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Stability of Switching Systems and Generalized Joint Spectral Radius

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Abstract— **This paper studies the mean stability of stochastic switching linear systems. We first show that the mean stability is characterized by an extended version of so called generalized joint spectral radius. Then it is shown that, under an invariance condition, the quantity can be computed as the spectral radius of a certain matrix associated with the given switching system. Also we show that the mean square stability is equivalent to the existence of a Lyapunov function. Our results are illustrated by numerical examples.**

I. Introduction

Switching systems have attracted much attention in systems theory in the last two decades and there are various results on the stability of such systems. The stability theory of switching systems can be divided into two parts. One is the stability analysis of switching systems with a deterministic switching signals, and the other is that with stochastic switching signals. The paper [15] by Lin and Antsaklis gives a detailed overview of the stability theory of deterministic switching systems.

For stochastic switching systems, so called almost sure stability, which requires that the trajectory of the system converges to 0 with probability 1, can be considered as one of the most natural notions of stability [13]. However this stability is difficult to check in practice because it is characterized by a quantity called the top Lyapunov exponent, whose computation is in general a hard problem [20]. Some attempts to compute the quantity can be found in [12] and the references therein.

It is mainly this difficulty that has been motivated many authors to study another notion of stability, called p-th mean stability, which requires that the expected value of the p-th power of the norm of the trajectory converges to 0 (for detail see Definition 2.6). It is known [8], [10], [6] that this stability gives a sufficient condition for the almost sure stability of a wide class of switching systems and also the sufficient condition becomes necessary in the limit $p \to 0$.

The most basic form of stochastic switching systems is given by

$$\Sigma : x(k+1) = A_k x(k), \quad x(0) \in \mathbb{R}^d \tag{1}$$

where A_1, A_2, … are random real $d \times d$ matrices. For the case when $\{A_k\}_{k=1}^{\infty}$ forms a Markov process with finite states, Fang et. al. [7] gave various conditions for the mean stability. Also Mariton [16] gave a necessary and sufficient condition of the mean stability. The mean square stability was extensively studied in [5]. However, for the case when each matrix A_k is allowed to take infinitely many values, there is no practical characterization of the mean stability. A partial result can be found in [9], where the author studied the mean square stability under the assumption that each A_k is a so called interval matrix.

The aim of this paper is to give a simple necessary and sufficient condition of the mean stability of the system Σ under a certain invariance condition, assuming that the matrices $\{A_k\}_{k=1}^{\infty}$ are independently and identically distributed. In contrast with existing results, we allow each matrix A_k to take infinitely many values. Our condition uses an extended version of so called generalized joint spectral radius [11]. We will also study the mean square stability, a special case of the mean stability. It will be shown that the mean square stability is equivalent to the existence of a Lyapunov function.

This paper is organized as follows. After preparing necessary mathematical notation and conventions, in Section II we first extend the notion of generalized joint spectral radius and also give a review of the mean and moment stability. Section III gives a characterization of the mean stability. The mean square stability is studied in Section IV.

A. Mathematical Preliminaries

Let $\mathbb{R}$ and $\mathbb{Q}$ denote the space of real and rational numbers, respectively. Let $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$. The vector space $\mathbb{R}^d$ of d-products of $\mathbb{R}$ becomes a Hilbert space with the inner product $(x,y) := y^\top x$ and the associated norm $\|x\| := \sqrt{(x,x)}$. Also $\mathbb{R}^d$ admits another norm $\|x\|_p := [\sum_{\ell=1}^d |x_\ell|^p]^{1/p}$ for $p \geq 1$ and $\|x\|_\infty := \max_{1\leq \ell \leq d}|x_\ell|$. It is well known that all the norms on $\mathbb{R}^d$ are equivalent, i.e., for any pair of two norms $\|\cdot\|$ and $|||\cdot|||$ on $\mathbb{R}^d$ there exist positive constants C_1 and C_2 such that, for every $x \in \mathbb{R}^d$ we have $C_1|||x||| \leq \|x\| \leq C_2|||x|||$. The standard basis of $\mathbb{R}^d$ is denoted by $e_1, \dots, e_d$. For a set X its cardinality is denoted by $|X|$.

A subset $K \subset \mathbb{R}^d$ is called a *cone* if K is closed under multiplication by nonnegative scalars. The cone is said to be *solid* if it possesses a nonempty interior. We say that a cone is *pointed* if it contains no line; i.e., if $x, -x \in K$ then $x = 0$. We say that K is *proper* if it is closed, convex, has nonempty interior, and pointed. In proving our main result the following lemma (see, e.g., [19]) plays an important role.

Lemma 1.1: Let K be a proper cone. Then there exists $f \in \mathbb{R}^d$ and a norm $|||\cdot|||$ on $\mathbb{R}^d$ such that

$$|||x||| = f^\top x \tag{2}$$

for every $x \in K$.

Let X be a subset of $\mathbb{R}^{d\times d}$. X is said to leave K invariant if, for every $M \in X$ and $v \in K$ we have $Mv \in K$. Define $\mathbb{R}_+ X := \{rM : r \geq 0,\ M \in X\}$.

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For a matrix $M \in \mathbb{R}^{m\times d}$ its maximal singular value is denoted by $\|M\|$, which gives a norm on $\mathbb{R}^{m\times d}$. For $p > 0$ define the norm $\|\cdot\|_p$ by $\|M\|_p := \left(\sum_{\ell=1}^{d} \|Me_\ell\|^p\right)^{1/p}$. If M is square we denote its spectral radius by $\rho(M)$. The *Kronecker product* [4] of the matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{m\times n}$ is defined as the block matrix

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix} \in \mathbb{R}^{(pm)\times(qn)}.$$

It is easy to check that the operator $\otimes$ is associative. For a positive integer p define the Kronecker power $A^{\otimes p}$ by $A^{\otimes 1} := A$ and $A^{\otimes (p)} = A^{\otimes (p-1)} \otimes A$ recursively for general p. It can be checked [4] that

$$(AB)^{\otimes p} = A^{\otimes p} B^{\otimes p}. \tag{3}$$

Lemma 1.2: Let $X \subset \mathbb{R}^{d\times d}$ and $\{M_k\}_{k=1}^{\infty}$ be a sequence in X. Let $\varepsilon > 0$ be arbitrary. Then there exists a pairwise distinct sequence $\{M'_k\}_{k=1}^{\infty} \subset \mathbb{R}_+ X$ such that

$$\|M_k - M'_k\| \le \varepsilon \tag{4}$$

for every k.

Proof: We use an induction. Let us set $M'_1 = M_1$. Then assume that there exist $M'_1, \dots, M'_{k_0}$ satisfying (4). To choose M'_{k_0+1} consider the set $S = \{M_{k_0+1}(1 + \ell^{-1}\varepsilon/\|M_{k_0+1}\|^{-1}\}_{\ell=1}^{k_0+1}$. Since $|S| = k_0 + 1$ there exists $s \in S$ that is different from every M'_k, $1 \le k \le k_0$. If we set $M'_{k_0+1} := s$ then, by definition of S, the inequality (4) holds for every $k \le k_0 + 1$. This completes the proof. ■

Let X be a subset of the normed space $\mathbb{R}^{d\times d}$. Let $P(X)$ denote the set of all probability measures on X. For a measurable function f on X we denote its expected value with respect to $\mu \in P(X)$ by $E_\mu[f]$. The subscript μ will be omitted when the underlying measure μ is obvious. The support [1] of μ, if it exists, is a closed set, denoted by $\operatorname{supp}\mu$, satisfying $\mu((\operatorname{supp}\mu)^c) = 0$, and if G is open and $G \cap \operatorname{supp}\mu \neq \emptyset$ then $\mu(G \cap \operatorname{supp}\mu) > 0$. Dirac's delta distribution on $x \in X$ is denoted by δ_x. For a positive integer k let μ^k denote the k-product of μ. The sequence $\{\mu_n\}_{n=1}^{\infty} \subset P(X)$ is said to converge to $\mu \in P(X)$ weakly if $\int_X f\,d\mu_n \to \int_X f\,d\mu$ for each bounded continuous function f on X. About the weak convergence the next lemma is basic.

Lemma 1.3 ([1, Theorem 15.10]): Let X be a subset of $\mathbb{R}^{d\times d}$. The set of probability measures on X with finite support is dense in $P(X)$.

Finally let us state a version of well-known Fekete's subadditive lemma [17, Ch. 3, Sect. 1].

Lemma 1.4: Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $a_{k+\ell} \le a_k a_\ell$ for all k and ℓ. Then the limit $\lim_{k\to\infty} a_k^{1/k}$ exists and equals $\inf_{k\ge 1} a_k^{1/k}$.

Proof: Apply Fekete's subadditive Lemma to the subadditive sequence $\{\log a_k\}_{k=1}^{\infty}$. ■

II. Generalized Joint Spectral Radius and Stability of Switching Systems

This section extends the notion of generalized joint spectral radius [14], [18] for matrices to probability distributions and then shows that the mean stability can be characterized by this quantity.

Throughout this paper μ denotes a probability distribution on $\mathbb{R}^{d\times d}$ and $A_1, A_2, \dots$ denote the random variables that follow μ independently.

Let us start from the following definition.

Definition 2.1: For $p > 0$ the *generalized joint spectral radius with exponent* p (*p-radius* for short) of μ is defined by

$$\rho_{p,\mu} := \lim_{k\to\infty} \left(E[\|A_k \cdots A_1\|^p]\right)^{1/pk}$$

Example 2.2: If μ is the uniform distribution on a finite subset $\mathscr{M}$ of $\mathbb{R}^{d\times d}$ then

$$\rho_{p,\mu} = \lim_{k\to\infty} \left[|\mathscr{M}|^{-k} \sum_{A_1,\dots,A_k \in \mathscr{M}} \|A_k \cdots A_1\|^p \right]^{1/(pk)},$$

which is the classical definition [14], [18] of generalized joint spectral radius for the finite set of matrices.

Throughout this paper we place the following assumption on the distribution μ.

Assumption 2.3: The support of μ is bounded.

This assumption ensures the existence of p-radius:

Lemma 2.4: The p-radius $\rho_{p,\mu}$ exists for all $p > 0$ and $\mu \in P(\mathbb{R}^{d\times d})$. Moreover if we let

$$h_{k,\mu} := E[\|A_k \cdots A_1\|^p] \tag{5}$$

then it holds that

$$\rho_{p,\mu}^p = \inf_{k\ge 1} h_{k,\mu}^{1/k}. \tag{6}$$

Proof: First notice that $h_{k,\mu}$ in (5) is well defined for every k because $\|A_k \cdots A_1\|$ is essentially bounded by Assumption 2.3. Since for every k and ℓ we have

$$\begin{aligned} h_{k+\ell,\mu} &\le E[\|A_{k+\ell} \cdots A_{1+\ell}\|^p \cdot \|A_\ell \cdots A_1\|^p] \\ &= E[\|A_{k+\ell} \cdots A_{1+\ell}\|^p] E[\|A_\ell \cdots A_1\|^p] = h_{k,\mu} h_{\ell,\mu}, \end{aligned}$$

Lemma 1.4 immediately implies that $\lim_{k\to\infty} h_{k,\mu}^{1/k} = \rho_{p,\mu}^p$ exists and the equation (6) holds. ■

The next lemma follows from the standard argument of the theory of L^p spaces.

Lemma 2.5: If $1 \le p \le q$ then $\rho_{p,\mu} \le \rho_{q,\mu}$.

A. Mean Stability

Consider the stochastic switching system Σ defined in (1) and assume that $\{A_k\}_{k\ge 1}$ follow μ independently. We denote the trajectory of Σ with the initial state $x(0) = x_0$ by $x(\cdot; x_0)$. To this system Σ we associate another stochastic switching system $\tilde{\Sigma}$ on $\mathbb{R}^{d\times d}$ defined by

$$\tilde{\Sigma} : X(k+1) = A_k X(k), \quad X(0) = I_d,$$

where I_d is the identity matrix with size d.

Definition 2.6:

- We say that Σ is *exponentially stable in p-th mean* (*p-th mean stable* for short) if there exist $M > 0$ and $\beta > 0$ such that, for any initial state x_0,

$$E[\|x(k; x_0)\|^p] \le M e^{-\beta k} \|x_0\|^p. \tag{7}$$

- We say that $\tilde{\Sigma}$ is *exponentially stable in p-th mean* (*p-th mean stable* for short) if there exist $M > 0$ and $\beta > 0$ such that

$$E[\|X(k)\|^p] \le Me^{-\beta k}. \quad (8)$$

Remark 2.7: By the equivalence of the norms on a finite dimensional vector space, we can replace the norm $\|\cdot\|$ in the definition above by any other norm on $\mathbb{R}^d$.

Checking the p-th mean stability can be reduced to checking the inequality (7) for only finitely many initial states x_0:

Lemma 2.8: Σ is p-th mean stable if and only if there exist $M, \beta > 0$ and a basis $\{v_\ell\}_{\ell=1}^d$ of $\mathbb{R}^d$ such that (7) holds for every $x_0 = v_\ell$, $\ell = 1, \ldots, d$.

Proof: The necessity is obvious. To prove the sufficiency let $x_0 \in \mathbb{R}^d$ be arbitrary and write $x_0 = \sum_{\ell=1}^d \alpha_\ell v_\ell$ with $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$. Since $x(k;x_0) = \sum_{\ell=1}^d \alpha_\ell x(k;v_\ell)$ it follows that

$$\|x(k;x_0)\|^p \le \left(\sum_{\ell=1}^d \|\alpha_\ell x(k;v_\ell)\|\right)^p \le C_1 \sum_{\ell=1}^d |\alpha_\ell|^p \|x(k;v_\ell)\|^p$$

for some $C_1 > 0$. Therefore

$$\begin{aligned} E[\|x(k;x_0)\|^p] &\le C_1 \sum_{\ell=1}^d |\alpha_\ell|^p E[\|x(k;v_\ell)\|^p] \\ &\le (C_1 M)e^{-\beta k} \sum_{\ell=1}^d |\alpha_\ell|^p \|v_\ell\|^p \quad \text{(by (7))} \\ &\le (C_1 C_2 M)e^{-\beta k} \|x_0\|^p \end{aligned}$$

for some $C_2 > 0$ because the function $|||x_0||| := (\sum_{\ell=1}^d |\alpha_\ell|^p \|v_\ell\|^p)^{1/p}$ gives a norm on $\mathbb{R}^d$. Therefore Σ is p-th mean stable. ∎

The p-radius characterizes p-th mean stability as follows.

Proposition 2.9: The following conditions are equivalent:

1) Σ is exponentially stable in p-th mean;
2) $\tilde{\Sigma}$ is exponentially stable in p-th mean;
3) $\rho_{p,\mu} < 1$.

Proof: We will show the cycle 1) ⇒ 2) ⇒ 3) ⇒ 1).

[1) ⇒ 2)]: If Σ is exponentially stable in p-th mean then

$$\begin{aligned} E[\|X(k)\|_p^p] &= E\left[\sum_{\ell=1}^d \|X(k)e_\ell\|^p\right] \\ &= \sum_{\ell=1}^d E[\|x(k;e_\ell)\|^p] \\ &\le \sum_{\ell=1}^d Me^{-\beta k}\|e_\ell\|^p \quad \text{(by (7))} \\ &= Mde^{-\beta k} \end{aligned}$$

for some $M > 0$ and $\beta > 0$. Therefore, by Remark 2.7, the system $\tilde{\Sigma}$ is exponentially stable in p-th mean.

[2) ⇒ 3)]: Assume that $\tilde{\Sigma}$ is exponentially stable in p-th mean. Then, by (8), there exist $M, \beta > 0$ such that $E[\|A_k \cdots A_1\|^p]^{1/pk} \le M^{1/pk} e^{-\beta/p}$ for every sufficiently large k and hence $\rho_{p,\mu} \le e^{-\beta/p} < 1$.

[3) ⇒ 1)]: Assume $\rho_{p,\mu} < 1$. Then there exists $\beta > 0$ such that if k is sufficiently large then $E[\|A_k \cdots A_1\|^p]^{1/pk} < e^{-\beta}$. Let $x_0 \in \mathbb{R}^d$ be arbitrary. For every sufficiently large k we have $E[\|x(k;x_0)\|^p] \le E[\|A_k \cdots A_1\|^p]\|x_0\|^p \le e^{-\beta pk}\|x_0\|^p$. This implies the exponential p-th mean stability of Σ. ∎

This proposition shows that the mean stability of Σ can be checked by computing the p-radius of the probability distribution associated with Σ. It is known [18], [21], [11] that, under a certain condition, the classical generalized joint spectral radius for finite sets of matrices can be computed easily. We can restate those results in out context as follows.

Theorem 2.10 ([18], [21], [11]): Let μ be the uniform distribution on a finite subset of $\mathbb{R}^{d\times d}$ and let p be an integer. If either

$$\operatorname{supp}\mu \text{ leaves a proper cone invariant} \quad \text{(A)}$$

or

$$p \text{ is even} \quad \text{(B)}$$

then

$$\rho_{p,\mu} = \left[\rho\left(|\operatorname{supp}\mu|^{-1} \sum_{A\in\operatorname{supp}\mu} A^{\otimes p}\right)\right]^{1/p} \quad (9)$$

holds.

B. Moment Stability

Before proceeding to the next section we introduce another stability notion called moment stability.

Definition 2.11: We say that the system Σ is *exponentially stable in p-th moment* (*p-th moment stable* for short) if there exist $M, \beta > 0$ such that, for every $x_0 \in \mathbb{R}^d$,

$$\|E[x(k;x_0)^{\otimes p}]\| \le Me^{-\beta k}\|x_0\|^p. \quad (10)$$

It is easy to derive the next sufficient condition for moment stability.

Proposition 2.12: If $\rho(E_\mu[X^{\otimes p}]) < 1$ then Σ is p-th moment stable.

Proof: Assume $\rho(E[X^{\otimes p}]) < 1$. Since (3) yields $x(k+1;x_0)^{\otimes p} = (A_k x(k;x_0))^{\otimes p} = A_k^{\otimes p} x(k;x_0)^{\otimes p}$, it follows that

$$E[x(k+1;x_0)^{\otimes p}] = E_\mu[X^{\otimes p}]E[x(k;x_0)^{\otimes p}]$$

and hence $E[x(k;x_0)^{\otimes p}] = E_\mu[X^{\otimes p}]^k x_0^{\otimes p}$. Therefore the assumption $\rho(E_\mu[X^{\otimes p}]) < 1$ gives the estimate of the form (10). ∎

III. Computation of Generalized Joint Spectral Radius

The aim of this section is to prove the next theorem, which extends Theorem 2.10 by removing the assumption that μ is the uniform distribution on a finite set.

Theorem 3.1: Let p be an integer. If either the condition (A) or (B) is true then

$$\rho_{p,\mu} = \rho(E_\mu[X^{\otimes p}])^{1/p}. \quad (11)$$

holds.

We start the proof of this theorem with showing

$$\rho_{p,\mu} \ge \rho(E_\mu[X^{\otimes p}])^{1/p}. \quad (12)$$

For the proof of this inequality we need the next lemma.

Lemma 3.2: Let $\mu \in P(\mathbb{R}^{d\times d})$. Then there exists a sequence $\{\mu_n\}_{n=1}^{\infty} \subset P(\mathbb{R}^{d\times d})$ such that

1) $\mu_n \to \mu$ weakly as $n \to \infty$ and
2) each μ_n is the uniform measure on a finite subset of $\mathbb{R}^{d\times d}$.

Moreover, if $\operatorname{supp}\mu$ leaves a cone K invariant then so does $\operatorname{supp}\mu_n$.

Proof: Let $X := \operatorname{supp}(\mu)$. Then the restriction $\mu|_X$ is in $P(X)$. By Lemma 1.3 there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of probability measures on X with finite support such that $\lambda_n \to \mu$ weakly. Each λ_n is a linear combination of Dirac's delta measures, and their coefficient can be assumed to be in $\mathbb{Q}$ without loss of generality because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then we can write $\lambda_n = \frac{1}{K_n}\sum_{k=1}^{K_n}\delta_{M_{k,n}}$ using a not necessarily pairwise distinct sequence $\{M_{k,n}\}_{k=1}^{K_n} \subset X$.

By Lemma 1.2 there exists a pairwise distinct sequence $\{M'_{k,n}\}_{k=1}^{K_n} \subset \mathbb{R}_+X$ such that

$$\max_{1\le k\le K_n} \|M_{k,n} - M'_{k,n}\| \le 1/n. \tag{13}$$

Now we define $\mu_n = \frac{1}{K_n}\sum_{k=1}^{K_n}\delta_{M'_{k,n}}$, which is the uniform measure on the pairwise distinct set $\{M'_{k,n}\}_{k=1}^{K_n}$.

Let us show that μ_n weakly converges to μ. Let f be a bounded and continuous function on $\mathbb{R}^{d\times d}$. Take a sufficiently large closed ball $B \subset \mathbb{R}^{d\times d}$ with center 0 that contains the points $M_{k,n}, M'_{k,n}$ for all k and n. This is possible by Assumption 2.3 and (13). Since f is uniformly continuous on B, from (13) it follows that

$$\begin{aligned}\left|\int f\,d\lambda_n - \int f\,d\mu_n\right| &\le \frac{1}{K_n}\sum_{k=1}^{K_n}|f(M_{k,n}) - f(M'_{k,n})| \\ &\le \sup_{1\le k\le K_n}|f(M_{k,n}) - f(M'_{k,n})| \to 0\end{aligned}$$

as $n \to \infty$. Thus the triangle inequality and the weak convergence $\lambda_n \to \mu$ readily shows that μ_n weakly converges to μ.

Finally if $X = \operatorname{supp}\mu$ leaves a cone invariant then $\operatorname{supp}\mu_n$ also leaves the cone invariant because each $M'_{k,n}$ is a positive scalar multiplication of $M_{k,n}$, which leaves the cone invariant. ■

Now we can prove the inequality (12).

Proof of the inequality (12): Let $\mu \in P(\mathbb{R}^{d\times d})$ and $p > 0$ satisfy the assumption of Theorem 3.1. Take the sequence $\{\mu_n\}_{n=1}^{\infty} \subset P(\mathbb{R}^{d\times d})$ given by Lemma 3.2. Then we can see that each pair (μ_n, p) satisfies the condition of Theorem 2.10 and therefore $\rho_{p,\mu_n} = \rho(E_{\mu_n}[X^{\otimes p}])^{1/p}$. Since the spectral radius of a matrix depends continuously on the matrix, from (6) it follows that

$$\begin{aligned}\limsup_{n\to\infty}\inf_k \frac{1}{k}h_{k,\mu_n} &= \limsup_{n\to\infty}\rho_{p,\mu_n}^p \\ &= \limsup_{n\to\infty}\rho(E_{\mu_n}[X^{\otimes p}]) \\ &= \rho(\lim_{n\to\infty}E_{\mu_n}[X^{\otimes p}]) \\ &= \rho(E_{\mu}[X^{\otimes p}]). \quad (\mu_n \to \mu \text{ weakly})\end{aligned} \tag{14}$$

On the other hand, since, for each fixed k, μ_n^k converges to μ^k weakly as $n \to \infty$ and hence $h_{k,\mu_n} \to h_{k,\mu}$ as $n \to \infty$, we have

$$\begin{aligned}\limsup_{n\to\infty}\inf_k h_{k,\mu_n}^{1/k} &\le \inf_k \limsup_{n\to\infty} h_{k,\mu_n}^{1/k} \\ &= \inf_k h_{k,\mu}^{1/k} = \rho_{p,\mu}^p. \quad (\text{by (6)})\end{aligned}$$

This inequality and (14) proves the inequality (12). ■

To complete the proof of the equality (11) we need the next result, which is itself of systems theoretical interest.

Proposition 3.3: Under the assumption of Theorem 3.1, if Σ is p-th moment stable then Σ is p-th mean stable.

Proof: Assume that Σ is p-th moment stable. Take $M, \beta > 0$ such that (10) holds for every x_0. First let us assume that p is an even integer. Let $x_0 \in \mathbb{R}^d$ be arbitrary. Then

$$\begin{aligned}E[\|x(k;x_0)\|_p^p] &= \sum_{\ell=1}^{d} E\left[x(k;x_0)_\ell^p\right] \quad (p \text{ is even}) \\ &\le \sum_{\ell=1}^{d}\|E[x(k;x_0)^{\otimes p}]\|_\infty \\ &\le dCMe^{-\beta k}\|x_0\|^p. \quad (\text{by (10)})\end{aligned}$$

Therefore Σ is exponentially stable in p-th mean.

Next assume that $\operatorname{supp}\mu$ leaves a proper cone K invariant and p is an integer. Since the cone K has a nonempty interior, we can take a basis $\{v_\ell\}_{\ell=1}^d$ of $\mathbb{R}^d$ from K. Then, since $\operatorname{supp}\mu$ leaves K invariant, for every k and ℓ we have $x(k;v_\ell) \in K$ with probability 1. Now take the vector $f \in \mathbb{R}^d$ and the norm $|||\cdot|||$ given by Lemma 1.1. Then (2) gives that

$$E[|||x(k;v_\ell)|||^p] = E\left[\left(f^\top x(k;v_\ell)\right)^p\right]. \tag{15}$$

Since $\left(f^\top x(k;v_\ell)\right)^p = \left(f^\top x(k;v_\ell)\right)^{\otimes p} = \left(f^\top\right)^{\otimes p} x(k;v_\ell)^{\otimes p}$,

$$\begin{aligned}E\left[\left(f^\top x(k;v_\ell)\right)^p\right] &= \left(f^\top\right)^{\otimes p} E[x(k;v_\ell)^{\otimes p}] \\ &\le \left\|(f^\top)^{\otimes p}\right\| \cdot \|E[x(k;v_\ell)^{\otimes p}]\| \\ &< \|f^{\otimes p}\| M e^{-\beta k}\|v_\ell\|^p. \quad (\text{by (10)})\end{aligned}$$

This inequality and (15) shows that the estimate of the form (7) holds for $x_0 = v_1, \dots, v_d$. Since $\{v_\ell\}_{\ell=1}^d$ forms a basis, by Lemma 2.8, Σ is p-th mean stable. ■

Now we can prove our main result of Theorem 3.1.

Proof of Theorem 3.1: By (12) it is sufficient to show $\rho_{p,\mu} \le \rho(E[X^{\otimes p}])^{1/p}$. Assume the contrary; i.e., that there exists $\mu \in P(\mathbb{R}^{d\times d})$ such that $\rho_{p,\mu} > \rho(E[X^{\otimes p}])^{1/p}$. For $\alpha > 0$ define $\mu_\alpha \in P(\mathbb{R}^{d\times d})$ by $\mu_\alpha(G) := \mu(\alpha^{-1}G)$ for all measureable subsets G of $\mathbb{R}^{d\times d}$. We can show that

$$\begin{gathered}\rho_{p,\mu_\alpha} = |\alpha|\rho_{p,\mu}, \\ \rho\left(E_{\mu_\alpha}[X^{\otimes p}]\right)^{1/p} = |\alpha|\rho\left(E_\mu[X^{\otimes p}]\right)^{1/p}.\end{gathered}$$

Therefore, by our assumption, there exists $\alpha > 0$ such that $\rho_{p,\mu_\alpha} > 1 > \rho(E_{\mu_\alpha}[X^{\otimes p}])^{1/p}$. The second part of this inequality implies that Σ is exponentially stable in p-th mean by Proposition 2.12 and Proposition 3.3. On the other hand the first part implies the contrary by Proposition 2.9. Thus we obtained a contradiction. ■

The next characterization of mean stability immediately follows from Proposition 2.9, Proposition 2.12, Proposition 3.3, and Theorem 3.1, extending [9] and [6, Corollary 2.7].

Theorem 3.4: Assume that $\mu \in P(\mathbb{R}^{d\times d})$ satisfies the assumption in Theorem 3.1. Then Σ is p-th mean stable if and only if $\rho\left(E_\mu[X^{\otimes p}]\right) < 1$.

A. Robust stability

The aim of this section is to study the robust stability of switching systems using Theorem 3.4. Let $A_1, \dots, A_m \in \mathbb{R}_+^{d\times d}$ be arbitrary . Take $p_1, \dots, p_m > 0$ such that $p_1 + \cdots + p_m = 1$. Define $\mu \in P(\mathbb{R}^{d\times d})$ by $\mu(\{A_n\}) = p_n$ for every n. Let Σ be the switching system associated with this μ. Let $\min A_n$ denote the minimum entry of A_n.

Let us define a perturbed version of Σ as follows. Let λ_n be the uniform measure on the closed ball $B_n := B(A_n; r_n)$ with $\lambda_n(B_n) = p_n$ and define $\lambda \in P(\mathbb{R}^{d\times d})$ by $\lambda := \lambda_1 + \cdots + \lambda_m$. Let us denote the corresponding switching system by Σ'. The next proposition shows the robustness of switching systems under such perturbations.

Proposition 3.5: If Σ is first mean stable and $r_n \leq \min A_n$ for every n then Σ' is first mean stable.

Proof: Notice that $\operatorname{supp}\mu$ leaves the proper cone $\mathbb{R}_+^d$ invariant. Since $r_n \leq \min A_n$, the support of each λ_n lies in $\mathbb{R}_+^{d\times d}$ and hence so does that of λ. Therefore $\operatorname{supp}\lambda$ also leaves $\mathbb{R}_+^d$ invariant. Therefore Theorem 3.1 gives that

$$\rho_{1,\lambda} = \rho\left(\sum_{n=1}^m p_n E[A_n + \lambda_n]\right) = \rho\left(\sum_{n=1}^m p_n E[A_n]\right) = \rho_{1,\mu} < 1,$$

as desired. This completes the proof. ∎

B. Stabilization

This section illustrates the stabilization of switching systems via an example using the stability criteria of Theorem 3.4. Consider the switching system Σ where μ is given by $\mu(\{A_1\}) = 0.9$ and $\mu(\{A_2\}) = 0.1$ with

$$A_1 = \begin{bmatrix} 1.1 & 1.6 \\ 0.9 & 1.2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4.3 & 2.1 \\ 2.5 & 3.9 \end{bmatrix}.$$

We can check $\rho_{1,\mu} = 2.6$ by Theorem 3.1 and hence, by Theorem 3.4, Σ is not 1st mean stable. Then let us consider the controlled switching system

$$\tilde{\Sigma} : x(k+1) = A_k x(k) - \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} u(k)$$

with the scalar valued state feedback $u(k) = f x(k)$ for some $f \in \mathbb{R}^{1\times 2}$. This yields the controlled system

$$\Sigma' : x(k+1) = (A_k + bf)x(k)$$

that is induced by the new probability measure ν of the random variable $A_k + bf$. We can check that, if $f = \begin{bmatrix} 1.2 & 1.7 \end{bmatrix}$ then $\operatorname{supp}\nu$ is still contained in $\mathbb{R}_+^{2\times 2}$ and has 1-radius 0.88 by Theorem 3.1. Therefore the state feedback control with the gain f stabilizes $\tilde{\Sigma}$ in the first mean. Fig. 1 shows the 20 sample paths of the original switching system and the stabilized switching system.

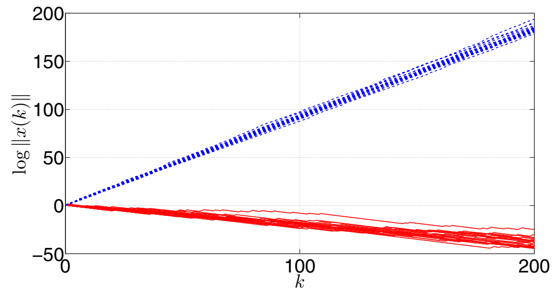

Fig. 1. Sample paths of switching systems. Dashed: Before stabilization. Solid: After stabilization

IV. Mean Square Stability

This section focuses on the special case of $p = 2$, i.e., the stability of the second mean (mean square stability) of the switching system (1). We will characterize the mean square stability by some conditions, which include the existence of a Lyapunov function.

For our characterization we need the notion of simultaneous contractibility of a probability measure, which is the stochastic counterpart of the simultaneous contractibility of a set of matrices studied in [2]. We say that $\mu \in P(\mathbb{R}^{d\times d})$ is *simultaneously contractible in square mean* if there exist $\gamma < 1$ and an invertible matrix $S \in \mathbb{R}^{d\times d}$ such that, for every $x \in \mathbb{R}^d$,

$$E\left[\frac{\|S^{-1}ASx\|^2}{\|x\|^2}\right] \leq \gamma.$$

The next theorem is the main result of this section.

Theorem 4.1: Let $\mu \in P(\mathbb{R}^{d\times d})$. The following conditions are equivalent:

1) $\rho_{2,\mu} < 1$;
2) The system Σ is mean square stable;
3) μ is simultaneously contractible in square mean;
4) (The existence of a Lyapunov function) There exists a positive definite matrix H and $\gamma \in [0, 1)$ such that
$$E[A^\top H A] \leq \gamma H. \tag{16}$$
5) Define $E_k := E[X_k]$ and $V_k := E[(X_k - E_k)(X_k - E_k)^\top]$ where $X_k = A_k \cdots A_1$. Then E_k and V_k converge to 0 exponentially fast as $k \to \infty$.

Proof: Since we already know the equivalence 1) ⇔ 2) by Proposition 2.9, it is sufficient to establish the two cycles 1) ⇒ 4) ⇒ 3) ⇒ 2) and 1) ⇒ 5) ⇒ 1).

[1) ⇒ 4)]: We use the idea of [2]. Assume $\rho_{2,\mu} < \alpha < 1$ for some α. Let $X_k = A_k \cdots A_1$. Then we can take a positive integer m such that

$$E[\|X_m\|^2] \leq \alpha^{2m}. \tag{17}$$

Define a positive definite matrix H by $H := I + \sum_{k=1}^{m-1} \alpha^{-2k} E[X_k^\top X_k]$, which induces a norm $|||\cdot|||$ on $\mathbb{R}^d$ as

$$|||x|||^2 := (Hx, x) = \|x\|^2 + \sum_{k=1}^{m-1} \alpha^{-2k} E[\|X_k x\|]^2. \tag{18}$$

Now if A follows μ then (17) yields that

$$E[|||Ax|||^2] \leq \sum_{k=0}^{m-2} \alpha^{-2k} E[\|X_{k+1}x\|^2] + \alpha^{-2(m-1)} \cdot \alpha^{2m} \|x\|^2 = \alpha^2 |||x|||^2. \tag{19}$$

Therefore, for every $x \in \mathbb{R}^d$, $(E[A^\top HA]x, x) = E(HAx, Ax) = E[|||Ax|||^2] \leq \alpha^2 E[|||x|||^2] = (\alpha^2 Hx, x)$ by (18) and (19), as desired.

[4) $\Rightarrow$ 3)]: Assume that $H > 0$ and $\gamma \in [0,1)$ satisfy (16). Let S be a square root of the positive definite matrix $H^{-\top}$, the inverse of the transpose of H. Then, for every $x \in \mathbb{R}^d$, (16) gives $E\left[\|S^{-1}ASx\|^2\right] = x^\top S^\top E[A^\top HA]Sx \leq x^\top S^\top (\gamma H)Sx = \gamma \|x^2\|$.

[3) $\Rightarrow$ 2)]: Assume that μ is simultaneously contractible in square mean with a constant $\gamma < 1$ and an invertible matrix S. Let us show the mean-square stability of Σ with respect to the norm $\|x\|_S := \|S^{-1}x\|$. It holds that

$$E[\|x(k+1;x_0)\|_S^2] = E[\|S^{-1}A_k x(k;x_0)\|^2] \leq \gamma E[\|S^{-1}x(k;x_0)\|^2] = \gamma E[\|x(k;x_0)\|_S^2]$$

and hence $E[\|x(k;x_0)\|_S^2] \leq \gamma^k \|x_0\|_S^2$. Since $0 \leq \gamma < 1$ the switching system Σ is mean square stable.

[1) $\Rightarrow$ 5)]: First notice that

$$V_k = E[X_k X_k^\top] - E_k E_k^\top. \tag{20}$$

Assume $\rho_{2,\mu} < 1$. Then $\|E[X_k X_k^\top]\| \leq E[\|X_k X_k^\top\|] \leq E[\|X_k\|^2] \to 0$ exponentially as $k \to \infty$. Also, by Lemma 2.5 we have $\rho_{1,\mu} \leq \rho_{2,\mu} < 1$ and therefore $\|E_k\| = \|E[X_k]\| \leq E[\|X_k\|] \to 0$ exponentially fast. Hence (20) implies that $V_k \to 0$ exponentially fast.

[5) $\Rightarrow$ 1)]: Assume that E_k and V_k converges exponentially to 0. Then, by (20), we can see that $E[X_k X_k^\top]$ converges to 0 exponentially fast. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Then, by the equivalence of the norms, there exists $C > 0$ such that $E[\|X_k\|^2] = E[\|X_k^\top\|^2] \leq E[(C\|X_k^\top\|_2)^2] = C^2 \sum_{n=1}^d e_n^\top E[X_k X_k^\top] e_n$, which converges to 0 exponentially fast. ∎

Example 4.2: Let us consider the switching system [3]

$$x(k+1) = \begin{bmatrix} [-0.8, 0] & [0.05, 0.35] \\ [0.050.35] & [0, 0.08] \end{bmatrix} x(k)$$

where the closed intervals in the matrix above indicate that each element follows a uniform distribution on the corresponding interval independently. Hibey [9] proved the mean square stability of the above system. Indeed we can easily see that $\rho_2 = 0.5212$. Therefore, by Theorem 3.4, this switching system admits a Lyapunov function. Indeed solving the linear matrix inequality (16) for H with $\gamma := 0.3 > \rho_2^2$ gives

$$H = \begin{bmatrix} 1.2193 & 0.0678 \\ 0.0678 & 1.3366 \end{bmatrix}.$$

Remark 4.3: Though (16) is not a linear matrix inequality when both H and γ are variables, it is easily solved by applying a bisection-like method for the scalar parameter γ with initial bound $[0,1]$ because (16) is a linear matrix inequality for each fixed γ.

V. Conclusion

We studied the mean stability of stochastic switching linear systems. An easy to check necessary and sufficient condition was obtained for the mean stability of a class of switching systems. In the condition an extended version of generalized joint spectral radius played a central role. Also we showed that mean square stability is equivalent to existence of a Lyapunov function.

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