

Approximation-Free Prescribed Performance Control for Unknown SISO Pure Feedback Systems

Charalampos P. Bechlioulis and George A. Rovithakis

Abstract—A universal control scheme is designed for unknown pure feedback systems, capable of guaranteeing, for any initial system condition, output tracking with prescribed performance and bounded signals in the closed loop. In this paper, by prescribed performance, it is meant that the output error converges to a predefined arbitrarily small residual set, with convergence rate no less than a certain prespecified value. The proposed approximation-free and low-complexity control scheme isolates the output performance from the control gains selection and exhibits strong robustness against model uncertainties. In fact, any system in pure feedback form obeying certain controllability assumptions can be controlled by the proposed scheme without altering either the controller structure or the control gain values. Finally, a simulation study clarifies and verifies the approach.

I. INTRODUCTION

During the past several years, adaptive control of systems possessing complex and unknown nonlinear dynamics has attracted considerable research effort. Significant progress has been achieved through adaptive feedback linearization [1], adaptive backstepping [2], control Lyapunov functions [3] and adaptive neural network/fuzzy logic control [4]. The aforementioned results were obtained for systems in affine form, that is, for plants linear in the control input variables. However, there exist practical systems such as chemical processes and flight control systems, which cannot be expressed in an affine form. The difficulty associated with the control design of such systems arises from the fact that an explicit inverting control design is, in general, impossible, even though the inverse exists. Initially, nonaffine systems in low triangular canonical form (i.e., system nonlinearities satisfy a matching condition) were considered. Subsequently, as the problem became more apparent, the significantly more complex as well as general class of pure feedback nonaffine systems (i.e., all system states and control inputs appear implicitly in the system nonlinearities) was tackled. In case of single-input single-output nonaffine systems with unknown nonlinearities, fuzzy systems and neural networks have been used to approximate an ‘ideal controller’, whose

existence is guaranteed by the Implicit Function Theorem. Works incorporating the Mean Value Theorem [5]–[13], the Taylor series expansion [14] and the contraction mapping method [15], [16] have been proposed to decompose the original nonaffine system into an affine in the control part and a nonaffine part representing generalized modeling errors. Subsequently, standard robust adaptive control tools were employed. However, approximating this “ideal controller” is a difficult task, leading also to complex neural network and fuzzy system structures. In [17], [18], instead of seeking a direct solution to the inverse problem, an analytically invertible model was introduced and a neural network was designed to compensate for the inversion error. Finally, in [19], singular perturbation theory was applied to derive an adaptive dynamical inversion method for uncertain nonaffine systems.

Despite the recent progress in the control of unknown nonaffine systems, certain issues still remain open. First, it should be noticed that all aforementioned works have resorted to approximation-based techniques to deal with the model uncertainties of the system. Unfortunately, this approach inherently introduces certain issues affecting closed loop stability and robustness. Specifically, even though the existence of a closed loop initialization set as well as of certain control gain values that guarantee closed loop stability is proven, the problem of proposing an explicit constructive methodology capable of a priori imposing the required stability properties is not discussed. As a consequence, the produced control schemes yield inevitably reduced levels of robustness against modeling imperfections. Moreover, the results are restricted to be local as they are valid only within the compact set where the capabilities of the universal approximators (i.e., neural networks, fuzzy systems, etc.) hold. Furthermore, the introduction of approximating structures increases the complexity of the proposed control schemes in the sense that extra adaptive parameters have to be updated (i.e., nonlinear differential equations have to be solved numerically) and extra calculations have to be conducted to output the control signal, thus making implementation difficult.

Finally, all aforementioned works guarantee convergence of the tracking error to a residual set, whose size depends on explicit design parameters and some unknown bounded terms. However, no systematic procedure exists to accurately compute the required upper bounds, thus making the a priori selection of the design parameters to satisfy certain steady

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state behavior practically impossible. Moreover, the convergence rate is difficult to be established even in the case of known nonlinearities. Such issue has been discussed only in terms of the \mathcal{L}_2 norm of the tracking error that is derived to be a function of explicit design parameters and initial estimation errors. However, the aforementioned performance index is connected only indirectly with the actual system response. Therefore, a reduction of the \mathcal{L}_2 norm of the tracking error results in an overall transient performance improvement, with, on the other hand, no specific connection to the trajectory-oriented metric of convergence rate. Thus, the problem of guaranteeing prescribed performance for nonaffine systems still remains. In this work, by prescribed performance, it is meant that the tracking error converges to a predefined arbitrarily small residual set with convergence rate no less than a prespecified value. Until recently, prescribed performance controllers have been proposed in [20]–[22] for specific classes of affine nonlinear systems (i.e., feedback linearizable and strict feedback) and in [23] for general affine in the control MIMO nonlinear system structures.

In this work, we extend the results derived in [22] to the problem of controlling unknown pure feedback systems with prescribed performance. In this direction, an approximation-free control scheme is proposed that achieves global results in the sense that given any initial system condition and any output performance specifications, regarding the output steady state error and convergence rate, the control objective is satisfied with bounded closed loop signals. Furthermore, output performance is isolated from control gains selection and robustness against model uncertainties is greatly extended. In fact, any system in pure feedback form obeying certain controllability assumptions can be controlled by the proposed scheme without altering its structure or its control gain values. Finally, only the desired trajectory and none of its higher order derivatives is required.

II. DEFINITIONS AND PRELIMINARIES

At this point, we recall some definitions and preliminary results that are necessary in the subsequent analysis.

A. Prescribed Performance

It will be clearly demonstrated in the Main Results Section, that the control design is heavily connected to the prescribed performance notion that was originally employed to design neuro-adaptive controllers, for various classes of nonlinear systems, namely feedback linearizable [20], strict feedback [21], [22] and general MIMO affine in the control [23], capable of guaranteeing output tracking with prescribed performance. In this work, by prescribed performance, it is meant that the output tracking error converges to a predefined arbitrarily small residual set with convergence rate no less than a certain prespecified value. For completeness and compactness of presentation, this subsection summarizes preliminary knowledge on prescribed performance. In that respect, consider a generic scalar tracking error $e(t)$. Prescribed performance is achieved if $e(t)$ evolves strictly within a predefined region that is bounded by decaying

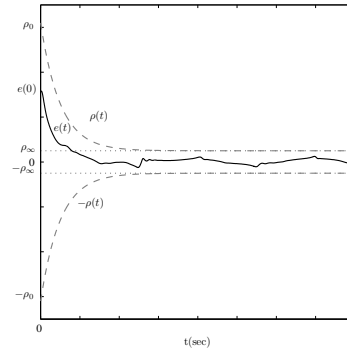


Fig. 1. Graphical illustration of the prescribed performance definition.

functions of time. The mathematical expression of prescribed performance is given, $\forall t \geq 0$, by the following inequalities:

$$-\rho(t) < e(t) < \rho(t) \quad (1)$$

where $\rho(t)$ is a smooth, bounded, strictly positive and decreasing function of time satisfying $\lim_{t \rightarrow \infty} \rho(t) > 0$, called performance function [20]. The aforementioned statements are clearly illustrated in Fig. 1 for an exponential performance function $\rho(t) = (\rho_0 - \rho_\infty) e^{-lt} + \rho_\infty$ with ρ_0, ρ_∞, l appropriately chosen strictly positive constants. The constant $\rho_0 = \rho(0)$ is selected such that $\rho_0 > |e(0)|$. The constant $\rho_\infty = \lim_{t \rightarrow \infty} \rho(t)$ represents the maximum allowable size of the tracking error $e(t)$ at the steady state, which may even be set arbitrarily small to a value reflecting the resolution of the measurement device, thus achieving practical convergence of $e(t)$ to zero. Moreover, the decreasing rate of $\rho(t)$, which is affected by the constant l in this case, introduces a lower bound on the required speed of convergence of $e(t)$. Therefore, the appropriate selection of the performance function $\rho(t)$ imposes performance characteristics on the tracking error $e(t)$.

B. Dynamical Systems

Consider the initial value problem:

$$\dot{\xi} = h(t, \xi), \quad \xi(0) = \xi^0 \in \Omega_\xi \quad (2)$$

with $h : \mathbb{R}_+ \times \Omega_\xi \rightarrow \mathbb{R}^n$ where $\Omega_\xi \subset \mathbb{R}^n$ is a non-empty open set.

Definition 1: [24] A solution $\xi(t)$ of the initial value problem (2) is maximal if it has no proper right extension that is also a solution of (2).

Theorem 1: [24] Consider the initial value problem (2). Assume that $h(t, \xi)$ is: a) locally Lipschitz on ξ for almost all $t \in \mathbb{R}_+$, b) piecewise continuous on t for each fixed $\xi \in \Omega_\xi$ and c) locally integrable on t for each fixed $\xi \in \Omega_\xi$. Then, there exists a maximal solution $\xi(t)$ of (2) on the time interval $[0, \tau_{\max})$ with $\tau_{\max} > 0$ such that $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$.

Proposition 1: [24] Assume that the hypotheses of Theorem 1 hold. For a maximal solution $\xi(t)$ on the time interval $[0, \tau_{\max})$ with $\tau_{\max} < \infty$ and for any compact set $\Omega'_\xi \subset \Omega_\xi$ there exists a time instant $t' \in [0, \tau_{\max})$ such that $\xi(t') \notin \Omega'_\xi$.

III. PROBLEM STATEMENT AND STANDING ASSUMPTIONS

Consider an n -th order pure feedback system described as follows:

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i, x_{i+1}), \quad i = 1, \dots, n-1 \\ \dot{x}_n &= f_n(\bar{x}_n, u) \\ y &= x_1 \end{aligned} \quad (3)$$

where $x_i \in \mathfrak{R}$, $i = 1, \dots, n$ are the states with initial conditions $x_i(0) = x_i^0$, $i = 1, \dots, n$ and $\bar{x}_i := [x_1 \ \dots \ x_i]^T$. Moreover, $u \in \mathfrak{R}$ is the control input, $y \in \mathfrak{R}$ is the output and $f_i : \mathfrak{R}^{i+1} \rightarrow \mathfrak{R}$, $i = 1, \dots, n$ are implicit functions. In what follows, we formulate the problem to be solved in this work.

Global Robust Approximation-free Prescribed Performance Control (GRAPPC) Problem: Design an approximation-free control scheme for system (3) involving unknown nonlinearities such that for any initial condition x_i^0 , $i = 1, \dots, n$ all signals in the closed loop system remain bounded and moreover the system output tracks a desired trajectory $y_d(t)$ with prescribed performance. By prescribed performance, we mean that the output tracking error $x_1(t) - y_d(t)$ should converge to a predefined arbitrarily small residual set with convergence rate no less than a certain prespecified value.

To solve the GRAPPC Problem, we make the following assumptions:

Assumption 1: The functions $f_i : \mathfrak{R}^{i+1} \rightarrow \mathfrak{R}$, $i = 1, \dots, n$ are smooth and there exist unknown positive constants b_i , $i = 1, \dots, n$ such that:

$$\begin{aligned} \left| \frac{\partial f_i(\bar{x}_i, x_{i+1})}{\partial x_{i+1}} \right| &\geq b_i > 0, \quad i = 1, \dots, n-1 \\ \left| \frac{\partial f_n(\bar{x}_n, u)}{\partial u} \right| &\geq b_n > 0, \quad \forall (\bar{x}_n, u) \in \mathfrak{R}^{n+1}. \end{aligned}$$

Assumption 2: The sign of $\frac{\partial f_i(\bar{x}_i, x_{i+1})}{\partial x_{i+1}}$, $i = 1, \dots, n-1$ and $\frac{\partial f_n(\bar{x}_n, u)}{\partial u}$ is considered known. Without loss of generality we assume that all signs are positive.

Assumption 3: The system state $\bar{x}_n = [x_1 \ \dots \ x_n]^T$ is available for measurement.

Assumption 4: The desired trajectory $y_d(t)$ is a known bounded function of time with bounded first derivative.

Remark 1: Assumption 1 is a sufficient global controllability condition for system (3). Moreover, as mentioned in Assumption 4, notice that we only request knowledge of the desired trajectory and none of its high order derivatives. Therefore, applications where the derivatives of $y_d(t)$ are not available can be considered. To the best of the author's knowledge, all works concerning the control of system (3) require high order time derivatives of $y_d(t)$.

IV. MAIN RESULTS

In this section, we shall first present an approximation-free prescribed performance control scheme and subsequently we shall prove that it leads to the solution of the GRAPPC Problem for system (3).

A. Control Scheme

Given a desired trajectory $y_d(t)$ and any initial system condition $\bar{x}_n(0) \in \mathfrak{R}^n$:

I-a. Select an output performance function $\rho_1(t)$ that i) satisfies $\rho_1(0) > |x_1(0) - y_d(0)|$ and ii) incorporates the desired performance specifications regarding the steady state error and the speed of convergence (p.e., a candidate function could be $\rho_1(t) = (\rho_{10} - \rho_{1\infty})e^{-l_1 t} + \rho_{1\infty}$ where $\rho_{10} > |x_1(0) - y_d(0)|$ and $l_1, \rho_{1\infty}$ are the required minimum exponential convergence rate and the maximum steady state error respectively).

I-b. Design the first intermediate control signal as:

$$a_1(x_1, t) = -k_1 \ln \left(\frac{1 + \frac{x_1 - y_d(t)}{\rho_1(t)}}{1 - \frac{x_1 - y_d(t)}{\rho_1(t)}} \right) \quad (4)$$

with any positive control gain k_1 .

II. Select a second performance function $\rho_2(t)$ that satisfies only $\rho_2(0) > |x_2(0) - a_1(x_1(0), 0)|$ and design the second intermediate control signal as:

$$a_2(\bar{x}_2, t) = -k_2 \ln \left(\frac{1 + \frac{x_2 - a_1(x_1, t)}{\rho_2(t)}}{1 - \frac{x_2 - a_1(x_1, t)}{\rho_2(t)}} \right) \quad (5)$$

with any positive control gain k_2 .

III. Repeat step II for all the remaining intermediate control signals:

$$a_i(\bar{x}_i, t) = -k_i \ln \left(\frac{1 + \frac{x_i - a_{i-1}(\bar{x}_{i-1}, t)}{\rho_i(t)}}{1 - \frac{x_i - a_{i-1}(\bar{x}_{i-1}, t)}{\rho_i(t)}} \right), \quad i = 3, \dots, n-1 \quad (6)$$

with any positive control gains k_i , $i = 3, \dots, n-1$ and any performance functions $\rho_i(t)$, $i = 3, \dots, n-1$ satisfying $\rho_i(0) > |x_i(0) - a_{i-1}(\bar{x}_{i-1}(0), 0)|$, $i = 3, \dots, n-1$.

IV. Finally, design the control input as:

$$u(\bar{x}_n, t) = -k_n \ln \left(\frac{1 + \frac{x_n - a_{n-1}(\bar{x}_{n-1}, t)}{\rho_n(t)}}{1 - \frac{x_n - a_{n-1}(\bar{x}_{n-1}, t)}{\rho_n(t)}} \right) \quad (7)$$

with any positive control gain k_n and any performance function $\rho_n(t)$ satisfying $\rho_n(0) > |x_n(0) - a_{n-1}(\bar{x}_{n-1}(0), 0)|$.

Remark 2: The proposed control scheme does not incorporate any prior knowledge of system nonlinearities or even of some corresponding upper bounding functions. Furthermore, no approximation structures (i.e., neural networks, fuzzy systems) have been employed to acquire such knowledge. Moreover, compared with the traditional backstepping-like approaches, the proposed methodology proves significantly less complex. Notice that no hard calculations are required to output the proposed control signal, thus making its implementation straightforward.

B. Stability Analysis

The main results of this work are summarized in the following theorem where it is proven that the aforementioned control scheme solves the GRAPPC Problem.

Theorem 2: Consider system (3) obeying Assumptions 1-3. Given any initial system condition $\bar{x}_n(0) \in \mathfrak{R}^n$ and any

desired trajectory obeying Assumption 4, the proposed control scheme (4)-(7) in Subsection IV-A, solves the GRAPPC Problem.

Proof: Let us define the normalized state errors:

$$\xi_1 = \frac{x_1 - y_d(t)}{\rho_1(t)}, \xi_i = \frac{x_i - a_{i-1}(\bar{x}_{i-1}(t))}{\rho_i(t)}, i = 2, \dots, n. \quad (8)$$

The intermediate control signals (4)-(6) and the control law (7) may be written as functions of the normalized errors ξ_i , $i = 1, \dots, n$ as follows:

$$a_i(\xi_i) = -k_i \ln\left(\frac{1 + \xi_i}{1 - \xi_i}\right), i = 1, \dots, n-1 \quad (9)$$

$$u(\xi_n) = -k_n \ln\left(\frac{1 + \xi_n}{1 - \xi_n}\right). \quad (10)$$

Differentiating the normalized errors (8) with respect to time and substituting the system equation (3) as well as the equations:

$$x_1 = \xi_1 \rho_1(t) + y_d(t) \quad (11)$$

$$x_i = \xi_i \rho_i(t) + a_{i-1}(\xi_{i-1}), i = 2, \dots, n, \quad (12)$$

which are directly derived from (8), we obtain:

$$\begin{aligned} \dot{\xi}_1 &= h_1(t, \xi_1, \xi_2) \\ &= \frac{1}{\rho_1(t)} (f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t) + a_1(\xi_1)) \\ &\quad - \dot{y}_d(t) - \xi_1 \dot{\rho}_1(t)) \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\xi}_2 &= h_2(t, \xi_1, \xi_2, \xi_3) \\ &= \frac{1}{\rho_2(t)} (f_2(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t) + a_1(\xi_1), \xi_3 \rho_3(t) \\ &\quad + a_2(\xi_2)) - \frac{d a_1}{d \xi_1} h_1(t, \xi_1, \xi_2) - \xi_2 \dot{\rho}_2(t)) \end{aligned} \quad (14)$$

$$\begin{aligned} \dot{\xi}_i &= h_i(t, \xi_1, \dots, \xi_{i+1}) \\ &= \frac{1}{\rho_i(t)} (f_i(\xi_1 \rho_1(t) + y_d(t), \dots, \xi_{i+1} \rho_{i+1}(t) + a_i(\xi_i)) \\ &\quad - \frac{d a_{i-1}}{d \xi_{i-1}} h_{i-1}(t, \xi_1, \dots, \xi_i) - \xi_i \dot{\rho}_i(t)), i = 3, \dots, n-1 \end{aligned} \quad (15)$$

$$\begin{aligned} \dot{\xi}_n &= h_n(t, \xi_1, \dots, \xi_n) \\ &= \frac{1}{\rho_n(t)} (f_n(\xi_1 \rho_1(t) + y_d(t), \dots, \xi_n \rho_n(t) \\ &\quad + a_{n-1}(\xi_{n-1}), u(\xi_n)) - \frac{d a_{n-1}}{d \xi_{n-1}} h_{n-1}(t, \xi_1, \dots, \xi_n) \\ &\quad - \xi_n \dot{\rho}_n(t)). \end{aligned} \quad (16)$$

In more compact form, the dynamical system of the overall normalized error vector $\xi = [\xi_1 \ \dots \ \xi_n]^T$ may be written as:

$$\dot{\xi} = h(t, \xi) = \begin{bmatrix} h_1(t, \xi_1, \xi_2) \\ \vdots \\ h_n(t, \xi_1, \dots, \xi_n) \end{bmatrix}. \quad (17)$$

Let us also define the open set:

$$\Omega_\xi = \underbrace{(-1, 1) \times \dots \times (-1, 1)}_{n\text{-times}}.$$

In the sequel, we proceed in two phases. First, the existence of a maximal solution $\xi(t)$ of (17) over the set Ω_ξ for a time interval $[0, \tau_{\max})$ (i.e., $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$) is ensured. Then, we prove that the proposed control scheme (9), (10) guarantees, for all $t \in [0, \tau_{\max})$: a) the boundedness of all closed loop signals of (17) as well as that b) $\xi(t)$ remains strictly within a compact subset of Ω_ξ , which leads by contradiction to $\tau_{\max} = \infty$ and consequently to the solution of the GRAPPC Problem.

Phase A. The set Ω_ξ is nonempty and open. Moreover, the performance functions $\rho_i(t)$ have been selected to satisfy $\rho_1(0) > |x_1(0) - y_d(0)|$ and $\rho_i(0) > |x_i(0) - a_{i-1}(\bar{x}_{i-1}(0), 0)|, i = 2, \dots, n$. As a consequence $|\xi_i(0)| < 1, i = 1, \dots, n$ which results in $\xi(0) \in \Omega_\xi$. Additionally, h is piecewise continuous on t and continuous on ξ over the set Ω_ξ . Therefore, the hypotheses of Theorem 1 stated in Subsection II-B hold and the existence and uniqueness of a maximal solution $\xi(t)$ of (17) on a time interval $[0, \tau_{\max})$ such that $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$ is ensured.

Phase B. We have proven in Phase A that $\xi(t) \in \Omega_\xi, \forall t \in [0, \tau_{\max})$ or equivalently that:

$$\xi_i(t) \in (-1, 1), i = 1, \dots, n \text{ for all } t \in [0, \tau_{\max}). \quad (18)$$

Therefore, the signals:

$$\varepsilon_i(t) = \ln\left(\frac{1 + \xi_i(t)}{1 - \xi_i(t)}\right), i = 1, \dots, n \quad (19)$$

are well defined for all $t \in [0, \tau_{\max})$.

Step 1. Consider the positive definite and radially unbounded function $V_1 = \frac{1}{2} \varepsilon_1^2$. Differentiating with respect to time and substituting (13), we obtain:

$$\begin{aligned} \dot{V}_1 &= \frac{2\varepsilon_1}{(1 - \xi_1^2) \rho_1(t)} (f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t) \\ &\quad + a_1(\xi_1)) - \dot{y}_d(t) - \xi_1 \dot{\rho}_1(t)). \end{aligned} \quad (20)$$

Moreover, the smoothness of $f_1(\cdot, \cdot)$ from Assumption 1 leads through the Mean Value Theorem to:

$$\begin{aligned} f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t) + a_1(\xi_1)) &= \\ f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t)) &+ a_1(\xi_1) \frac{\partial f_1(\xi_1 \rho_1(t) + y_d(t), z)}{\partial z} \Big|_{z=z^*} \end{aligned} \quad (21)$$

where:

$$z^* = \lambda(\xi_2 \rho_2(t) + a_1(\xi_1)) + (1 - \lambda) \xi_2 \rho_2(t)$$

for some $\lambda \in (0, 1)$. Hence, substituting (21) and (9) as well as incorporating (19) in (20), \dot{V}_1 becomes:

$$\begin{aligned} \dot{V}_1 &= \frac{2\varepsilon_1}{(1 - \xi_1^2) \rho_1(t)} (f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t)) - \dot{y}_d(t) \\ &\quad - \xi_1 \dot{\rho}_1(t) - k_1 \varepsilon_1 \frac{\partial f_1(\xi_1 \rho_1(t) + y_d(t), z)}{\partial z} \Big|_{z=z^*}). \end{aligned}$$

Furthermore, utilizing (18), the fact that $\rho_1(t), \dot{\rho}_1(t), \rho_2(t), y_d(t), \dot{y}_d(t)$ are bounded by construction and by Assumption 4 and employing the Extreme Value Theorem owing to the smoothness of $f_1(\cdot, \cdot)$, we arrive at:

$$|f_1(\xi_1 \rho_1(t) + y_d(t), \xi_2 \rho_2(t)) - \dot{y}_d(t) - \xi_1 \dot{\rho}_1(t)| \leq \bar{F}_1,$$

$\forall t \in [0, \tau_{\max})$, for an unknown positive constant \bar{F}_1 . Moreover, Assumptions 1 and 2 dictate:

$$\frac{\partial f_1(\xi_1 \rho_1(t) + y_d(t), z)}{\partial z} \Big|_{z=z^*} \geq b_1.$$

Furthermore, owing to (18) it holds that $\frac{1}{(1 - \xi_1^2)} > 1$, whereas $\rho_1(t) > \lim_{t \rightarrow \infty} \rho_1(t) > 0$ by construction. Therefore, we conclude that \dot{V}_1 is negative when $|\varepsilon_1(t)| > \frac{\bar{F}_1}{k_1 b_1}$ and subsequently that:

$$|\varepsilon_1(t)| \leq \bar{\varepsilon}_1 = \max\left\{|\varepsilon_1(0)|, \frac{\bar{F}_1}{k_1 b_1}\right\}, \forall t \in [0, \tau_{\max}), \quad (22)$$

which from (19), taking the inverse logarithmic function leads to:

$$-1 < \frac{e^{-\bar{\varepsilon}_1} - 1}{e^{-\bar{\varepsilon}_1} + 1} = \underline{\xi}_1 \leq \xi_1(t) \leq \bar{\xi}_1 = \frac{e^{\bar{\varepsilon}_1} - 1}{e^{\bar{\varepsilon}_1} + 1} < 1 \quad (23)$$

for all $t \in [0, \tau_{\max})$. As a result, the first intermediate control signal remains bounded (i.e., $|a_1(\xi_1)| \leq k_1 \bar{\varepsilon}_1$) for all $t \in [0, \tau_{\max})$. Furthermore, differentiating $a_1(\xi_1)$ with respect to time, substituting (13) and utilizing (23), it is straightforward to deduce the boundedness of $\dot{a}_1(\xi_1)$, $\forall t \in [0, \tau_{\max})$. Finally, invoking (12) for $i = 2$ we also conclude the boundedness of x_2 for all $t \in [0, \tau_{\max})$.

Step i ($2 \leq i \leq n$): Applying recursively for the remaining steps the aforementioned line of proof, considering $V_i = \frac{1}{2} \varepsilon_i^2$, we conclude that:

$$|\varepsilon_i(t)| \leq \bar{\varepsilon}_i = \max \left\{ |\varepsilon_i(0)|, \frac{\bar{F}_i}{k_i b_i} \right\}, \quad \forall t \in [0, \tau_{\max}) \quad (24)$$

where the constants $\bar{F}_i > 0$, $i = 2, \dots, n$ satisfy:

$$\begin{aligned} & |f_i(\xi_1 \rho_1(t) + y_d(t), \dots, \xi_{i+1} \rho_{i+1}(t)) - \xi_i \dot{\rho}_i(t) \\ & - \frac{d a_{i-1}}{d \xi_{i-1}} h_{i-1}(t, \xi_1, \dots, \xi_i) | \leq \bar{F}_i, \quad i = 2, \dots, n-1 \\ & |f_n(\xi_1 \rho_1(t) + y_d(t), \dots, \xi_n \rho_n(t) + a_{n-1}(\xi_{n-1}), 0) \\ & - \xi_n \dot{\rho}_n(t) - \frac{d a_{n-1}}{d \xi_{n-1}} h_{n-1}(t, \xi_1, \dots, \xi_n) | \leq \bar{F}_n \end{aligned}$$

for all $t \in [0, \tau_{\max})$. Correspondingly, (19) leads also in:

$$-1 < \frac{e^{-\bar{\varepsilon}_i} - 1}{e^{-\bar{\varepsilon}_i} + 1} = \underline{\xi}_i \leq \xi_i(t) \leq \bar{\xi}_i = \frac{e^{\bar{\varepsilon}_i} - 1}{e^{\bar{\varepsilon}_i} + 1} < 1 \quad (25)$$

for $i = 2, \dots, n$ and $\forall t \in [0, \tau_{\max})$. As a consequence, all intermediate control signals $a_i(\xi_i)$ and system states x_{i+1} , $i = 2, \dots, n-1$ as well as the control law $u(\xi_n)$ remain bounded for all $t \in [0, \tau_{\max})$.

Up to this point, what remains to be shown is that $t_f = \infty$. Notice that (23) and (25) imply that $\xi(t) \in \Omega'_\xi$, $\forall t \in [0, \tau_{\max})$, where the set $\Omega'_\xi = \prod_{i=1}^n \left[\frac{e^{-\bar{\varepsilon}_i} - 1}{e^{-\bar{\varepsilon}_i} + 1}, \frac{e^{\bar{\varepsilon}_i} - 1}{e^{\bar{\varepsilon}_i} + 1} \right]$ is nonempty and compact. Moreover, it can be easily verified that $\Omega_\xi \subset \Omega'_\xi$. Hence, assuming $\tau_{\max} < \infty$ and since $\Omega'_\xi \subset \Omega_\xi$, Proposition 1 in Subsection II-B dictates the existence of a time instant $t' \in [0, \tau_{\max})$ such that $\xi(t') \notin \Omega'_\xi$, which is a clear contradiction. Therefore, $\tau_{\max} = \infty$. Hence, all closed loop signals remain bounded and moreover $\xi(t) \in \Omega_\xi \subset \Omega'_\xi$, $\forall t \geq 0$. Finally, from (11) we conclude that:

$$-\rho_1(t) < \frac{e^{-\bar{\varepsilon}_1} - 1}{e^{-\bar{\varepsilon}_1} + 1} \rho_1(t) \leq x_1 - y_d(t) \leq \frac{e^{\bar{\varepsilon}_1} - 1}{e^{\bar{\varepsilon}_1} + 1} \rho_1(t) < \rho_1(t)$$

for all $t \geq 0$ and consequently that output tracking with prescribed performance is achieved which completes the proof. ■

Remark 3: From the aforementioned proof, it is worth noticing that the proposed control scheme achieves its goals without resorting to the need of rendering $\bar{\varepsilon}_i$, $i = 1, \dots, n$ (see (22) and (24)) arbitrarily small, through extreme values of the control gains k_i , $i = 1, \dots, n$. In this spirit, large unknown system nonlinearities $f_i(\cdot, \cdot)$, $i = 1, \dots, n$ compatible with Assumption 1, can be compensated, as they affect only the size of $\bar{\varepsilon}_i$, $i = 1, \dots, n$ but leave unaltered the achieved stability properties. In fact, the actual output tracking performance, which is solely determined by the output performance function $\rho_1(t)$, becomes isolated against

the model uncertainties thus extending greatly the robustness of the proposed control scheme. Thus, contrary to what is the common practice in the relevant literature, the selection of the control gains k_i , $i = 1, \dots, n$ is significantly simplified to adopting those values that lead to reasonable control effort. The same reasoning applies also to the selection of the performance functions $\rho_i(t)$, $i = 2, \dots, n$ as the only constraint is the satisfaction of $\rho_i(0) > |x_i(0) - a_{i-1}(x_{i-1}(0), 0)|$, $i = 2, \dots, n$.

Remark 4: The proposed control scheme achieves global results in the sense that given any initial system condition $\bar{x}_n(0) \in \mathfrak{R}^n$ and any performance specifications for the output, regarding the steady state error and the speed of convergence, the control objective is satisfied.

Remark 5: Interestingly, the proposed control scheme is independent of the time derivatives of $y_d(t)$. Certainly, the first intermediate control signal $a_1(x_1, t)$ depends on $y_d(t)$. However, $\dot{a}_1(x_1, t)$ which involves $\dot{y}_d(t)$ is proven bounded and therefore we do not compensate for it through the design of the second intermediate control signal $a_2(\bar{x}_2, t)$. Correspondingly, the same holds for all intermediate control signals, thus isolating the appearance of high order derivatives of $y_d(t)$ in the control scheme.

V. SIMULATION RESULTS

We consider the system:

$$\begin{aligned} \dot{x}_1 &= \frac{1 - e^{-x_1}}{1 + e^{-x_1}} + x_2^3 + x_2 e^{-(1+x_1^2)} + 0.1 \\ \dot{x}_2 &= x_1^2 + 0.15u^3 + 0.1(1 + x_2^2)u + \sin(0.1u) \\ y &= x_1 \end{aligned} \quad (\Sigma)$$

with initial condition $x_1(0) = 1$, $x_2(0) = 2$ as well as the desired trajectory $y_d(t) = 0.5 \cos(t) + \sin(2t)$. Clearly, system (Σ) is in pure feedback form and Assumptions 1-4 are satisfied. For the output tracking error $x_1(t) - y_d(t)$ we require a steady state error of no more than 0.01 and a minimum speed of convergence as obtained by the exponential e^{-2t} . The aforementioned transient and steady state error bounds are incorporated in the output performance function $\rho_1(t) = (\rho_{10} - 0.01)e^{-2t} + 0.01$. Moreover, since $x_1(0) - y_d(0) = 0.5$ we select $\rho_{10} = 1$ to guarantee $\rho_1(0) > |x_1(0) - y_d(0)|$. Thus, we design the first intermediate control signal $a_1(x_1, t)$ as in (4). We also select the second performance function $\rho_2(t) = (\rho_{20} - 0.01)e^{-2t} + 0.01$ with $\rho_{20} = 6.2$ to satisfy $\rho_2(0) > |x_2(0) - a_1(x_1(0), 0)| \simeq 3.1$. Hence, the control law is formulated as in (7). Finally, to produce reasonable control effort, the control gains are selected as $k_1 = 1$, $k_2 = 4$. The aforementioned control scheme, is applied to system (Σ) . The output tracking is shown in Fig. 2 and the required control input is pictured in Fig. 3. Obviously, prescribed performance output tracking with bounded closed loop signals is achieved, as it was predicted by the theoretical analysis, despite the presence of unknown system nonlinearities.

VI. CONCLUSIONS

In this paper, we have established a general framework to handle the GRAPPC Problem for unknown pure feedback systems. An approximation-free and low-complexity

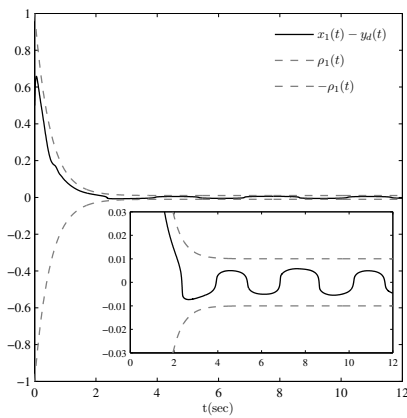


Fig. 2. Output tracking. The evolution of the tracking error $x_1(t) - y_d(t)$ along with the transient and steady state performance bounds imposed by $\rho_1(t)$. The subplot gives details during the steady state.

control scheme is designed that yields global results in the sense that given any initial system condition and any output performance requirements, regarding the steady state error and the speed of convergence, the control objective is satisfied. The actual output tracking performance, which is solely determined by the output performance function $\rho_1(t)$, contrary to the relevant literature, becomes isolated from: i) the selection of the control gains, simplifying thus further the control design procedure, as well as ii) the model uncertainties, extending thus greatly the robustness of the proposed control scheme. Additionally, only the desired trajectory and none of its higher order derivatives is required.

REFERENCES

- [1] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Transactions on Automatic Control*, vol. 34, no. 11, pp. 1123–1131, 1989.
- [2] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.
- [3] E. D. Sontag, "A 'universal' construction of artstein's theorem on nonlinear stabilization," *Systems and Control Letters*, vol. 13, no. 2, pp. 117–123, 1989.
- [4] J. A. Farrell and M. M. Polycarpou, *Adaptive Approximation Based Control: Unifying Neural, Fuzzy and Traditional Adaptive Approximation Approaches*. New York, NY: Wiley, 2006.
- [5] S. S. Ge and T. Zhang, "Neural-network control of nonaffine nonlinear system with zero dynamics by state and output feedback," *IEEE Transactions on Neural Networks*, vol. 14, no. 4, pp. 900–918, 2003.
- [6] S. Labiod and T. M. Guerra, "Adaptive fuzzy control of a class of siso nonaffine nonlinear systems," *Fuzzy Sets and Systems*, vol. 158, no. 10, pp. 1126–1137, 2007.
- [7] W. Wang, Y. Chien, and T. Lee, "Observer-based ts fuzzy control for a class of general nonaffine nonlinear systems using generalized projection-update laws," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 3, pp. 493–504, 2011.
- [8] S. S. Ge and C. Wang, "Adaptive nn control of uncertain nonlinear pure-feedback systems," *Automatica*, vol. 38, no. 4, pp. 671–682, 2002.
- [9] C. Wang, D. J. Hill, S. S. Ge, and G. Chen, "An iss-modular approach for adaptive neural control of pure-feedback systems," *Automatica*, vol. 42, no. 5, pp. 723–731, 2006.
- [10] T. Zhang, H. Wen, and Q. Zhu, "Adaptive fuzzy control of nonlinear systems in pure feedback form based on input-to-state stability," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 1, pp. 80–93, 2010.
- [11] W. Wang, Y. Chien, Y. Leu, and T. Lee, "Adaptive t-s fuzzy-neural modeling and control for general mimo unknown nonaffine nonlinear systems using projection update laws," *Automatica*, vol. 46, no. 5, pp. 852–863, 2010.

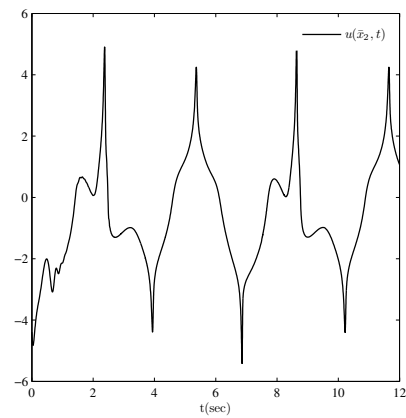


Fig. 3. The evolution of the required control signal $u(\bar{x}_2, t)$.

- [12] T. Zhang, Q. Zhu, and Y. Yang, "Adaptive neural control of non-affine pure-feedback non-linear systems with input nonlinearity and perturbed uncertainties," *International Journal of Systems Science*, vol. 43, no. 4, pp. 691–706, 2012.
- [13] Y. Chien, W. Wang, Y. Leu, and T. Lee, "Robust adaptive controller design for a class of uncertain nonlinear systems using online t-s fuzzy-neural modeling approach," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 2, pp. 542–552, 2011.
- [14] Y. Leu, W. Wang, and T. Lee, "Observer-based direct adaptive fuzzy-neural control for nonaffine nonlinear systems," *IEEE Transactions on Neural Networks*, vol. 16, no. 4, pp. 853–861, 2005.
- [15] J. Park, S. Huh, S. Kim, S. Seo, and G. Park, "Direct adaptive controller for nonaffine nonlinear systems using self-structuring neural networks," *IEEE Transactions on Neural Networks*, vol. 16, no. 2, pp. 414–422, 2005.
- [16] J. Park, G. Park, S. Kim, and C. Moon, "Direct adaptive self-structuring fuzzy controller for nonaffine nonlinear system," *Fuzzy Sets and Systems*, vol. 153, no. 3, pp. 429–445, 2005.
- [17] N. Hovakimyan, F. Nardi, and A. J. Calise, "A novel error observer-based adaptive output feedback approach for control of uncertain systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 8, pp. 1310–1314, 2002.
- [18] B. Yang and A. J. Calise, "Adaptive control of a class of nonaffine systems using neural networks," *IEEE Transactions on Neural Networks*, vol. 18, no. 4, pp. 1149–1159, 2007.
- [19] N. Hovakimyan, E. Lavretsky, and C. Cao, "Adaptive dynamic inversion via time-scale separation," *IEEE Transactions on Neural Networks*, vol. 19, no. 10, pp. 1702–1711, 2008.
- [20] C. P. Bechlioulis and G. A. Rovithakis, "Robust adaptive control of feedback linearizable mimo nonlinear systems with prescribed performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2090–2099, 2008.
- [21] —, "Adaptive control with guaranteed transient and steady state tracking error bounds for strict feedback systems," *Automatica*, vol. 45, no. 2, pp. 532–538, 2009.
- [22] —, "Robust partial-state feedback prescribed performance control of cascade systems with unknown nonlinearities," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2224–2230, 2011.
- [23] —, "Prescribed performance adaptive control for multi-input multi-output affine in the control nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1220–1226, 2010.
- [24] E. D. Sontag, *Mathematical Control Theory*. London, U.K.: Springer, 1998.