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Synchronization Control of Fractional-Order Discrete-Time Chaotic Systems

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Abstract— **This paper investigates the bifurcation phenomenon of fractional-order discrete-time chaotic systems and proposes the nonlinear control to synchronize two fractional-order discrete-time chaotic systems. By taking a finite truncation, fractional-order discrete-time chaotic systems are constructed. Chaotic phenomenon is also determined by the fractional-order. By adjusting the fractional order appropriately, the chaotic areas can be controlled. The synchronization control is designed for the same and different fractional-orders in the drive and response systems. Finally, the bifurcation phenomenon and synchronization of fractional-order Logistic system are studied. The proposed method can be extended to synchronize other fractional-order discrete-time systems.**

I. INTRODUCTION

The concept of fractional-order calculus was proposed early 300 years ago, but in recent years, the fractional-order calculus has been applied to a variety of practical projects such as the fractional-order PID controllers [1], the fractional-order signal processing [2], the fractional-order modeling. Compared with integer-order systems, fractional-order systems can describe the dynamics of the viscoelasticity and diffusion better [3]. Recently, the chaotic phenomena of fractional-order systems have attracted much attention, and various behaviors of fractional-order chaotic systems have been studied such as the fractional-order Lonenz system [4], the fractional-order Chen system [5], the fractional-order Rösser system [6], the fractional-order Newton-Leipnikd system [7]. Some circuits were designed to research chaotic phenomena in fractional-order systems in [8]-[10].

The synchronization of chaotic systems plays an important role in science and technology. As for fractional-order systems, the synchronization of fractional-order systems has also been studied widely. In [11], the phase and anti-phase synchronization of fractional-order chaotic systems was investigated using the active control theory. In [12], an adaptive fuzzy sliding mode control was proposed to synchronize two different uncertain fractional-order time-delay systems. In [13], a scheme for the synchronization of two perturbed fractional-order Chen systems was proposed. In [14], an adaptive feedback control scheme was presented for the synchronization of two coupled chaotic fractional-order systems with different fractional-orders. In [15], a new adaptive synchronization scheme by pragmatical asymptotical stability theorem was designed and applied on the synchronization of two completely different nonlinear systems with different orders.

To our best knowledge, the synchronization of fractional-order systems is generally studied for continuous-time chaotic systems. As for the fractional-order discrete-time chaotic systems, the researches are not very fruitful. For the synchronization problem in the integer-order discrete-time chaotic systems, the observer-based controller was designed in [16]. In [17], the sliding mode controller was applied to achieve the synchronization of two Hénon maps. In [18], the problem of the reliable impulsive lag synchronization for a class of nonlinear discrete chaotic systems was investigated.

As few researches on the fractional-order discrete-time chaotic systems have been done, we investigate the bifurcation and synchronization control for fractional-order discrete-time systems. In this study, we propose a method to construct a fractional-order discrete-time system. Taking the fractional-order Logistic system as an example, we study the influence of fractional-order α for chaotic areas. Then, nonlinear controllers are designed for the synchronization of two fractional-order discrete-time chaotic systems with the different and same orders, respectively. Using the theorem given in [19], we can simplify the expression of the controller for the $0<\alpha<1$ case. Finally, the synchronization of two fractional-order Logistic systems are simulated to demonstrate the effectiveness of nonlinear controllers.

The rest of this paper is organized as follows. In Section 2, a construction method of fractional-order systems is proposed by finite truncation and the fractional-order difference concept. The bifurcation of the fractional-order Logistic system is studied in Section 3. In Section 4, we design nonlinear controllers to synchronize fractional-order discrete-time systems with the same order and different order. Meanwhile, these methods are applied on the synchronization of fractional-order Logistic systems in Section 5. Conclusions are given in Section 6.

II. FRACTIONAL-ORDER DISCRETE-TIME CHAOTIC SYSTEMS

We assume the sample time of fractional-order systems is T, then the α order difference operation for the state vector can be represented as follows [20]:

$$\bigtriangledown^{\alpha}x(k)=\sum_{j=0}^{k}(-1)^{j}\begin{pmatrix}\alpha\\ j\end{pmatrix}x(k-j), \tag{1}$$

where $\alpha>0$ is the fractional-order, $x(k)\in\mathbb{R}^{n}$ is the state vector at the current time, $x(k-j)$ is the delayed state vector

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with the time jT and

$$\begin{pmatrix} \alpha \\ j \end{pmatrix} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for} \quad j > 0 \end{cases}, \tag{2}$$

Consider the following nonlinear discrete-time system as:

$$x(k+1) = f(x(k)), \tag{3}$$

where $f(\cdot)$ is a smooth nonlinear function.

We denote the 1-order difference for the integer-order discrete-time system (3) as:

$$\bigtriangledown^1 x(k+1) = x(k+1) - x(k) = f(x(k)) - x(k). \tag{4}$$

Resembling the 1-order difference, we denote the α-order difference as [21]:

$$\bigtriangledown^\alpha x(k+1) = f(x(k)) - x(k). \tag{5}$$

Noting the α-order difference (1), we have

$$\bigtriangledown^\alpha x(k+1) = x(k+1) - \alpha x(k) + \sum_{j=2}^{k+1} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} x(k-j+1). \tag{6}$$

Let $p = j - 1$, then we get

$$\begin{aligned} \bigtriangledown^\alpha x(k+1) =& x(k+1) - \alpha x(k) + \\ & \sum_{p=1}^{k} (-1)^{p+1} \begin{pmatrix} \alpha \\ p+1 \end{pmatrix} x(k-p). \end{aligned} \tag{7}$$

Denoting the parameter $C_p = (-1)^p \begin{pmatrix} \alpha \\ p+1 \end{pmatrix}$ and substituting (7) into (5) yields

$$x(k+1) = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^{k} C_p x(k-p). \tag{8}$$

If we set $\alpha = 1$, the coefficients C_p, $p = 1, 2, \ldots, k$ become 0 by the definition of fractional-order difference. That is to say, the fractional-order discrete-time system for $\alpha = 1$ is converted into an integer-order system, considering that all the state delays do not exist. Therefore, we can regard an integer-order system as a special case of a fractional-order system.

In a practical computing process, we cannot save all the states of a fractional-order discrete-time system since this process needs a great lot of calculation and space for each iteration computing.

By the definition of C_p, we set $\alpha = 0.2, 0.4, \ldots, 1.4$, the figure of C_p is depicted in Fig. 1.

As shown in Fig. 1, all the absolute values of the coefficients decrease as the iteration p increases for each fractional-order α. Hence, we can use a finite truncation to approximate a fractional-order discrete-time system. The truncation length is denoted by L and it is selected appropriately according to a practical problem. For the fractional-order discrete-time system (8), the finite truncation is described as:

$$x(k+1) = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^{L} C_p x(k-p). \tag{9}$$

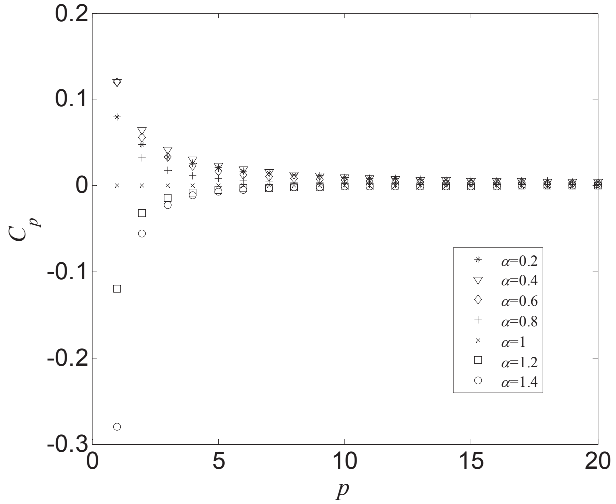

Fig. 1. Curves of C_p for $\alpha = 0.2, 0.4, \ldots, 1.4$

In the following section, taking the fractional-order discrete-time Logistic system as an example, we analyze the effect of the fractional-order α on the bifurcation phenomenon.

III. BIFURCATION PHENOMENON OF FRACTIONAL-ORDER DISCRETE-TIME LOGISTIC SYSTEM

Considering the balance between the computing amount and approximate error, we set $L = 20$ to research the bifurcation phenomenon of fractional-order Logistic systems.

The difference equation of the integral-order Logistic system is described as:

$$x(k+1) = \mu x(k)(1 - x(k)), \tag{10}$$

where μ is a parameter determining the bifurcation phenomenon. If it is larger than a special value, the chaotic phenomenon will appear.

The nonlinear function in (10) is $f(x(k)) = \mu x(k)(1 - x(k))$. Adopting the finite truncation L, the fractional-order discrete-time Logistic system is presented as:

$$x(k+1) = \mu x(k)(1 - x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^{L} C_p x(k-p). \tag{11}$$

The dynamic behavior of (11) is determined by two parameters α and μ. Next, we depict the bifurcation figures for (11), and the parameter μ is selected as $\mu = 3.4, 3.5, 3.8, 4.2$, respectively. The results are given in Fig. 2-Fig. 5 for the fractional-order $\alpha \in [0, 2]$.

As shown in Fig. 2 and Fig. 3, if $\alpha = 1$, the dynamic behavior of the system (11) performs 2-period orbits and 4-period orbits, respectively. These two cases corresponds to the dynamic behaviors of the integer-order Logistic system.

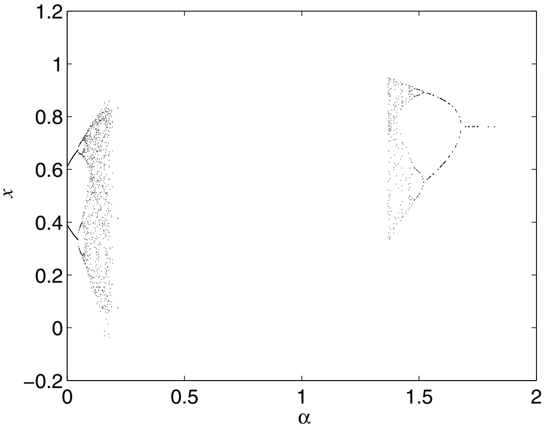

Fig. 2. Bifurcation of fractional-order discrete-time Logistic system for $\mu = 3.4$

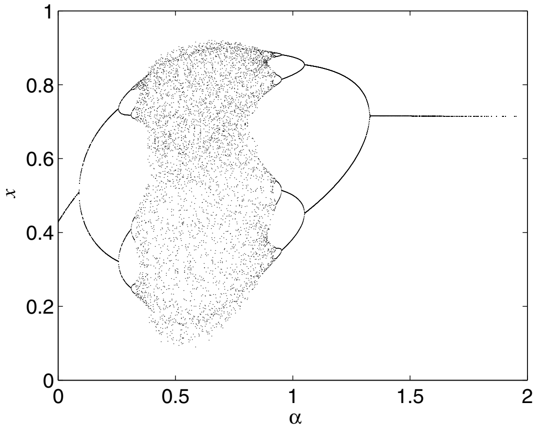

Fig. 3. Bifurcation of fractional-order discrete-time Logistic system for $\mu = 3.5$

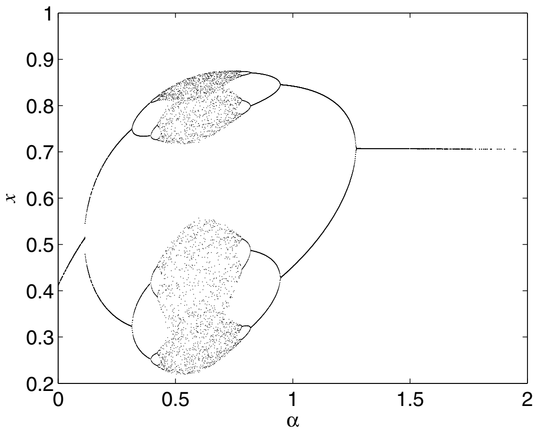

Fig. 4. Bifurcation of fractional-order discrete-time Logistic system for $\mu = 3.8$

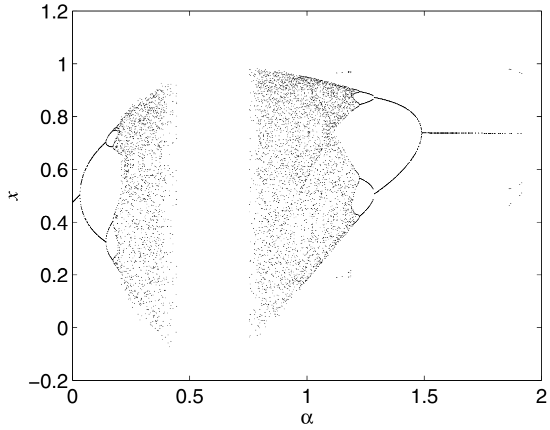

Fig. 5. Bifurcation of fractional-order discrete-time Logistic system for $\mu = 4.2$

However, if we reduce α, some new chaotic areas appear. Hence, we can also adjust the fractional-order α to control the chaos and the range of the chaotic phenomenon.

If we continue to increase the parameter μ, some unstable areas will appear in Fig. 4 and Fig. 5 for $\mu = 3.8, 4.2$ respectively. From Fig. 4, some branches appear in the 1-period orbit area and the chaotic area. These phenomena demonstrate that the behaviors of fractional-order discrete-time are more complicated than integer-order systems. For $\mu = 4.2$, the integer-order Logistic system is unstable. But we can obtain some chaotic areas when we adjust the fractional-order appropriately according to Fig. 5.

IV. DESIGN OF SYNCHRONIZATION CONTROL

In this section, we discuss two kinds of synchronization problems. Namely, the drive and response systems have the same fractional-orders with different initial values and different fractional-orders with different initial values.

A. Synchronization of fractional-order discrete-time systems with same fractional-orders

Consider the drive and response systems with the same fractional orders as:

Drive system:

$$x(k+1) = f(x(k)) + (\alpha - 1)x(k) + \sum_{p=1}^{L} C_p x(k-p), \tag{12}$$

Response system:

$$y(k+1) = f(y(k)) + (\alpha - 1)y(k) + \sum_{p=1}^{L} C_p y(k-p) + u(k), \tag{13}$$

where $x(k) \in \mathbb{R}^n$ and $y(k) \in \mathbb{R}^n$ are the state vectors of the drive and response systems, $u(k) \in \mathbb{R}^n$ is the synchronization control.

Denote the synchronization error $e(k) = y(k) - x(k)$, then we have

$$e(k+1) = f(y(k)) - f(x(k)) + (\alpha - 1)e(k) + \sum_{p=1}^{L} C_p e(k-p) + u(k). \tag{14}$$

Letting the control be $u(k) = Ke(k) - \sum_{p=1}^{L} C_p e(k-p) - f(y(k)) + f(x(k))$ and substituting $u(k)$ into (14) yields

$$e(k+1) = (\alpha - 1 - K)e(k), \tag{15}$$

where K is a scalar. Therefore, if $|\alpha - 1 - K| < 1$ holds, the error converges to zero, the synchronization between (14) and (15) is achieved. The condition $|\alpha - 1 - K| < 1$ is equivalent to $K \in (\alpha - 2, \alpha)$, that is to say, the feedback parameter K depends on the fractional-order α.

B. Synchronization of fractional-order discrete-time systems with different fractional-orders

Viewing the definition of C_p, we know that C_p is related with the fractional-order α. If the fractional-orders of the drive and response systems are set as different values, the difference equations are not the same just for state delays.

Let the fractional-order of the drive system be α_1 and the fractional-order of the response system be α_2, then we have

Drive system:

$$x(k+1) = f(x(k)) + (\alpha_1 - 1)x(k) + \sum_{p=1}^{L} C_{1p} x(k-p), \tag{16}$$

Response system:

$$y(k+1) = f(y(k)) + (\alpha_2 - 1)y(k) + \sum_{p=1}^{L} C_{2p} y(k-p) + u(k), \tag{17}$$

where C_{1p} and C_{2p} are the coefficients of the drive and response systems, respectively.

Denote the synchronization error $e(k) = y(k) - x(k)$, then we get

$$e(k+1) = f(y(k)) - f(x(k)) - e(k) + (\alpha_2 y(k) - \alpha_1 x(k)) + \sum_{p=1}^{L} C_{2p} y(k-p) - \sum_{p=1}^{L} C_{2p} x(k-p) + u(k). \tag{18}$$

Divide $\alpha_2 y(k) = (\alpha_2 - \alpha_1)y(k) + \alpha_1 y(k)$ and $C_{2p} y(k-p) = (C_{2p} - C_{1p})y(k-p) + C_{1p} y(k-p)$, then (18) can be rewritten as:

$$e(k+1) = f(y(k)) - f(x(k)) + \alpha_1 e(k) - e(k) + (\alpha_2 - \alpha_1)y(k) + \sum_{p=1}^{L} C_{1p} e(k-p) + \sum_{p=1}^{L} (C_{2p} - C_{1p}) y(k-p) + u(k). \tag{19}$$

Choose the nonlinear control as follows:

$$u(k) = -f(y(k)) + f(x(k)) + (\alpha_1 - \alpha_2)y(k) + Ke(k) - \sum_{p=1}^{L} C_{1p} e(k-p) - \sum_{p=1}^{L} (C_{2p} - C_{1p}) y(k-p). \tag{20}$$

Substituting the controller (20) into (19) yields

$$e(k+1) = (\alpha_1 - 1 + K)e(k). \tag{21}$$

Therefore, if $|\alpha_1 - 1 + K| < 1$ holds, the error $e(k)$ converges to zero, hence the synchronization is achieved. It means that $K \in (\alpha_1 - 2, \alpha_1)$ should be fulfilled.

Remark 1. We can also divide $\alpha_1 x(k) = (\alpha_1 - \alpha_2)x(k) + \alpha_2 x(k)$ and $C_{1p} x(k-p) = (C_{1p} - C_{2p})x(k-p) + C_{2p} x(k-p)$, then the feedback K depends on the fractional-order α_2.

Remark 2. If the fractional-order satisfies $0 < \alpha < 1$, each coefficient C_p is positive illustrated in Fig. 1, and the fractional-order system is defined as the positive fractional-order system.

For positive fractional-order systems, we have the following lemma to test the stability.

Lemma 1. [19] Consider the following positive fractional-order system with a finite truncation L:

$$x(k+1) = A_\alpha x(k) + \sum_{j=1}^{L} C_p x(k-j). \tag{22}$$

The system (22) is stable only if the system $x(k+1) = A_\alpha x(k)$ is stable, where $x \in \mathbb{R}^n$ and $A_\alpha \in \mathbb{R}^{n \times n}$.

Therefore, if the fractional-order satisfies $0 < \alpha_1 < 1$, the sum of the error delays $\sum_{p=1}^{L} C_{1p} e(k-p)$ can be deleted in the controller $u(k)$ as:

$$u(k) = -f(y(k)) + f(x(k)) + (\alpha_1 - \alpha_2)y(k) - \sum_{p=1}^{L} (C_{2p} - C_{1p}) y(k-p) + Ke(k), \tag{23}$$

and the error equation becomes

$$e(k+1) = (\alpha_1 - 1 + K)e(k) + \sum_{p=1}^{L} C_{1p} e(k-p). \tag{24}$$

Since each coefficient C_{1p}, $i = 1, 2, \ldots$ is positive, if $\alpha_1 - 2 < K < \alpha_1$ holds, the synchronization error under the controller (23) also converges to zero.

Remark 3. The above conclusion can be applied for the synchronization of the chaotic systems with the same fractional- orders which are all set between 0 and 1. The controller can be redesign as

$$u(k) = Ke(k) - f(y(k)) + f(x(k)). \tag{25}$$

V. SYNCHRONIZATION OF FRACTIONAL-ORDER DISCRETE-TIME LOGISTIC SYSTEM

In this section, we investigate the synchronization of the fractional-order discrete-time Logistic system.

If the fractional-orders in the drive and response systems satisfy $\alpha_1 = \alpha_2 = \alpha$, then $C_{1p} = C_{2p} = C_p$. It means that if we

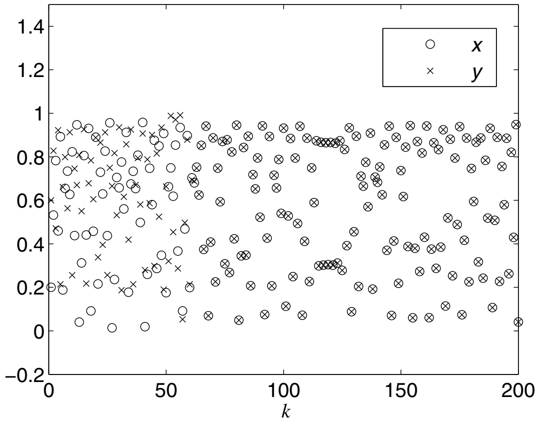

Fig. 6. Responses of $x(k)$ and $y(k)$ for $\alpha_1 = 0.7$, $\alpha_2 = 0.9$ and $\xi = 0$

set $\alpha_1 = \alpha_2$, the controller (20) and (23) can be also applied for the synchronization of fractional-order Logistic systems with the same orders. Hence, in the section we just study the synchronization of the fractional-order systems with different orders.

Drive system:

$$x(k+1) = \mu x(k)(1-x(k)) + (\alpha_1 - 1)x(k) + \sum_{p=1}^{L} C_{p1} x(k-p). \tag{26}$$

Response system:

$$\begin{aligned} y(k+1) =& \mu y(k)(1-y(k)) + (\alpha_2 - 1)y(k) + \\ & \sum_{p=1}^{L} C_{p2} y(k-p) + u(k). \end{aligned} \tag{27}$$

The length of the truncation is also set as $L = 20$ and the controller is designed as follows:

$$\begin{aligned} u(k) =& -\mu e(k) + \mu e(k)(x(k)+y(k)) + (\alpha_1 - \alpha_2)y(k) + \\ & Ke(k) - \xi \sum_{p=1}^{L} C_{1p} e(k-p) - \sum_{p=1}^{L} (C_{2p} - C_{1p}) y(k-p), \end{aligned} \tag{28}$$

where $\alpha_1 - 2 < K < \alpha_1$, $\xi = 1$ for $\alpha_1 > 1$ and $\xi = 0$ for $0 < \alpha_1 < 1$.

Let $\alpha_1 = 0.7$, $\alpha_2 = 0.9$ and $\mu = 3.7$, then we have $\xi = 0$. The initial values of the drive and response systems are set as $x(0) = 0.2$ and $y(0) = 0.6$. The controller is active on the response system for $k > 50$ by setting $K = \alpha_1 - 1 = -0.3$. The synchronization performance and the absolute value of $e(k)$ are depicted in Fig. 6 and Fig. 7, respectively.

Let $\alpha_1 = 1.1$, $\alpha_2 = 1.2$ and $\mu = 3.7$, then we have $\xi = 1$. The initial values are also set as $x(0) = 0.2$ and $y(0) = 0.6$. The controller is active on the response system for $k > 50$ by setting $K = \alpha_1 - 1 = 0.05$, then the synchronization performance and the absolute value of $e(k)$ are depicted in Fig. 8 and Fig. 9, respectively.

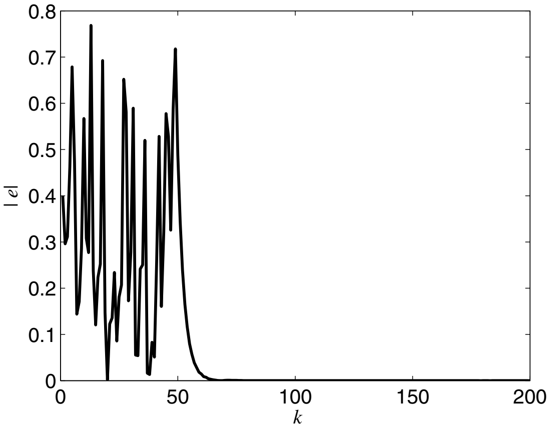

Fig. 7. Responses of $|e(k)|$ for $\alpha_1 = 0.7$, $\alpha_2 = 0.9$ and $\xi = 0$

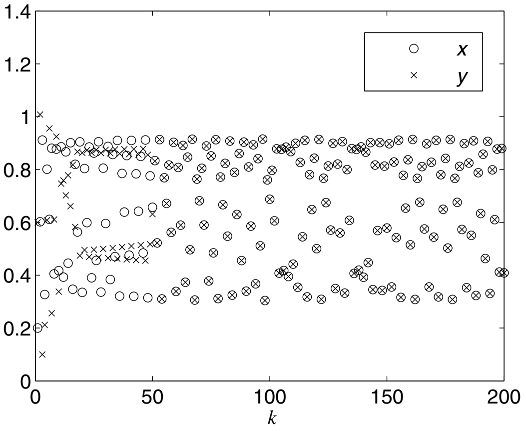

Fig. 8. Responses of $x(k)$ and $y(k)$ for $\alpha_1 = 1.05$, $\alpha_2 = 1.2$ and $\xi = 1$

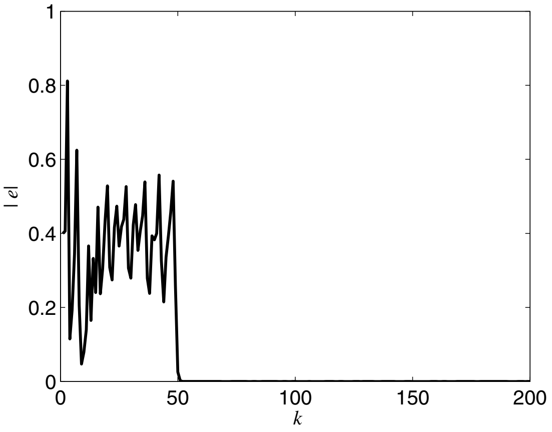

Fig. 9. Responses of $|e(k)|$ for $\alpha_1 = 1.05$, $\alpha_2 = 1.2$ and $\xi = 1$

As shown in Fig. 6 and Fig. 8, the synchronization of the fractional-order Logistic systems with the different orders is achieved. The absolute value of synchronization errors $e(k)$ for $\alpha_1 = 0.7$ and $\alpha_1 = 1.2$ converges to zero by the nonlinear controller (28) illustrated in Fig. 7 and Fig. 9, respectively.

VI. CONCLUSIONS

Based on the concept of the fractional-order difference, we present a method to construct fractional-order discrete-time chaotic systems. By adjusting the fractional-order α appropriately, the chaotic phenomenon of the fractional-order system appears or disappears. Taking the fractional-order Logistic system as an example, we analyze the bifurcation behavior. The emergency of chaos in the fractional-order Logistic system is determined by the parameter μ and the fractional-order α. Hence, the fractional-order is also a key factor to affect the dynamic behavior. The synchronization of fractional-order discrete-time systems with the same and different orders are studied and the controllers are designed for these cases. For the fractional-order Logistic system with different orders, we apply the nonlinear control on the response system for $k > 50$, and the error converges to zero. It means that the synchronization of two Logistic systems is achieved.

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