2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Resonance stability in Electrical Railway Systems – a dissipativity approach

I. Pendharkar
Bombardier Transportation Switzerland, Brown-Boveri Strasse 5, 8050 Zurich, Switzerland.
E-mail :ishan.pendharkar@ch.transport.bombardier.com

Abstract— **We propose a theory for stability analysis of railway systems based on energy exchange principles between the infrastructure and rolling stock. We show that if the rolling stock and infrastructure is dissipative with respect to specially constructed power functions (more general than voltage$\times$ current) stability is guaranteed. We thereby provide a proof of the Input Admittance Criterion (IAC) – a widely used empirical criterion in the railway industry – as special case of our results. Application of the theory is demonstrated with an example from a metro train system.**

I. Introduction

The term "Resonance stability" is used in the railway industry to mean stability of the electrical infrastructure–rolling stock system, in the face of resonances in the infrastructure. The stability problem is fundamental to electrical railway operations and stems from the energy exchange between rolling stock and infrastructure. Traditionally electrical traction power was provided by traction vehicles (locomotives) with DC and later AC motors controlled via stepped transformers, stepped resistances, rotating converter machinery, etc [6]. The use of such linear passive control schemes implied that the locomotive as a whole behaved as a passive system for every frequency. The passive behavior througout the frequency spectrum simplified the stability problem considerably – it was sufficient guarantee stability to low-frequency region around the fundamental frequency (16.7 hz, 50 hz, or 60 hz in most cases).

Since the early 1990s chopper based drives started being used for railway application. Today these are by far the most commonly used systems for supplying and controlling electrical energy to the traction vehicles. From a frequency response and stability point of view, choppers are fundamentally different than e.g. stepped transformers, in that they operate by switching and are therefore highly nonlinear. Due to the generated harmonics, such vehicles can potentially feed back energy at any arbitrary frequency into the infrastructure. Since modern traction vehicles can consume energy, as well as feed back energy into the infrastructure, the distinction between railway rolling stock and power supply infrastructure is diminishing. Indeed, because of precisely this diminishing distinction between sources and loads, a deeper study of interaction of traction vehicles with their environment is an important engineering problem. Failure to account for the energy exchange amongst vehicles and power supply sub-stations led to instability issues in Switzerland in the 1990s [1],[14]. That this occured synchronously with the onset of using PWM Inverter drives in railway operations is noteworthy.

An intensive study of interaction between vehicles and their environment was undertaken culminating with the pioneering work by Meyer et al [12]. They propose a frequency domain criterion, known in the railway industry as the Input Admittance Criterion, based on empirical principles to address the stability problem. The criterion was addressed and improved upon, mainly during the initial years of the present century, see e.g. [4] for some early results. The key aspect of the criterion is a differentiation between specifications for vehicle manufacturers and the infrastructure – which helps address the stability problem in an organizationally tractable manner. The criterion can be stated as follows: the Input Admittance Criterion (henceforth abbreviated as **IAC**) requires that the infrastructure have no resonant peaks at frequencies where a vehicle is active in order for the overall system (vehicle and environment) to be stable. Meyer [13] also showed that for the problem of stability analyis, a linear frequency domain model of the infrastructure and the vehicle is fairly accurate in the frequency range 50-1000 hz, which is a justification for the techniques used in this paper, through the underlying physical system is highly nonlinear.

The IAC and its implications are today well understood and used especially in western Europe, and a part of the Standard [5]. For vehicles manufacturers the criterion is usually interpreted as a "cut off frequency" above which a vehicle is not allowed to be active [2]. On the down side, the IAC as it currently exists, provides no bounds on permissible system resonances. Therefore, the IAC, though based on sound engineering insights, is not easy to implement as a stability-test without extensive modelling and simulation. Moreover a theoretical investigation or proof of the IAC is absent.

Contributions: This paper shows that the Input Admittance Criterion (IAC) can be viewed as a special case of a class of stability results obtained by considering the rolling stock and infrastructure as systems dissipative with respect to generalized power-supply functions (i.e. functions more general than Voltage$\times$Current, which is the physical definition of power) In this process we provide a proof for the IAC. We provide concrete upper bounds on permissible resonant peaks and thus weaken the criterion. We use the ideas in this paper to investigate a real-life stability problem encountered with a Metro Rail system.

This paper is organized as follows: in Section II we provide a brief theoretical background necessary for the results obtained in this paper. In Section III we prove a theorem for establishing stability of a infrastructure-rolling stock system. Section IV is about applications of the stability theory - in Section IV-A we consider the input admittance criterion ; Section IV-B is about the application of the stability theory to a typical Metro system.The analysis presented in this paper is based on the behavioral approach to systems theory [8]. One of the corner stones of the behavioral approach is a *partition free* treatment of system variables (i.e. inputs and outputs are not considered as being linked by a cause-effect relationship). The partition free treatment is of particular interest in electrical systems where no natural cause-effect order between voltages and currents exists.

II. BACKGROUND

We use the following notation in this paper: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^{\mathtt{n}}$ denotes the $\mathtt{n}$-dimensional vector space over $\mathbb{R}$. $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ denotes the set of $\mathtt{n}\times\mathtt{n}$ matrices over $\mathbb{R}$. $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}[D]$ denotes the set of $\mathtt{n}\times\mathtt{n}$ real polynomial matrices in the variable "D". Likewise $\mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\zeta,\eta]$ denotes the set of $\mathtt{n}\times\mathtt{n}$ real polynomial matrices in the variables "ζ" and "η". As mentioned before, the treatment in this paper is based on the behavioral approach to systems theory. In order to provide a short and not too technically cumbersome introduction to the behavioral approach, mathematical rigour in this section has, at some places, been sacrificed in the favour of principles and ideas – readers are requested to consult references e.g. [8] about the finer mathematical aspects of the theory. The behavior $\mathfrak{B}$ of a linear time-invariant electrical system with terminal voltages V and terminal currents I is defined as the set of solutions of a system of differential equations

$$\begin{bmatrix} R(\frac{d}{dt}) & S(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} = 0 \qquad (1)$$

where V,I are $\mathtt{n}$-valued functions of t. Since the object of our analysis are electrical networks with exponential (includig sinusoidal) voltages and currents we restrict the solutions to $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$, the space of infinitely continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$ The operators R and S are polynomial differential operators with real coefficients. The behavior $\mathfrak{B}$ is called controllable if one can "steer" an arbitrary trajectory $(V_1,I_1)\in\mathfrak{B}$ to any other trajectory $(V_2,I_2)\in\mathfrak{B}$ along a trajectory $(V_3,I_3)\in\mathfrak{B}$. See [8] for the exact technical definition. Controllability implies that V,I can be expressed as the image of a differential operator acting on a free variable:

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} Q(\frac{d}{dt}) \\ P(\frac{d}{dt}) \end{bmatrix} \ell,\ \ell\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}}) \text{ and free} \qquad (2)$$

In the sequel, unless explicitly mentioned otherwise, we assume that a behavior is controllable.

If two systems having behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$ given defined by equations of the form (1) are interconnected with one another, one obtains a behavior $\mathfrak{B}$ which is percisely $\mathfrak{B}_1\cap\mathfrak{B}_2$. Thus, $\mathfrak{B}$ is in general smaller than both $\mathfrak{B}_1,\mathfrak{B}_2$. In the context of electrical systems two types of interconnections are commonly used:

Definition 2.1: Given $\mathfrak{B}_1=(V_1,I_1)$ and $\mathfrak{B}_2=(V_2,I_2)$, the parallel connection is defined as $\mathfrak{B}=\mathfrak{B}_1\parallel\mathfrak{B}_2:=(V,I)$ where $V=V_1=V_2$ and $I=I_1+I_2$. A Serial connection is defined as $\mathfrak{B}=\mathfrak{B}_1-\mathfrak{B}_2:=(V,I)$ where $V=V_1+V_2$ and $I=I_1=I_2$. □

Given a behavior $\mathfrak{B}$ one can define the concept of *stability* in terms of the stability of equilibrium of the system defined by setting $I=0$ in Equation (1):

Definition 2.2: Given $\mathfrak{B}=(V,I)$, $\mathfrak{B}$ is defined to be asymptotically stable if

$$I=0\Rightarrow\lim_{t\to\infty}\|V(t)\|=0,\|\frac{d^kV(t)}{dt^k}\|=0,k=1,2,\cdots$$

for arbitrary initial conditions. Physically stability implies that when $I=0$ voltage V (and all its derivatives) reduce asymptotically to zero. □

The notion of *passivity* is well understood in the context of electrical networks. Passive systems are those in which the total electrical power is non-negative at all times:

Definition 2.3: A behavior $\mathfrak{B}=(V,I)$ is called passive if $\int_{-\infty}^t V^T\cdot I dt\geq 0\,\forall(V,I)\in\mathfrak{B}$ with compact support. Further, $\mathfrak{B}$ is called strictly passive if there exists $\varepsilon>0$ such that $\int_{-\infty}^t V^T\cdot I dt\geq\varepsilon\int_{-\infty}^t V^TV+I^TI dt>0$. □

Passive systems are commonly encountered in electrical installations - systems constructed out of resistors, capacitors, inductors and transformers are passive [7].

From a stability point of view, passivity is a very desirable property, since arbitrary interconnections of passive systems remain passive, and therefore stable [7]. Since however, as mentioned before, modern vehicles are not passive due to the harmonics generated during operation, and also due to the possibility of regenerative braking. Therefore it is of interest to investigate "passivity like" properties which, among others, help ensure stability of interconnected systems. In order to extend the motivation of passivity we introduce functionals called *Quadratic Differential Forms* or QDFs [11], which are quadratic functionals of voltages, currents, and their derivatives. For the sake of notational simplicity we combine the voltages and currents in a single variable w:

$$w=\begin{bmatrix} V \\ I \end{bmatrix} \qquad (3)$$

and define a QDF Q_Φ as a map

$$Q_\Phi:\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{2\mathtt{n}})\to\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \qquad (4)$$

defined by

$$(Q_\Phi(w))(t)=\sum_{k,l}(\frac{d^kw(t)}{dt^k})^T\Phi_{kl}(\frac{d^lw(t)}{dt^l}) \qquad (5)$$

where $\Phi_{kl}\in\mathbb{R}^{2\mathtt{n}\times2\mathtt{n}}$. The QDF Q_Φ can be associated with a two variable polynomial matrix $\Phi(\zeta,\eta)\in\mathbb{R}^{2\mathtt{n}\times2\mathtt{n}}[\zeta,\eta]$ defined by

$$\Phi(\zeta,\eta)=\sum_{k,l}\Phi_{kl}\zeta^k\eta^l \qquad (6)$$

In the two variable polynomial notation, ζ^k denotes the k-th derivative of w from the left while η^l denotes the l-th derivative of w from the right. Notice that for passive systems Q_Φ corresponds to the case $\zeta = 0, \eta = 0$

$$J := \Phi_{00} = \begin{bmatrix} 0 & \frac{I^{n}}{2} \\ \frac{I^{n}}{2} & 0 \end{bmatrix} \tag{7}$$

Analogous to passivity, where the power is defined by the QDF Q_J defined by Equation (7) one defines the notion of dissipativity for generalized power supply functions defined by a QDF Q_Φ:

Definition 2.4: Consider a controllable behavior $\mathfrak{B}$ with terminal signals $w := \begin{bmatrix} V \\ I \end{bmatrix}$. Then $\mathfrak{B}$ is said to be dissipative with respect to the power supply functional Q_Φ if $\int_{-\infty}^{t} Q_\Phi(w)dt \geq 0$ along all trajectories $w \in \mathfrak{B}$ having compact support. We say then, that $\mathfrak{B}$ is Φ-dissipative. Further, $\mathfrak{B}$ is called *strictly* Φ-dissipative if there exists $\varepsilon > 0$ such that $\int_{-\infty}^{t} Q_\Phi(w)dt \geq \varepsilon \int_{-\infty}^{t} w^T w dt > 0$ □

Clearly strict Φ-dissipativity is a stronger condition than Φ-dissipativity. We shall need to invoke it later in the context of stability analysis. As we shall see in the sequel, in the context of interaction of vehicles and infrastructure, generalized power supply functionals defined by a QDF Q_Φ arise in a natural way. There exists an algebraic condition for checking whether a given controllable behavior $\mathfrak{B}$ is Φ-dissipative: In the sequel we consider supply functions Q_Φ having the form $\Phi(\zeta,\eta) = K^T(\zeta)JK(\eta)$ with $K(D) \in \mathbb{R}^{n\times n}[D]$ and nonsingular. Such supply functions allow for a simple characterization of Φ-dissipative behaviors in terms of passive systems.

Theorem 2.5: Consider the behavior $\mathfrak{B}$ defined in Equation (2) and Q_Φ with $\Phi(\zeta,\eta) = K^T(\zeta)JK(\eta)$ with $K(D) \in \mathbb{R}^{n\times n}[D]$ and nonsingular. Consider the associated behavior $\mathfrak{B}_1\{(V_1, I_1)\}$ defined by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = K(\frac{d}{dt}) \begin{bmatrix} V \\ I \end{bmatrix}$$

Define the associated matrices

$$\begin{bmatrix} Q_1(D) \\ P_1(D) \end{bmatrix} = K(D) \begin{bmatrix} Q(D) \\ P(D) \end{bmatrix}$$

Let $\max\deg(Q_1, P_1) \geq \max\deg(Q, P)$ and Q_1, P_1 right coprime. Then the following statements are equivalent:

1) $\mathfrak{B}$ is strictly Φ-dissipative in the sense of Definition 2.4.
2) The Behavior $\mathfrak{B}_1$ is strictly passive in the sense of Definition 2.3
3) The rational function $Z := P_1 Q_1^{-1}$ is strictly positive real (SPR), i.e. Q_1 and P_1 are Hurwitz and $\exists \varepsilon > 0$ such that $Z(i\omega) + Z^T(-i\omega) > \varepsilon \forall \omega \in \mathbb{R}$
4) There exists a positive definite matrix $K > 0$ such that
$$\frac{d}{dt}(x^T K x) < Q_\Phi(w), w \in \mathfrak{B}$$
where x denote a minimal set of states for the behavior $\mathfrak{B}$

□

Theorem 2.5 is a generalization of the celebrated Kalman-Yakubovich-Popov (KYP) lemma for dissipative behaviors. The Theorem relates the notions of dissipativity Q_Φ and the existence of certain positive definite forms K satisfying a so called "dissipation inequality" $\frac{d}{dt}(x^T K x) < Q_\Phi(w)$. Functionals K are known as "storage function". The inequality means physically that the rate of change of a storage function doesn't exceed the power supplied. Storage functions are interesting since they can be used to construct Lyapunov functions and establish stability, as the next Section demonstrates, see [10] for further information on storage functions.

III. Stability of Interconnected Systems

In this Section we provide a result for establishing the stability of equilibrium of two interconnected systems based on dissipativity ideas. Intuitively the result shows that if a Φ-dissipative system is interconnected with another system which is $-\Phi$-dissipative (in an appropriate sense) the resulting system loses energy with time and is therefore stable. In this Section, we formulate and prove this intuitive idea in a rigorous manner.

Consider two behaviors $\mathfrak{B}_1 := (V_1, I_1)$ and $\mathfrak{B}_2 := (V_2, I_2)$ interconnected in parallel ($\mathfrak{B} = \mathfrak{B}_1 \parallel \mathfrak{B}_2$, Definition 2.1). Consider a supply function Q_Φ defined by

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & \Phi_{22} \end{bmatrix} \tag{8}$$

We associate with the supply function Q_Φ another supply function Q_Θ defined by by

$$\Theta = \begin{bmatrix} -\Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & -\Phi_{22} \end{bmatrix} \tag{9}$$

We stipulate that matrices Φ, and Θ be factorizable as $K^T(\zeta)JK(\eta)$. Then we have the following stability theorem:

Theorem 3.1: Let $\mathfrak{B}_1$ be strictly Φ-dissipative and $\mathfrak{B}_2$ be strictly Θ-dissipative. Let $\mathfrak{B} := \mathfrak{B}_1 \parallel \mathfrak{B}_2$ be well posed and controllable. Then $\mathfrak{B}$ is stable in the sense of Definition 2.2

IV. Application of Stability Results

We now revisit the problem of stability analysis arising out of energy exchange between vehicles and the infrastructure. Notice that an immediate corollary of Theorem 3.1 is that stability is guaranteed if two behaviors $\mathfrak{B}_1, \mathfrak{B}_2$ are passive and are interconnected as $\mathfrak{B}_1 \parallel \mathfrak{B}_2$. However as mentioned in the Introduction, with the use of chopper drives, passivity is generally speaking not guaranteed. We consider a system of finitely many traction vehicles with behaviors $\mathfrak{B}_1, \mathfrak{B}_2, \ldots$ that are connected in parallel with the infrastructure. The system consisting of interconnected vehicles is fed by a substation $\mathfrak{S}$ which forms a part of the infrastructure. As we shall show in the sequel, the relative positions of vehicles are unimportant from a stability point of view. We investigate under what conditions such a parallel connection still yields a stable system.

A. Input Admittance Criterion

As mentioned in the Introduction, the Input Admittance Criterion (IAC) is a criterion, popular predominantly in western Europe for ensuring compatibility between vehicles and the infrastructure. On the vehicles side, the IAC proposes a "cut off frequency" above which vehicles may not be active, and on the infrastructure side, there exist qualitative conditions on existence of resonance peaks. A rigorous mathematical proof or investigation of the IAC does not exist till date. We show in this Subsection how the IAC arises as a special case of the results presented in Theorem 3.1.

Consider a vehicle with behavior $\mathfrak{B}$ (i.e behavior with scalar valued terminals V and I) which is allowed to be "active" in the interval $(0,\omega_1]$ and passive in the interval (ω_1,∞). Mathematically the passivity constraint can be expressed as a frequency domain inequality using the Admitance function Y of the vehicle, and a weighting function involving parameters k,a:

$$\text{Real}\left[Y(i\omega)+\frac{k}{i\omega+a}\right]>\varepsilon\geq 0\;\forall\omega\in\mathbb{R} \tag{10}$$

We assume that Y is analytic in the closed right half complex plane.

Notice that the real part of the term $\frac{k}{i\omega+a}$ is $\frac{ka}{\omega^2+a^2}$, which monotonically decreases with ω. Therefore as $\omega\to\infty$ real part of $Y(i\omega)$ must be positive in order to ensure that the sum is positive for all $\omega\in(0,\infty)$ (passive behavior). On the other hand, at $\omega<<a$, the real part of $Y(i\omega)$ can be negative (active behavior) and yet the sum be positive. The inequality (10) can be viewed as a dissipativity condition on $\mathfrak{B}$ with respect to a special supply function, as the following theorem shows:

Theorem 4.1: The Admittance function of $\mathfrak{B}=(V,I)$ satisfies inequality (10) if and only if $\mathfrak{B}$ is Φ-dissipative with

$$\Phi=\begin{bmatrix}\frac{k(\zeta+\eta+2a)}{2} & \frac{(\zeta+a)(\eta+a)}{2}\\ \frac{(\zeta+a)(\eta+a)}{2} & 0\end{bmatrix}$$

,i.e. $\int_{-\infty}^{t}(\frac{d}{dt}V+aV)(kV+\frac{d}{dt}I+aI)dt>0$

The supply Q_Φ has an interesting property regarding parallel interconnections with passive systems, as the following lemma shows:

Lemma 4.2: Given $\mathfrak{B}$ Φ-dissipative (with Φ as defined in Theorem 4.1) the interconnection of $\mathfrak{B}_2=\mathfrak{B}\parallel\mathfrak{B}_1$ remains Φ-dissipative if $\mathfrak{B}_1$ is passive.

Lemma 4.2 means physically that interconnection of a Φ-dissipative behavior $\mathfrak{B}$ with a passive system with arbitrary parameters continues to be Φ-dissipative. We now investigate a general parallel interconnection of identical vehicles, all having behaviors $\mathfrak{B}$:

Lemma 4.3: Given $\mathfrak{B}$ Φ-dissipative, the parallel connection of n identical vehicles $\underbrace{\mathfrak{B}\parallel\mathfrak{B}\ldots\parallel\mathfrak{B}}_{n-\text{times}}$ is dissipative with respect to a QDF Q_Σ where

$$\Sigma(\zeta,\eta)=\begin{bmatrix}\frac{nk(\zeta+\eta+2a)}{2} & \frac{(\zeta+a)(\eta+a)}{2}\\ \frac{(\zeta+a)(\eta+a)}{2} & 0\end{bmatrix}$$

i.e. Σ is obtained from Φ by replacing k with nk.

Lemmas 4.2 and 4.3 enable us to study dissipativity properties of a network of n-vehicles, interconnected with one-another by passive components, predominantly high voltage transmission lines, having arbitrary parameters.

In order to establish stability of a parallel combination with infrastructure, $\mathfrak{B}\parallel\mathfrak{S}$, we only need invoke Theorem 3.1. Notice in particular that the Φ obtained in Theorem 4.1 meets all the required conditions of Theorem 3.1, in particular that of factorizability. We interconnect $\mathfrak{B}$ and the infrastructure $\mathfrak{S}$ in parallel $\mathfrak{B}\parallel\mathfrak{S}$ and obtain a supply function Q_Θ as defined in Theorem 3.1:

Lemma 4.4: With Φ as in Theorem 4.1 define Θ as follows

$$\Theta=\begin{bmatrix}-\frac{k(\zeta+\eta+2a)}{2} & \frac{(\zeta+a)(\eta+a)}{2}\\ \frac{(\zeta+a)(\eta+a)}{2} & 0\end{bmatrix}$$

If the behavior $\mathfrak{B}$ of the vehicle is strictly Φ-dissipative, and that of the infrastructure $\mathfrak{S}$ is strictly Θ-dissipative, then $\mathfrak{B}\parallel\mathfrak{S}$ is asymptotically stable.

We now obtain concrete frequency domain inequalities which illustrate the meaning of Lemma 4.4. For the purpose of illustration we work with the Impedance function of $\mathfrak{S}$ which we denote by Z. Dissipativity of $\mathfrak{S}$ implies that the following inequality is satisfied:

$$\underbrace{\frac{-ka}{a^2+\omega^2}\left[Z(-i\omega)Z(i\omega)\right]}_{\text{Term1}}+\underbrace{\frac{1}{2}\left[Z(-i\omega)+Z(i\omega)\right]}_{\text{Term2}}\geq\gamma>0\forall\omega\in\mathbb{R} \tag{11}$$

Since Term1 $\leq 0\forall\omega$, we must have that Term2 is positive, i.e.

$$\exists\varepsilon>0\text{ such that} Z(-i\omega)+Z(i\omega)\geq 2\varepsilon\forall\omega\in\mathbb{R} \tag{12}$$

Further Inequality (11) implies that

$$Z(-i\omega)Z(i\omega)<\frac{a^2+\omega^2}{2ka}\left[Z(-i\omega)+Z(i\omega)\right] \tag{13}$$

The conditions (12) and (13) imposed on Z can be summarized as follows:

Corollary 4.5: If the impedance function Z of the infrastructure $\mathfrak{S}$ satisfies the conditions

1) $Z(i\omega)+Z(-i\omega)\geq 2\varepsilon>0\forall\omega\in\mathbb{R}$, which implies that the phase of Z remains in the interval $(-\pi/2,\pi/2)\forall\omega$, and
2) $Z(-i\omega)Z(i\omega)<\frac{a^2+\omega^2}{ka}\varepsilon$

Then $\mathfrak{B}\parallel\mathfrak{S}$ is asymptotically stable. □

Corollary 4.5 implies that the magnitude of $Z(i\omega)$ is bounded by a "frequency dependent" weight $\frac{a^2+\omega^2}{ka}\varepsilon$. Notice that k,a remaining constant, the bound increases with increasing ω. We call the curve defined by plotting $\frac{a^2+\omega^2}{ka}\varepsilon$ against ω for fixed k,a,ε the "limiting curve" for 1 vehicle. Notice that the parameter a which defines the shape of the limiting curve (k,ε are only scaling factors) depends only on the vehicle properties (Inequality 10), and not on the properties of the infrastructure. Since as $\omega\to\infty$, the magnitude of the limiting curve is unbounded, it implies that high frequency

”resonant peaks” are allowed, but low frequency resnonances are only allowed if the inequalities in Corollary 4.5 are satisfied. Notice that on a logrithmic scale the magnitude of the limiting curve increases linearly with increasing ω, $\omega > a$

Remark 4.6: The Input Admittance Criterion (IAC) as it is currently known and used in the industry stipulates that there be no ”resonant peaks” in the infrastructure around frequencies in which a vehicle is active. Due to empirical origins of the criterion, no information about the theoretically permissible damping of these resonant peaks, or their frequencies is known. Corollary 4.5, while proving the essential principle behind the IAC, provides a stronger result, viz. it provides an explicit upper bound on the impedance function of the infrastructure. Thus at any given frequency one can determine whether resonant peaks (if these exist) lie below the limiting curve. However one must highlight the fact that Corollary 4.5 (and for that matter also the IAC) is a *sufficient* condition for stability; i.e. if Corollary 4.5 is not satisfied, it does not imply instability. □

The ideas in Corollary 4.5 are further illustrated by the following case study.

B. Case study - A typical Metro System

We demonstrate a problem of stability analysis encountered in a typical metro system. Modern vehicles have the possibility to actively cancel certain harmonics in the current (usually 3rd and/or 5th harmonics). The cancellation is required to ensure stable operation in the presence of older passively controlled vehicles which produce these harmonics of the fundamental current in the catenary. The cancellation is implemented by generating a out-of-phase current of the desired frequency. Note that active cancellation of current of a certain frequency necessarily implies that the vehicle is active around that frequency. In our case study we consider two cases, viz,

1) Case 1 - All vehicles without active harmonic cancellation,
2) Case 2 - All vehicles with active harmonic cancellation for the 3rd harmonic (150 hz).

The admittance Y of a single vehicle is measured (behavior $\mathfrak{B}_1$) as a small signal magnitude-phase relationship between sinusoidal voltages and currents having the same frequency. See [13] for measurement methods. From this frequency domain data, and parameters k and a are determined (e.g. by plotting the Nyquist diagram). Notice that an explicit identification of the vehicle model, i.e. Transfer function $Y(s)$ in Inequality (10) is not necessary for determination of k and a. These parameters are of course non-unique.

1) Stability without active harmonic cancellation : Using measurements of magnitude and phase, we select $a = 2\pi 100$ and $k = 0.3$. We consider a model of the infrastructure $\mathfrak{S}$ with the lowest resonance at about $2\pi 400$ rad/sec, see the Magnitude Plot in Figure 1. The impedance function Z of the infrastructure model satisfies $Z(-i\omega)+Z(i\omega) \geq 0.39$, and we define $\varepsilon = 0.39$. We see that the limiting curve for 1 vehicle in Figure 1, defined by $\frac{a^2+\omega^2}{ka}\varepsilon$, is well above the magnitude of Z. Therefore the inequality in Corollary 4.5 is satisfied, the parallel combination $\mathfrak{S} \parallel \mathfrak{B}_1$ is asymptotically stable.

Notice that if n vehicles without active harmonic cancellation are interconnected in parallel with the infrastructure $\mathfrak{S}$, the limiting curve is defined by $\frac{a^2+\omega^2}{2nka}$ (See Lemma 4.3). We see that the limiting curves till $n = 150$ lie above the magnitude plot of Z. Therefore the inequality in Corollary 4.5 is satisfied for $n = 1, 2 \ldots, 150$, and hence Therefore the parallel combination of 150 identical vehicles wih the infrastructure, $\mathfrak{S} \parallel \mathfrak{B}_1 \parallel \ldots \parallel \mathfrak{B}_{150}$, is asymptotically stable. As a consequence of Lemma 4.2, the 150 vehicles can be interconnected with one another using passive elements having arbitrary parameters. This theoretical result can be interpreted as the relative positions of vehicles is unimportant for stability; of course in practice other problems not considered in this analysis (e.g. power transfer) could also occur if, for instances, some of the high voltage cables are extremely long.

2) Stability with active harmonic cancellation 150 hz : We now consider the case when all vehicles actively cancel harmonics of 150 hz. Due to active harmonic cancellation, vehicles are now active around a higher frequency, than those in Section IV-B.1. The parameters k and a have to be necessarily larger so as to satisfy inequality (10). We select $a = 2\pi 170$ and $k = 1.8$. We connect vehicles all satisfying inequality (10) with the same infrastructure in Section IV-B.1. Note that ε, depending solely on the infrastructure remains unchanged, 0.39. Limiting curves for vehicles are shown in Figure 2. Note that the limiting curve for 1 vehicle lies well above the magnitude plot of Z. Therefore inequality in Corollary 4.5 is satisfied for 1 vehicle. The limiting curve for 150 vehicles however intersects the magnitude plot of Z, hence the inequality in Corollary 4.5 is no longer satisfied and stability can't be guaranteed. By plotting limiting curves for successively reducing number of vehicles (starting from 150) one can see that stability can be guaranteed only with about 25 vehicles, which is considerably less than the number obtained in Section IV-B.1. Though for $n > 25$ stability can't be guaranteed, this loss of guarantee in no way implies that the resulting system is unstable – indeed one of the shortcomings of Theorem 3.1 is that no information about possible instability is available.

V. Results and Conclusions

In this paper we propose a theory for addressing the problem of resonance stability in railway operations based on energy exchange between the rolling stock and the infrastructure. We show that if the rolling stock and infrastructure are dissipative with respect to a generalized power supply function consisting of voltages, currents, and their derivatives, an interconnection of the rolling stock and infrastructure yield a stable system. We obtain a proof for Input Admittance Criterion (IAC)– an empirical criterion widely used in the railway industry – as a special case of our theory. Further, we provide concrete upper bounds on the magnitude

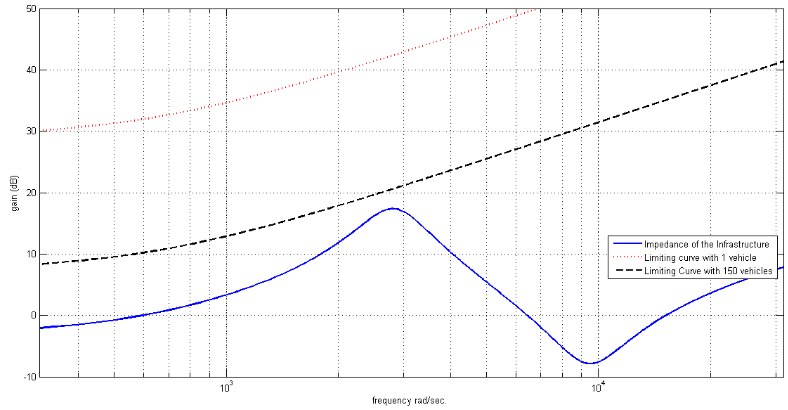

Fig. 1. Metro System - vehicles have no active harmonic cancellation

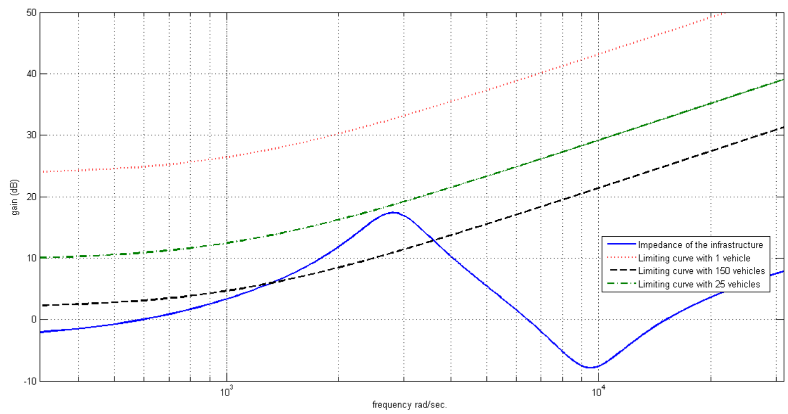

Fig. 2. Metro System - vehicles have active harmonic cancellation around 150 hz

of the Impedance function for the infrastructure, which is new, and serves to strengthen the IAC. As an application of the theory proposed in this paper, we consider the stability problem in a typical Metro System. Using measurements of vehicle admittance, we show how energy exchange amongst vehicles and the infrastructure effect stability. We obtain thereby an estimate of the number of maximum vehicles in the network which guarantee stability when operating with a given infrastructure. Further we consider the case of vehicles with active harmonic cancellation – a widely used technique in the train industry for suppressing currents of certain harmonic frequencies. We show how, if the infrastructure has low resonance frequencies, the number of permissible vehicles can be dramatically reduced.

REFERENCES

[1] "Raetselhafte Lokomotivstoerungen im Raum Zuerich" *Schweizer Eisenbahn-Revue* , 5(1995) pp 171. (in German)

[2] M. Aeberhard "Anforderungen an die Eingangs-Admittanz von Umrichtertriebfahrzeugen" Swiss Federal Railways, I-20005 (2009) (in German)

[3] M. Aeberhard "Spezifikation fuer Triebfahrzeug-Frequenzgangmessungen" Swiss Federal Railways, 47.10.002 (2007) (in German)

[4] Andrew D.B. Paice, M. Meyer, "Rail network modelling and stability: the input admittance criterion" *In proceedings of MTNS Perpignan, 2000*

[5] Cenelec "50388-Chapter 10: Railway applications Compatibility between rolling stock and train detection systems" (Final draft for vote) *European Technical Specification, 2011*

[6] Z. Filipovic *"Elektrische Bahnen"* Springer Verlag, 1989 (in German)

[7] R.W. Newcomb, *"Linear Multiport Synthesis"* McGraw Hill, New York, 1966.

[8] J.W. Polderman, J.C. Willems, *"Introduction to mathematical systems theory: A behavioral approach"* Springer-Verlag, 1997.

[9] I. Pendharkar, Harish K. Pillai *A parametrization for dissipative behaviors – the matrix case* International Journal of Control, 82(2009), pp. 1006-1017.

[10] J.C. Willems, "Dissipative dynamical systems, Part 1: General theory" *Archives for Rational Mechanics and Analysis*, 45 (1972), pp 321-351.

[11] J.C. Willems and H.L. Trentelman, 1998, "On Quadratic Differential Forms", *SIAM Journal of Control and Optimization*, 36 (1998), pp 1703-1749.

[12] M.Meyer, M. Aeberhard *et al* "Messung des Frequenzgangs von Triebfahrzeugen" *Oldenburg Industriverlag, 2008* (in German)

[13] M.Meyer, M. Stadelmann *et al* "Auswirkungen der Netzresonanzen im Loetschberg-Basistunnel" *Oldenburg Industriverlag, 2008* (in German)

[14] M.Meyer, J. Schoening "Netzstabilitaet in grosseren Bahnnetzen" *Schweizer Eisenbahn-Revue*, 7-8(1999), pp 312-317 (in German)