systems. Specifically, we consider $\mathcal{F} = \mathcal{F}_{\mathrm{c}}$ to be the set of continuous time linear uncertain singular time delay dynamical systems, with

$$\begin{aligned} &A_{\mathrm{c}}x(t) + A_{\mathrm{d}}x(t-\tau)\mathcal{F}_{\mathrm{c}} = \{(A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) \\ &+(A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-\tau) : x \in \mathbb{R}^n, A_{\mathrm{c}} \in \mathbb{R}^{n\times n}, \\ &\Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{A_{\mathrm{c}}}, \Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{A_{\mathrm{d}}}, \end{aligned} \qquad (25)$$

where $\mathbf{\Delta}_{A_{\mathrm{c}}} \subset \mathbb{R}^{n\times n}$, $\mathbf{\Delta}_{A_{\mathrm{d}}} \subset \mathbb{R}^{n\times n}$ are given bounded uncertainty sets of uncertain perturbations ΔA_{c}, ΔA_{d} of the nominal asymptotically stable system matrices A_{c}, A_{d} such that $0 \in \mathbf{\Delta}_{A_{\mathrm{c}}}$, $0 \in \mathbf{\Delta}_{A_{\mathrm{d}}}$. For simplicity of exposition, we also define $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{Ac} = \mathbf{\Delta}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{Ad} = \mathbf{\Delta}$.

Corollary 2.2: Let $R_{\mathrm{c}} \in \mathbb{P}^n$, $R_{\mathrm{d}} \in \mathbb{P}^n$. Consider the continuous time linear state-dependent uncertain singular time delay dynamical system

$$\begin{aligned} E_{\mathrm{c}}\dot{x}(t) = (A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-\tau), \\ x(0) = x_0,\; t \geq 0, \end{aligned} \qquad (26)$$

with performance functional

$$J_{\Delta A_{\mathrm{c}},\Delta A_{\mathrm{d}}\tau}(E_{\mathrm{c}}x_0) \;=\; \int_0^\infty x^{\mathrm{T}}(t)E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}E_{\mathrm{c}}x(t)\mathrm{d}t \qquad (27)$$

where $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}$. Let $\Omega_{\mathrm{c}} : \mathcal{N}_{\mathrm{P}} \subseteq \mathbb{S}^n \to \mathbb{N}^n$, $P \in \mathcal{N}_{\mathrm{P}}$, be such that

$$\begin{aligned} &x^{\mathrm{T}}(t)(\Delta A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}P\Delta A_{\mathrm{c}})x(t) + x^{\mathrm{T}}(t-\tau)(\Delta A_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{c}} \\ &+E_{\mathrm{c}}^{\mathrm{T}}P\Delta A_{\mathrm{d}})x(t-\tau) \leq x(t)^{\mathrm{T}}E_{\mathrm{c}}^{\mathrm{T}}\Omega_{\mathrm{c}}(P)E_{\mathrm{c}}x(t) \\ &+x(t-\tau)^{\mathrm{T}}E_{\mathrm{c}}^{\mathrm{T}}\Omega_{\mathrm{d}}(P)E_{\mathrm{c}}x(t-\tau), \\ &\Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{Ac}, \Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{Ad} \end{aligned} \qquad (28)$$

Furthermore, suppose there exist $P \in \mathbb{P}^n$ satisfying

$$0 \;=\; x^{\mathrm{T}}(t)(A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}} + \Omega_{\mathrm{c}}(P) + E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}})E_{\mathrm{c}}x(t), \qquad (29)$$

$$0 \;=\; x^{\mathrm{T}}(t-\tau)(A_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}}(P) + E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{d}})E_{\mathrm{c}}x(t-\tau), \qquad (30)$$

Then the zero solution $x(t) \equiv 0$ to (26), is globally asymptotically stable for all $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}$,and the performance functional (**??**) satisfies

$$\begin{aligned} \sup\nolimits_{\Delta A_{\mathrm{c}} \in \mathbf{\Delta}} J_{\Delta A_{\mathrm{c}}}(E_{\mathrm{c}}x_0, \tau) &\leq \mathcal{J}(E_{\mathrm{c}}x_0, \tau) \\ &= x_0^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x_0, x_0 \in \mathbb{R}^n, \end{aligned} \tag{31}$$

$$\begin{aligned} \sup\nolimits_{\Delta A_{\mathrm{d}} \in \mathbf{\Delta}} J_{\Delta A_{\mathrm{d}}}(E_{\mathrm{c}}x_0, \tau) &\leq \mathcal{J}(E_{\mathrm{c}}x_0, \tau) \\ &= x_0^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x_0, x_0 \in \mathbb{R}^n, \end{aligned} \tag{32}$$

where

$$\mathcal{J}(E_{\mathrm{c}}x_0, \tau) = \int_0^\infty x^{\mathrm{T}}(t)(\Omega_{\mathrm{c}}(P) + E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}} E_{\mathrm{c}}) x(t) \mathrm{d}t \tag{33}$$

$$\mathcal{J}(E_{\mathrm{c}}x_0, \tau) = \int_0^\infty x^{\mathrm{T}}(t-\tau)(\Omega_{\mathrm{d}}(P) + E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{d}} E_{\mathrm{c}}) x(t-\tau) \mathrm{d}t \tag{34}$$

and where $x(t)$, $t \geq 0$, is a solution to (26), with $\Delta A_{\mathrm{c}} = 0$, $\Delta A_{\mathrm{d}} = 0$, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: The result is direct consequence of Theorem 2.1 with $f_{\mathrm{c}}(x,\tau) = (A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-\tau)$, $f_{\mathrm{c}0}(x,\tau) = A_{\mathrm{c}}x(t) + A_{\mathrm{d}}x(t-\tau)$, $L_{\mathrm{c}}(E_{\mathrm{c}}x,\tau) = x^{\mathrm{T}}(t)E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}E_{\mathrm{c}}x(t) + x(t-\tau)^{\mathrm{T}}E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}E_{\mathrm{c}}x(t-\tau)$, $\Gamma_{\mathrm{c}}(x,\tau) = x^{\mathrm{T}}\Omega_{\mathrm{c}}(P)x + x(t-\tau)^{\mathrm{T}}\Omega_{\mathrm{d}}(P)x(t-\tau)$, $V(E_{\mathrm{c}}x) = x^{\mathrm{T}}E_{\mathrm{c/d}}^{\mathrm{T}}PE_{\mathrm{c/d}}x$, with arguments $E_{\mathrm{c}}x$ and $E_{\mathrm{d}}x$, and $\mathcal{D} = \mathbb{R}^n$. Specifically, conditions (3) and are trivially satisfied. Now, for the argument $E_{\mathrm{c}}x$, $V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}}(x,\tau) = x^{\mathrm{T}}(t)(A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}})x(t) + x^{\mathrm{T}}(t)(\Delta A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}P\Delta A_{\mathrm{c}})x(t) + x^{\mathrm{T}}(t)(A_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{d}})x(t) + x^{\mathrm{T}}(t-\tau)(\Delta A_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}P\Delta A_{\mathrm{d}})x(t-\tau)$, for all $x \neq 0$, and $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{Ac}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{Ad}$, and hence it follows from (28) that $V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}}(x,\tau) \leq V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) = x^{\mathrm{T}}(t)(A_{\mathrm{c}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{c}} + \Omega_{\mathrm{c}}(P))x(t) + x^{\mathrm{T}}(t-\tau)(A_{\mathrm{d}}^{\mathrm{T}}PE_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}}PA_{\mathrm{d}} + \Omega_{\mathrm{d}}(P))x(t-\tau)$, for all $x \neq 0$, $x \notin \mathcal{Z}$. Furthermore, it follows from (29, 30) that $L_{\mathrm{c}}(E_{\mathrm{c}}x,\tau) + V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) = 0$, $x \notin \mathcal{Z}$ and hence $V'(E_{\mathrm{c}}x,\tau)f_{\mathrm{c}0}(x,\tau) + \Gamma_{\mathrm{c}}(x,\tau) < 0$, for all $x \neq 0$. Finally, since $V(E_{\mathrm{c}}x,\tau)$, $x \in \mathbb{R}^n$, is radially unbounded, (26), is globally asymptotically stable for all $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}$. ■

III. OPTIMAL ROBUST CONTROL FOR NONLINEAR UNCERTAIN SINGULAR CONTINUOUS TIME DYNAMICAL SYSTEMS

In this section we consider a control problem for continuous time nonlinear uncertain singular time delay dynamical systems involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic cost criterion over a prescribed uncertainty set. The optimal robust time-invariant feedback controllers are derived as a direct consequence of Theorem 2.1 and provide a generalization of the Hamilton-Jacobi-Bellman conditions for continuous time state-dependent singular time delay dynamical systems with optimality notions over the infinite horizon, for addressing robust feedback controllers of continuous time nonlinear uncertain singular time delay dynamical systems. To address robust optimal control problem let $\mathcal{D} \subset \mathbb{R}^n$ be an open set with $0 \in \mathcal{D}$, and let $\mathcal{C}_{\mathrm{c}} \subset \mathbb{R}^{m_{\mathrm{c}}}$, where $0 \in \mathcal{C}_{\mathrm{c}}$. Furthermore, let $\mathcal{F}_{\mathrm{c}} \subset \{F_{\mathrm{c}} : \mathcal{D} \times \mathcal{C}_{\mathrm{c}} \to \mathbb{R}^n : F_{\mathrm{c}}(0,0) = 0\}$. For simplicity of exposition, we also define $F_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}_{\mathrm{c}} \triangleq \mathcal{F}$. Next, consider the continuous time nonlinear uncertain singular time delay controlled dynamical system

$$\begin{aligned} E_{\mathrm{c}}\dot{x}(t) &= F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), \tau), \quad x(0) = 0, \\ & \quad u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}}, \end{aligned} \tag{35}$$

where $t \geq 0$, $\tau \geq 0$, $x(t) \in \mathcal{D}$ is the state vector, $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}} \subset \mathcal{C}_{\mathrm{c}}$, is the control input, where the control constraint sets $\mathcal{U}_{\mathrm{c}}$ are given. We assume $0 \in \mathcal{U}_{\mathrm{c}}$, $F_{\mathrm{c}} : \mathcal{D} \times \mathbb{R} \times \mathcal{U}_{\mathrm{c}} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_{\mathrm{c}}(0,0) = 0$. To address the robust optimal nonlinear feedback control problem let $\phi_{\mathrm{c}} : \mathcal{D} \to \mathcal{U}_{\mathrm{c}}$ be such that $\phi_{\mathrm{c}}(0,0) = 0$. If $u_{\mathrm{c}}(t) = \phi_{\mathrm{c}}(E_{\mathrm{c}}x(t),\tau)$, where $x(t)$, $t \geq 0$, satisfies (35) then $u_{\mathrm{c}}(\cdot)$ is a *feedback control*. Given the feedback control $u_{\mathrm{c}}(t) = \phi_{\mathrm{c}}(E_{\mathrm{c}}x(t),\tau)$, the closed-loop continuous time state-dependent singular time delay dynamical system has the form

$$\begin{aligned} E_{\mathrm{c}}\dot{x}(t) = F_{\mathrm{c}}(x(t), \phi_{\mathrm{c}}(E_{\mathrm{c}}x(t)), \tau), \qquad x(0) = x_0, \\ t \geq 0, \end{aligned} \tag{36}$$

for all $F_{\mathrm{c}}(\cdot,\cdot) \in \mathcal{F}$.

Next we present sufficient conditions for characterizing robust nonlinear feedback controllers that guarantee robust stability over a class of continuous time nonlinear uncertain singular time delay dynamical systems and minimize an auxiliary performance functional. For the statement of this result let $L_{\mathrm{c}} : \mathcal{D} \times \mathcal{U}_{\mathrm{c}} \times \mathbb{R} \to \mathbb{R}$, and define the set of asymptotically stabilizing controllers for the nominal continuous time nonlinear singular time delay dynamical system $(F_{\mathrm{c}0}(\cdot,\cdot)$ by

$$\begin{aligned} \mathcal{C}(x_0) \triangleq \{ & u_{\mathrm{c}}(\cdot) : (u_{\mathrm{c}}(\cdot) \text{ is admissible} \\ & \text{and the zero solution} x(t) \equiv 0 \text{to } (35), \\ & \text{is asymptotically stable with} \\ & F_{\mathrm{c}}(\cdot,\cdot,\cdot) = F_{\mathrm{c}0}(\cdot,\cdot,\cdot)\}. \end{aligned} \tag{37}$$

Theorem 3.1: Consider the continuous time nonlinear uncertain singular time delay dynamical system (35) with performance functional

$$J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \tau) = \int_0^\infty L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u(t), \tau) \mathrm{d}t, \tag{38}$$

where $\mathcal{F}_{\mathrm{c}}(\cdot,\cdot,\cdot) \in \mathcal{F}$ and $u_{\mathrm{c}}(\cdot)$ is an admissible control. Assume there exist functions $V : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, $\Gamma_{\mathrm{c}} : \mathcal{D} \times \mathcal{U}_{\mathrm{c}} \times \mathbb{R} \to \mathbb{R}$, and a control law $\phi_{\mathrm{c}} : \mathcal{D} \times \mathbb{R} \to \mathcal{U}_{\mathrm{c}}$, where $V(\cdot,\cdot)$ is a C^1 function, such that

$$V(0,0) = 0, \tag{39}$$

$$V(E_{\mathrm{c}}x,\tau) \geq 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{40}$$

$$\phi_{\mathrm{c}}(0,0) = 0, \tag{41}$$

$$\begin{aligned} V'(E_{\mathrm{c}}x,\tau)F_{\mathrm{c}}(x,\phi_{\mathrm{c}}(x),\tau) \leq V'(E_{\mathrm{c}}x,\tau)F_{\mathrm{c}0}(x,\phi_{\mathrm{c}}(x),\tau) \\ +\Gamma_{\mathrm{c}}(x,\phi_{\mathrm{c}}(x),\tau),\ F_{\mathrm{c}}(\cdot,\cdot,\cdot) \in \mathcal{F}_{\mathrm{c}}, \end{aligned} \tag{42}$$

$$V'(E_{\mathrm{c}}x,\tau)F_{\mathrm{c}0}(x,\phi_{\mathrm{c}}(x),\tau) + \Gamma_{\mathrm{c}}(x,\phi_{\mathrm{c}}(x),\tau) < 0,\ x \neq 0, \tag{43}$$

$$H_{\mathrm{c}}(E_{\mathrm{c}}x, \phi_{\mathrm{c}}(x), \tau) = 0, \tag{44}$$

$$H_{\mathrm{c}}(E_{\mathrm{c}}x, u_{\mathrm{c}}(x), \tau) \geq 0, \quad u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}, \tag{45}$$

where $F_{\mathrm{c}0}(\cdot,\cdot,\cdot) \in \mathcal{F}$ defines the nominal continuous time singular time delay dynamical system and

$$H_{\mathrm{c}}(E_{\mathrm{c}}x, u_{\mathrm{c}}, \tau) \triangleq L_{\mathrm{c}}(E_{\mathrm{c}}x, u_{\mathrm{c}}, \tau) + V'(E_{\mathrm{c}}x, \tau)F_{\mathrm{c}0}(x, u_{\mathrm{c}}, \tau) + \Gamma_{\mathrm{c}}(x, u_{\mathrm{c}}, \tau), \tag{46}$$

Then, with the feedback control $u_{\mathrm{c}}(\cdot), = \phi_{\mathrm{c}}(E_{\mathrm{c}}x(\cdot), \tau)$, there exists a neighborhood of the origin $\mathcal{D}_0 \subset \mathcal{D}$ such that if $x_0 \in \mathcal{D}_0$, the zero solution $x(t) \equiv 0$ of the closed-loop system (36) is locally asymptotically stable for all $F_{\mathrm{c}}(\cdot,\cdot,\cdot) \in \mathcal{F}$. Furthermore,

$$\sup_{F_{\mathrm{c}}(\cdot,\cdot,\cdot)\in\mathcal{F}} J(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(E_{\mathrm{c}}x(\cdot)), \cdot) \leq \mathcal{J}(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(\cdot), \cdot) = V(E_{\mathrm{c}}x_0, \tau), \quad x_0 \in \mathcal{D}_0, \tag{47}$$

where

$$\mathcal{J}(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \tau) \triangleq \int_0^\infty [L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau) + \Gamma_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), \tau)]\mathrm{d}t \tag{48}$$

and where $u_{\mathrm{c}}(\cdot)$ is an admissible control and $x(t)$, $t \geq 0$, is a solution of (35) with $F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), \tau) = F_{\mathrm{c}0}(x(t), u_{\mathrm{c}}(t), \tau)$. In addition, if $x_0 \in \mathcal{D}_0$ then the feedback control $u_{\mathrm{c}}(\cdot) = (\phi_{\mathrm{c}}(E_{\mathrm{c}}x(\cdot), \tau)$, minimizes $J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \tau)$ in the sense that

$$J(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(E_{\mathrm{c}}x(\cdot), \tau) = \min_{(u_{\mathrm{c}}(\cdot))\in\mathcal{C}(x_0)} J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \tau). \tag{49}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, and

$$V(E_{\mathrm{c}}x, \tau) \to \infty \quad \text{as} \quad \|x\| \to \infty, \tag{50}$$

then the zero solution $x(t) \equiv 0$ of the closed-loop system (36) is globally asymptotically stable for all $F_{\mathrm{c}}(\cdot,\cdot,\cdot) \in \mathcal{F}$, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: Local and global asymptotic stability is a direct consequence of (39)–(43) by applying Theorem 2.1 to the closed-loop system (36). Furthermore, using (44) condition (47) is a restatement of (8) as applied to the closed-loop system (36). Next, let $u_{\mathrm{c}}(\cdot) \in \mathcal{C}(x_0)$ and let $x(\cdot)$ be the solution of (35) with $F_{\mathrm{c}}(\cdot,\cdot,\cdot) = F_{\mathrm{c}0}(\cdot,\cdot,\cdot)$.

Then it follows that

$$0 = -\dot{V}(E_{\mathrm{c}}x(t), \tau) + V'(E_{\mathrm{c}}x(t), \tau)F_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), \tau) \tag{51}$$

Hence,

$$\begin{aligned} L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau) + \Gamma_{\mathrm{c}}(E_{\mathrm{c}}\tilde{x}(t), u_{\mathrm{c}}(t), \tau) = \\ -\dot{V}(E_{\mathrm{c}}x(t), \tau) + L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau) \\ +V'(E_{\mathrm{c}}x(t), \tau)F_{\mathrm{c}0}(x(t), u_{\mathrm{c}}(t), \tau) \\ +\Gamma_{\mathrm{c}}(E_{\mathrm{c}}\tilde{x}(t), u_{\mathrm{c}}(t), \tau) \\ = -\dot{V}(E_{\mathrm{c}}x(t), \tau) + H_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau), \end{aligned} \tag{52}$$

Now, over the interval $[0, t)$ yields

$$\begin{aligned} &\int_0^t [L_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau) + \Gamma_{\mathrm{c}}(\tilde{x}(t), u_{\mathrm{c}}(t), \tau)]\mathrm{d}t \\ &= + \int_0^t [-\dot{V}(E_{\mathrm{c}}x(t), \tau) + H_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), \tau)]\mathrm{d}t \\ &= -V(E_{\mathrm{c}}x(t), \tau) + V(E_{\mathrm{c}}x_0, \tau) \\ &+ \int_0^t H_{\mathrm{c}}(E_{\mathrm{c}}x(t), u_{\mathrm{c}}(t), \tau)\mathrm{d}t \\ &\geq V(E_{\mathrm{c}}x_0, \tau) \\ &= \mathcal{J}(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(x(\cdot), \tau)). \end{aligned} \tag{53}$$

Letting $t \to \infty$ and noting that $V(E_{\mathrm{c}}x(t), \tau) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields (49). ■

Remark 3.1: If $\mathcal{F}$ consists of only the nominal continuous time nonlinear singular time delay closed-loop system $F_{\mathrm{c}0}(\cdot,\cdot,\cdot)$, then $\Gamma_{\mathrm{c}}(x, u_{\mathrm{c}}, \cdot) = 0$ for all $x \in \mathcal{D}$, $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$ satisfy (42) and hence $J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \tau) = \mathcal{J}(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot), \cdot)$. In this case Theorem 3.1 specializes to Chapter 4, Theorem 3.2.3.

Remark 3.2: Theorem 3.1 guarantees optimality with respect to the set of admissible stabilizing controllers $\mathcal{C}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{C}(x_0)$ is not required. In addition, the optimal robustly stabilizing feedback control law $u_{\mathrm{c}} = \phi_{\mathrm{c}}(x)$ is independent of the initial condition x_0.

Next, we specialize Theorem 3.1 to continuous time linear uncertain singular time delay dynamical systems. Specifically, in this case we consider $\mathcal{F} \triangleq \mathcal{F}_{\mathrm{c}}$ to be the set of uncertain continuous time linear singular time delay dynamical systems, where

$$\begin{aligned} \mathcal{F}_{\mathrm{c}} = \{&(A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t-\tau) \\ &+B_{\mathrm{c}}u_{\mathrm{c}}(t) : x \in \mathbb{R}^n, A_{\mathrm{c}} \in \mathbb{R}^{n\times n}, A_{\mathrm{d}} \in \mathbb{R}^{n\times n}, \\ &B_{\mathrm{c}} \in \mathbb{R}^{n\times m_{\mathrm{c}}}, \Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{A_{\mathrm{c}}}, \Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{A_{\mathrm{d}}}\}, \end{aligned} \tag{54}$$

where $\mathbf{\Delta}_{A_{\mathrm{c}}}, \subset \mathbb{R}^{n\times n}$, $\mathbf{\Delta}_{A_{\mathrm{d}}}, \subset \mathbb{R}^{n\times n}$, are given bounded uncertainty sets of uncertain perturbations ΔA_{c}, ΔA_{d} of the nominal system matrices A_{c}, A_{d}, such that $0 \in \mathbf{\Delta}_{A_{\mathrm{c}}}$, $0 \in \mathbf{\Delta}_{A_{\mathrm{d}}}$,. For simplicity of exposition, we also define $\Delta A_{\mathrm{c}} \in \mathbf{\Delta}_{Ac} \times \triangleq \mathbf{\Delta}$, $\Delta A_{\mathrm{d}} \in \mathbf{\Delta}_{Ad} \times \triangleq \mathbf{\Delta}$. For the following result let $R_{\mathrm{c}1} \in \mathbb{P}^n$, $R_{\mathrm{c}2} \in \mathbb{P}^{m_{\mathrm{c}}}$ be given.

Corollary 3.1: Consider the continuous time linear state-dependent uncertain singular time delay controlled dynamical system

$$\begin{aligned} E_{\mathrm{c}}\dot{x}(t) = (A_{\mathrm{c}} + \Delta A_{\mathrm{c}})x(t) + (A_{\mathrm{d}} + \Delta A_{\mathrm{d}})x(t) \\ +B_{\mathrm{c}}u_{\mathrm{c}}(t), \qquad x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{55}$$

with performance functional

$$\begin{aligned} &J_{\Delta A_{\mathrm{c}}, \Delta A_{\mathrm{d}}}(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot),) \triangleq \\ &\int_0^\infty [x^{\mathrm{T}}(t)E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}1}E_{\mathrm{c}}x(t) + x^{\mathrm{T}}(t-\tau)E_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{d}1}E_{\mathrm{c}}x(t-\tau) \\ &+u_{\mathrm{c}}^{\mathrm{T}}(t)R_{\mathrm{c}2}u_{\mathrm{c}}(t)]\mathrm{d}t \end{aligned} \tag{56}$$

where $u_{\mathrm{c}}(\cdot)$ is admissible, $\Delta A_{\mathrm{c}} \in \boldsymbol{\Delta}$, $\Delta A_{\mathrm{d}} \in \boldsymbol{\Delta}$. Furthermore, assume there exist $P \in \mathbb{P}^n$, $\Omega_{\mathrm{c}} : \mathbb{P}^n \to \mathbb{N}^n$, $\Omega_{\mathrm{d}} : \mathbb{P}^n \to \mathbb{N}^n$, such that

$$\begin{aligned} x^{\mathrm{T}}(t)(\Delta A_{\mathrm{c}}^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P + P \Delta A_{\mathrm{c}} E_{\mathrm{c}}) x(t) \leq \\ x^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} \Omega_{\mathrm{c}}(P) E_{\mathrm{c}} x(t), \quad \Delta A_{\mathrm{c}} \in \boldsymbol{\Delta}_{A\mathrm{c}}, \end{aligned} \tag{57}$$

$$\begin{aligned} x^{\mathrm{T}}(t-\tau)(\Delta A_{\mathrm{d}}^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P + P \Delta A_{\mathrm{d}} E_{\mathrm{c}}) x(t-\tau) \leq \\ x(t-\tau)^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} \Omega_{\mathrm{d}}(P) E_{\mathrm{c}} x(t-\tau), \quad \Delta A_{\mathrm{c}} \in \boldsymbol{\Delta}_{A\mathrm{c}}, \end{aligned} \tag{58}$$

Furthermore, suppose there exists $P \in \mathbb{P}^n$ satisfying

$$\begin{aligned} 0 = x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}1} E_{\mathrm{c}} + \Omega_{\mathrm{c}}(P) - \\ P B_{\mathrm{c}} {R_{\mathrm{c}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P) E_{\mathrm{c}} x, \end{aligned} \tag{59}$$

$$\begin{aligned} 0 = x^{\mathrm{T}}(t-\tau)(A_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{d}} + E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{d}1} E_{\mathrm{c}} + \Omega_{\mathrm{d}}(P) - \\ P B_{\mathrm{c}} {R_{\mathrm{d}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P) E_{\mathrm{c}} x(t-\tau), \end{aligned} \tag{60}$$

Then, with feedback control $u_{\mathrm{c}}(t) = \phi_{\mathrm{c}}(x(t), \tau) = (-{R_{\mathrm{c}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t) - -{R_{\mathrm{d}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t-\tau)$, the zero solution $x(t) \equiv 0$ to (55), is globally asymptotically stable for all $x_0 \in \mathbb{R}^n$, $\Delta A_{\mathrm{c}} \in \boldsymbol{\Delta}_{A\mathrm{c}}$, $\Delta A_{\mathrm{d}} \in \boldsymbol{\Delta}_{A\mathrm{d}}$ and

$$\begin{aligned} \sup\nolimits_{(\Delta_{\mathrm{c}}) \in \boldsymbol{\Delta}} J_{\Delta A_{\mathrm{c}}}(E_{\mathrm{c}} x_0) \leq \mathcal{J}(E_{\mathrm{c}} x_0, \phi_{\mathrm{c}}(\cdot)) \\ = x_0^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x_0, \quad x_0 \in \mathbb{R}^n, \end{aligned} \tag{61}$$

$$\begin{aligned} \sup\nolimits_{(\Delta_{\mathrm{d}}) \in \boldsymbol{\Delta}} J_{\Delta A_{\mathrm{d}}}(E_{\mathrm{c}} x_0) \leq \mathcal{J}(E_{\mathrm{c}} x_0, \phi_{\mathrm{c}}(\cdot)) \\ = x_0^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x_0, \quad x_0 \in \mathbb{R}^n, \end{aligned} \tag{62}$$

where

$$\begin{aligned} \mathcal{J}(E_{\mathrm{c}} x_0, u_{\mathrm{c}}(\cdot), \tau) \triangleq \int_0^\infty & [x^{\mathrm{T}}(t) E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}1} E_{\mathrm{c}} x(t) \\ & + x^{\mathrm{T}}(t-\tau) E_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{d}1} E_{\mathrm{c}} x(t-\tau) \\ & + u_{\mathrm{c}}^{\mathrm{T}}(t) R_{\mathrm{c}2} u_{\mathrm{c}}(t) + x^{\mathrm{T}}(t) \Omega_{\mathrm{c}}(P) x(t) \\ & + x^{\mathrm{T}}(t-\tau) \Omega_{\mathrm{d}}(P) x(t-\tau)] \mathrm{d}t \end{aligned} \tag{63}$$

and where u_{c} is admissible and $x(t)$, $t \geq 0$, is a solution to (55) with $\Delta A_{\mathrm{c}} = 0$, $\Delta A_{\mathrm{d}} = 0$. Furthermore,

$$\begin{aligned} \mathcal{J}(E_{\mathrm{c}} x_0, \phi_{\mathrm{c}}(x(\cdot)), \tau) = \\ \min_{(u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot), \tau) \in \mathcal{C}(x_0)} \mathcal{J}(E_{\mathrm{c}} x_0, u_{\mathrm{c}}(\cdot), \tau), \end{aligned} \tag{64}$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing controllers for the nominal continuous time singular time delay dynamical system and $x_0 \in \mathbb{R}^n$, Haddad, Chellaboina, and Kablar (2001b) and Kablar (2005a).

Proof: The result is direct consequence of Theorem 3.1 and is omitted due to space limitations. ■

Remark 3.3: Note that since $\mathcal{U}_{\mathrm{c}} = \mathbb{R}^{m_{\mathrm{c}}}$, the robust feedback control $u_{\mathrm{c}} = \phi_{\mathrm{c}}(x, \tau)$ is globally optimal since it minimizes $(H_{\mathrm{c}}(E_{\mathrm{c}} x, u_{\mathrm{c}}, \tau)$, and satisfies (44). Specifically, setting

$$(\frac{\partial}{\partial u_{\mathrm{c}}} H_{\mathrm{c}}(E_{\mathrm{c}} x, u_{\mathrm{c}}, \tau) = 0, \tag{65}$$

yields the robust control $\phi_{\mathrm{c}}(x, \tau)$, where

$$\phi_{\mathrm{c}}(x, \tau) = -{R_{\mathrm{c}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t) - -{R_{\mathrm{d}}}_2^{-1} B_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x(t-\tau), \tag{66}$$

Now, since

$$\frac{\partial^2}{\partial u_{\mathrm{c}}^2} H_{\mathrm{c}}(E_{\mathrm{c}} x, u_{\mathrm{c}}, \tau) = R_{\mathrm{c}2} + R_{\mathrm{d}2} > 0, \tag{67}$$

it follows that for all $x \in \mathbb{R}^n$ the robust feedback law $\phi_{\mathrm{c}}(x, \tau)$ given by (66) minimizes $H_{\mathrm{c}}(E_{\mathrm{c}} x, u_{\mathrm{c}}, \tau)$.

IV. CONCLUSION

In this paper we have developed robust stability analyze results for the class of continuous time nonlinear uncertain singular time delay dynamical systems, based on Haddad et al. (2001a,b). Results are based on Lyapunov and asymptotic stability theorems developed in Kablar (2003b).

V. FUTURE WORK

Further work will specialize results of this paper to nonnegative, compartmental and large scale dynamical systems. Results will extend to discrete time systems. Sensitivity and robustness are very important qualitative properties to be conserved in biological systems and we will focus in establishing results applicable for particular examples of biological systems.

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