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Sufficiency of a Necessary Condition for Local Observability of Discrete-Time Polynomial Systems

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***Abstract*— We consider local observability of discrete-time polynomial systems. When testing local observability of nonlinear systems, the observability rank condition is commonly used. However, the rank condition is only a sufficient condition in general. For this problem, recently, in terms of commutative algebra and algebraic geometry, a necessary condition for local observability of the polynomial systems has been derived. In this paper, we show the necessary condition is a sufficient condition for local observability for almost all initial states. Based on this result, we derive a necessary and sufficient condition for local observability.**

I. Introduction

In this paper, we consider local observability of discrete-time polynomial systems. Observability is one of the most fundamental properties, along with controllability and stability, in modern control theory. For nonlinear systems, there are various definitions of observability [1]–[4]. Mainly, global and local observability have been defined, and two types of each observability have been defined. One is called global or local observability at *an initial state*. The other is simply called global or local observability and means that a system is globally or locally observable for *all initial states*. In this paper, we focus on local observability and adopt weak observability [1] as its definition. Note that weak observability is sometimes adopted as local observability [4], [7].

Local observability is worth researching in its own right. Global observability is a too strong property when we design state feedback controllers for nonlinear systems in practice. Sometimes, local observability at possible initial states is sufficient for controlling dynamical systems even if they are not globally observable. For local observability, the observability rank condition [3], [4] has been derived. This is only a sufficient condition, and for necessity, this is a necessary condition for *almost all* initial states. That is, the rank condition is not a necessary and sufficient condition for *all* initial states. Therefore, there is no necessary and sufficient condition for local observability for all initial states.

For polynomial systems, observability is studied in terms of commutative algebra and algebraic geometry [5]–[7]. Especially, [7] shows a local observability condition at an initial state but it is difficult to check this condition for all initial states. In [8], based on this local observability condition, a necessary condition for local observability for *all* initial states has been derived. In this study, we show that this necessary condition is also a sufficient condition for *almost all* initial states. Note that the observability rank condition is a necessary condition for *almost all* initial states and sufficient condition for *all* initial states. In contrast, our condition is a necessary condition for *all* initial states and sufficient condition for *almost all* initial states. Finally, by utilizing the necessary condition, we derive a necessary and sufficient condition for local observability. It is not always easy to check the necessary and sufficient condition. However, there exists a polynomial system whose local observability is verified with our condition while the system does not satisfy the rank condition.

The remainder of this paper is organized as follows. In Section II, we show a necessary condition for local observability and discuss sufficiency of the necessary condition. Moreover, we derive a necessary and sufficient condition for local observability. In Section IV, an example illustrates that local observability can be tested by using our condition. Concluding remarks are given in Section V, and some lemmas are summarized in Appendix.

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II. Local Observability at an Initial State

A. Notations

Throughout the paper, $\mathbf{R}$ and $\mathbf{N}$ denote the field of real numbers and the set of non-negative integers, respectively. The polynomial ring over $\mathbf{R}$ with variables ξ_i, η_i $(i = 1, 2, \ldots, n)$ is denoted by $\mathbf{R}[\xi, \eta] (:= \mathbf{R}[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n])$. The ideal generated by $a_1, a_2, \ldots, a_s \in \mathbf{R}[\xi, \eta]$ is denoted as $\langle a_1, \ldots, a_s \rangle \subset \mathbf{R}[\xi, \eta]$. The set $\mathbf{V}(I) \subset \mathbf{R}^n \times \mathbf{R}^n$ represents an affine variety, which is the set of common zeros of all elements in an ideal $I \subset \mathbf{R}[\xi, \eta]$. For more details, see Appendix.

B. Polynomial Systems

Let us consider a discrete-time polynomial system described by

$$\Sigma_D \begin{cases} x[t+1] = f(x[t]),\ x[0] = x_0, \\ y[t] = h(x[t]), \end{cases}$$

where $t \in \mathbf{N}$, $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$ denote the time step, the state and the output of the system, respectively. Moreover, $f : \mathbf{R}^n \to \mathbf{R}^n$ and $h : \mathbf{R}^n \to \mathbf{R}$ are polynomial mappings. The identity mapping of $\mathbf{R}^n$ is denoted by f^0, and we recursively define a composite mapping $f^{i+1} := f \circ f^i$ $(i \in \mathbf{N})$. For discrete-time systems, the output at time $N \in \mathbf{N}$ starting from $x_0 \in \mathbf{R}^n$ can be expressed as $h(f^N(x_0))$.

Observability of a system Σ_D is defined as follows.

Definition 1: A pair of initial states $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n$ of a system Σ_D is distinguishable if there exists a time step $N \in \mathbf{N}$ that yields $h(f^N(\xi)) \neq h(f^N(\eta))$.

Definition 2: A system is locally observable at an initial state $\sigma \in \mathbf{R}^n$ if there exists a neighborhood of σ, denoted as $U(\sigma) \subset \mathbf{R}^n$, such that for all $\xi \in U(\sigma) \setminus \{\sigma\}$, pairs of (σ, ξ) are distinguishable. Moreover, a system is simply called locally observable if the system is locally observable for all initial states.

The following proposition [3], [4] is commonly used when testing local observability of discrete-time nonlinear systems.

Proposition 1: (Observability rank condition) System Σ_D is locally observable if there exists $N \in \mathbf{N}$ such that

$$\operatorname{rank} \frac{\partial \Phi}{\partial x}(\sigma) = n, \ \Phi(x) = \begin{bmatrix} h(x) \\ h(f(x)) \\ \vdots \\ h(f^N(x)) \end{bmatrix} \tag{1}$$

holds for all $\sigma \in \mathbf{R}^n$. Conversely, if a system is locally observable, then there exists $N \in \mathbf{N}$ such that (1) holds for almost all $\sigma \in \mathbf{R}^n$.

The observability rank condition (1) is only a sufficient condition for local observability. Even for the following simple and locally observable polynomial system, the rank condition does not hold.

Example 1: Consider a system:

$$\begin{cases} x[t+1] = x[t], \\ y[t] = x^3[t]. \end{cases}$$

First, we check the rank condition for this system. The mapping Φ is computed as

$$\Phi(x) = \begin{bmatrix} x^3 & \cdots & x^3 \end{bmatrix}^{\mathrm{T}},$$

and consequently,

$$\frac{\partial \Phi}{\partial x}(x) = \begin{bmatrix} 3x^2 & \cdots & 3x^2 \end{bmatrix}^{\mathrm{T}}.$$

Therefore, we obtain

$$\operatorname{rank} \frac{\partial \Phi}{\partial x}(0) = 0,$$

which implies that the observability rank condition does not hold at the origin.

However, it is possible to show that the system is locally observable based on its definition. The set of pairs of indistinguishable initial states $(\xi, \eta) \in \mathbf{R} \times \mathbf{R}$ satisfies

$$\xi^3 = \eta^3.$$

By fixing η on each point $\sigma \in \mathbf{R}^n$, we have $\xi^3 = \sigma^3$. By solving this equation, we obtain $\xi = \sigma$. Therefore, any initial state $\sigma \in \mathbf{R}^n$ is only indistinguishable from itself. That is, the system is locally observable in the sense of Definition 2.

Example 1 shows that the observability rank condition is only a sufficient condition even for a one dimensional polynomial system. Therefore, it is necessary to derive another local observability condition. For polynomial systems, a necessary condition has been given [8]. In this paper, we discuss the sufficiency of the necessary condition. Moreover, by using the necessary condition, we derive a necessary and sufficient condition for local observability.

C. Pairs of Indistinguishable Initial States

To show a necessary condition for local observability, we obtain pairs of indistinguishable initial states. Here, we show that the set of pairs of indistinguishable initial states is characterized by a finite set of polynomials.

We generate the following ideals J_i $(i \in \mathbf{N})$ in $\mathbf{R}[\xi, \eta]$:

$$J_i = \langle h(\xi) - h(\eta), \ldots, h(f^i(\xi)) - h(f^i(\eta)) \rangle.$$

From the definition of J_i, we have $J_0 \subset J_1$, and consequently an ascending chain of ideals $J_0 \subset J_1 \subset \cdots$. According to Hilbert's basis theorem, the chain will stabilize. In other words, there exists a certain ideal $\mathcal{J} \subset \mathbf{R}[\xi, \eta]$ satisfying

$$J_0 \subset J_1 \subset \cdots \subset J_N = J_{N+1} = \cdots = \mathcal{J} \tag{2}$$

for $N \in \mathbf{N}$. In particular, it is readily shown that the chain (2) has the following strict inclusion property [7]:

$$J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_N = J_{N+1} = \cdots = \mathcal{J}. \tag{3}$$

Then, the variety $\mathbf{V}(\mathcal{J}) \subset \mathbf{R}^n \times \mathbf{R}^n$ represents the set of pairs of indistinguishable initial states $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n$.

To describe observability conditions, we define an ideal $\mathcal{I} \subset \mathbf{R}[\xi, \eta]$ as

$$\mathcal{I} = \langle \xi_1 - \eta_1, \ldots, \xi_n - \eta_n \rangle. \tag{4}$$

D. Local Observability Condition at an Initial State

Here, we show a local observability condition at an initial state $\sigma \in \mathbf{R}^n$ of a system Σ_D derived in [7]. Based on this condition, a necessary condition for local observability has been obtained in [8].

Suppose that $p \subset \mathbf{R}[\xi]$ is a prime ideal. Then a localization $\mathbf{R}[\xi]_p$ of $\mathbf{R}[\xi]$ at p is defined as

$$\mathbf{R}[\xi]_p = \{a/b : a, b \in \mathbf{R}[\xi], b \notin p\},$$

and for an ideal $I \subset \mathbf{R}[\xi]$, an ideal $I_p \subset \mathbf{R}[\xi]_p$ is defined as

$$I_p = \{a/b : a \in I, b \in \mathbf{R}[\xi], b \notin p\}.$$

Based on the fact that $I_p \cap \mathbf{R}[\xi]$ is an ideal in $\mathbf{R}[\xi]$, an affine variety $\mathbf{V}(I_p) \subset \mathbf{V}(I) \subset \mathbf{R}^n$ is defined as

$$\mathbf{V}(I_p) := \mathbf{V}(I_p \cap \mathbf{R}[\xi]).$$

To describe a local observability condition at an initial state $\sigma \in \mathbf{R}^n$, it is necessary to obtain the set of initial states indistinguishable from σ. Let $\varphi_\sigma : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\xi]$ be a mapping that substitutes $\eta = \sigma \in \mathbf{R}^n$ for an element of $\mathbf{R}[\xi, \eta]$, which is a surjective ring homomorphism over $\mathbf{R}[\xi, \eta]$. Then, $\mathbf{V}(\varphi_\sigma(\mathcal{J})) \subset \mathbf{R}^n$ denotes the set of initial states indistinguishable from σ. For $\varphi_\sigma(\mathcal{I})$, we have the following.

Lemma 1: The ideal $\varphi_\sigma(\mathcal{I}) \subset \mathbf{R}[\xi]$ is a maximal ideal.

Lemma 1 holds because of $\varphi_\sigma(\mathcal{I}) = \langle \xi_1 - \sigma_1, \ldots, \xi_n - \sigma_n \rangle$. Moreover, we obtain

$$\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathcal{I}))) = \varphi_\sigma(\mathcal{I}). \tag{5}$$

Since a maximal ideal is prime, $\mathbf{R}[\xi]_{\varphi_\sigma(\mathcal{I})}$ is a local ring. From Proposition 9 in Appendix, $\varphi_\sigma(\mathcal{I})_{\varphi_\sigma(\mathcal{I})} \cap \mathbf{R}[\xi] = \varphi_\sigma(\mathcal{I})$ holds. Thus, we have

$$\mathbf{V}\left(\varphi_\sigma(\mathcal{I})_{\varphi_\sigma(\mathcal{I})}\right) = \mathbf{V}(\varphi_\sigma(\mathcal{I})). \tag{6}$$

A local observability condition at an initial state is obtained as follows [7].

Theorem 1: A discrete-time polynomial system Σ_D is locally observable at $\sigma \in \mathbf{R}^n$ if and only if

$$\mathbf{V}\left(\varphi_\sigma(\mathcal{J})_{\varphi_\sigma(\mathcal{I})}\right) = \mathbf{V}(\varphi_\sigma(\mathcal{I})) \tag{7}$$

holds.

Theorem 1 is a necessary and sufficient condition for local observability at a certain initial state while the observability rank condition (1) is only a sufficient condition.

III. Local Observability

It is difficult to directly check (7) for all initial states because we need to compute the ideal $\varphi_\sigma(\mathcal{J})_{\varphi_\sigma(\mathcal{I})}$ at each $\sigma \in \mathbf{R}^n$. On the other hand, based on Theorem 1, a necessary condition for local observability has been presented [8]. In this paper, we show that the necessary condition is a sufficient condition for almost all initial states.

The following properties of $\mathcal{I}$ shown in [8] are used in this paper.

Lemma 2: For the ideal $\mathcal{I} \subset \mathbf{R}[\xi, \eta]$ in (4), we have the followings.

1) $\mathcal{I}$ is prime.
2) $\mathbf{V}(\mathcal{I}) \subset \mathbf{R}^n \times \mathbf{R}^n$ is irreducible.

The kernel $\mathrm{Ker}\varphi_\sigma \subset \mathbf{R}[\xi, \eta]$ of the surjective ring homomorphism $\varphi_\sigma : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\eta]$ is generated as

$$\mathrm{Ker}\varphi_\sigma = \langle \eta_1 - \sigma_1, \eta_2 - \sigma_2, \ldots, \eta_n - \sigma_n \rangle. \tag{8}$$

From (8), $\mathbf{I}(\mathbf{V}(\mathrm{Ker}\varphi_\sigma)) = \mathrm{Ker}\varphi_\sigma$ holds.

Furthermore, $\mathcal{I} + \mathrm{Ker}\varphi_\sigma \subset \mathbf{R}[\xi, \eta]$ is generated as

$$\begin{aligned} &\mathcal{I} + \mathrm{Ker}\varphi_\sigma \\ &\quad = \langle \xi_1 - \sigma_1, \ldots, \xi_n - \sigma_n, \eta_1 - \sigma_1, \ldots, \eta_n - \sigma_n \rangle, \end{aligned} \tag{9}$$

which implies

$$\mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma) = \{(\sigma, \sigma)\}, \tag{10}$$
$$\mathbf{I}(\mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma)) = \mathcal{I} + \mathrm{Ker}\varphi_\sigma. \tag{11}$$

A necessary condition for local observability is derived as follows.

Theorem 2: [8] If a discrete-time polynomial system Σ_D is locally observable, then

$$\mathbf{V}(\mathcal{J}_\mathcal{I}) = \mathbf{V}(\mathcal{I}) \tag{12}$$

holds.

In this paper, we show necessary condition (12) is a sufficient condition for almost all initial states.

Theorem 3: If (12) holds, we have the followings.

1) The minimal decomposition of $\mathbf{V}(\mathcal{J})$ is
$$\mathbf{V}(\mathcal{J}) = \mathbf{V}(\mathcal{I}) \cup Y_1 \cup \cdots \cup Y_t, \tag{13}$$
where $Y_i \subset \mathbf{R}^n$ $(i = 1, \ldots, t)$ is an irreducible variety or the empty set.
2) A polynomial system Σ_D is locally observable at any initial state $\sigma \in \mathbf{R}^n$ such that
$$\{(\sigma, \sigma)\} \subset \mathbf{V}(\mathcal{I}) \setminus (Y_1 \cup \cdots \cup Y_t) \tag{14}$$
holds.
3) The Zariski closure of $\mathbf{V}(\mathcal{I}) \setminus (Y_1 \cup \cdots \cup Y_t)$ is $\mathbf{V}(\mathcal{I})$.

Proof: If (12) holds then from Proposition 10 in Appendix, $\mathbf{V}(\mathcal{J})$ has the minimal decomposition as (13). The minimal decomposition has a property $\mathbf{V}(\mathcal{I}) \not\subset Y_1 \cup \cdots \cup Y_t$, and $\mathbf{V}(\mathcal{I}) = \bigcup_{\sigma \in \mathbf{R}^n} \{(\sigma, \sigma)\}$ implies the existence of an initial state $\sigma \in \mathbf{R}^n$ such that (14) holds. Let $\varphi_\sigma : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\xi]$ be a mapping that substitutes $\sigma \in \mathbf{R}^n$ satisfying (14) into $\eta \in \mathbf{R}[\xi, \eta]$. From (14), we have

$$\{(\sigma, \sigma)\} \not\subset Y_1 \cup \cdots \cup Y_t,$$

and consequently

$$\{(\sigma, \sigma)\} \not\subset (Y_1 \cup \cdots \cup Y_t) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma).$$

The left hand side can be computed as follows with (10).

$$\mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma) \not\subset (Y_1 \cup \cdots \cup Y_t) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma).$$

From Propositions 3.1) and 4 in Appendix, the right hand side satisfies

$$\begin{aligned} &(Y_1 \cup \cdots \cup Y_t) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma) \\ &\quad = \mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t)) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma), \\ &\quad = \mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma), \end{aligned} \tag{15}$$

which yields

$$\mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma) \not\subset \mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma).$$

From Proposition 2.3) and (11) in Appendix, we have

$$\begin{aligned} \mathbf{I}(\mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma)) &\not\supset \mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma)), \\ \mathcal{I} + \mathrm{Ker}\varphi_\sigma &\not\supset \mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma)), \end{aligned}$$

and, from Propositions 5 and 12 in Appendix,

$$\begin{aligned} \varphi_\sigma(\mathcal{I}) &\not\supset \varphi_\sigma(\mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma))), \\ \varphi_\sigma(\mathcal{I}) &\not\supset \mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathbf{I}(Y_1 \cup \cdots \cup Y_t)))). \end{aligned} \tag{16}$$

On the other hand, by computing intersections between $\mathbf{V}(\mathrm{Ker}\varphi_\sigma)$ and each side of the minimal decomposition (13), we have

$$\begin{aligned} &\mathbf{V}(\mathcal{J}) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma) \\ &\quad = (\mathbf{V}(\mathcal{I}) \cup (Y_1 \cup \cdots \cup Y_t)) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma), \\ &\quad = (\mathbf{V}(\mathcal{I}) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma)) \cup ((Y_1 \cup \cdots \cup Y_t) \cap \mathbf{V}(\mathrm{Ker}\varphi_\sigma)). \end{aligned}$$

From Proposition 3.1) in Appendix and (15), we obtain

$$\begin{aligned} &\mathbf{V}(\mathcal{J} + \mathrm{Ker}\varphi_\sigma) \\ &\quad = \mathbf{V}(\mathcal{I} + \mathrm{Ker}\varphi_\sigma) \cup \mathbf{V}(\mathbf{I}(Y_1 \cup \cdots \cup Y_t) + \mathrm{Ker}\varphi_\sigma), \end{aligned}$$

and, from Proposition 3.3) in Appendix,

$$\begin{aligned}&\mathbf{I}(\mathbf{V}(\mathcal{J}+\mathrm{Ker}\varphi_\sigma))\\&=\mathbf{I}(\mathbf{V}(\mathcal{I}+\mathrm{Ker}\varphi_\sigma)\cup\mathbf{V}(\mathbf{I}(Y_1\cup\cdots\cup Y_t)+\mathrm{Ker}\varphi_\sigma),\\&=\mathbf{I}(\mathbf{V}(\mathcal{I}+\mathrm{Ker}\varphi_\sigma))\cap\mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1\cup\cdots\cup Y_t)+\mathrm{Ker}\varphi_\sigma)).\end{aligned}$$

From (11), we have

$$\begin{aligned}&\mathbf{I}(\mathbf{V}(\mathcal{J}+\mathrm{Ker}\varphi_\sigma))\\&=(\mathcal{I}+\mathrm{Ker}\varphi_\sigma)\cap\mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1\cup\cdots\cup Y_t)+\mathrm{Ker}\varphi_\sigma)),\end{aligned}$$

and thus

$$\begin{aligned}&\varphi_\sigma(\mathbf{I}(\mathbf{V}(\mathcal{J}+\mathrm{Ker}\varphi_\sigma)))\\&=\varphi_\sigma((\mathcal{I}+\mathrm{Ker}\varphi_\sigma)\cap\mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1\cup\cdots\cup Y_t)+\mathrm{Ker}\varphi_\sigma))).\end{aligned}$$

Proposition 6 in Appendix implies

$$\begin{aligned}&\varphi_\sigma(\mathbf{I}(\mathbf{V}(\mathcal{J}+\mathrm{Ker}\varphi_\sigma)))\\&=\varphi_\sigma(\mathcal{I}+\mathrm{Ker}\varphi_\sigma)\cap\varphi_\sigma(\mathbf{I}(\mathbf{V}(\mathbf{I}(Y_1\cup\cdots\cup Y_t)+\mathrm{Ker}\varphi_\sigma))).\end{aligned}$$

Moreover, from Proposition 12 in Appendix, we have

$$\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathcal{J})))=\varphi_\sigma(\mathcal{I})\cap\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathbf{I}(Y_1\cup\cdots\cup Y_t)))).\quad(17)$$

From Lemma 1, $\varphi_\sigma(\mathcal{I})$ is prime, and thus $\mathbf{R}[\xi]_{\varphi_\sigma(\mathcal{I})}$ is a local ring. Equation (17) implies

$$\begin{aligned}&\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathcal{J})))_{\varphi_\sigma(\mathcal{I})}\\&=(\varphi_\sigma(\mathcal{I})\cap\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathbf{I}(Y_1\cup\cdots\cup Y_t)))))_{\varphi_\sigma(\mathcal{I})}.\end{aligned}$$

The right hand side can be computed as follows with (16) and Proposition 8.

$$\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathcal{J})))_{\varphi_\sigma(\mathcal{I})}=\varphi_\sigma(\mathcal{I})_{\varphi_\sigma(\mathcal{I})},$$

Then, we have

$$\mathbf{V}\left(\mathbf{I}(\mathbf{V}(\varphi_\sigma(\mathcal{J})))_{\varphi_\sigma(\mathcal{I})}\right)=\mathbf{V}\left(\varphi_\sigma(\mathcal{I})_{\varphi_\sigma(\mathcal{I})}\right).$$

From Proposition 11 in Appendix and (6), we have

$$\mathbf{V}\left(\varphi_\sigma(\mathcal{J})_{\varphi_\sigma(\mathcal{I})}\right)=\mathbf{V}(\varphi_\sigma(\mathcal{I})).$$

Thus, (7) holds, and the system Σ_D is locally observable at σ satisfying (14).

Finally, we show that the Zariski closure of $\mathbf{V}(\mathcal{I})\setminus(Y_1\cup\cdots\cup Y_t)$ is $\mathbf{V}(\mathcal{I})$, i.e.,

$$\mathbf{V}(\mathbf{I}(\mathbf{V}(\mathcal{I})\setminus(Y_1\cup\cdots\cup Y_t)))=\mathbf{V}(\mathcal{I})\quad(18)$$

holds. When $Y_1\cup\cdots\cup Y_t$ is the empty set, from Proposition 4 in Appendix, (18) holds. Thus, assume that $Y_1\cup\cdots\cup Y_t$ is not the empty set. Let $Z_1\cup\cdots\cup Z_s$ be the minimal decomposition of $\mathbf{V}(\mathcal{I})\cap(Y_1\cup\cdots\cup Y_t)$. Then, we have $\mathbf{V}(\mathcal{I})\setminus(Y_1\cup\cdots\cup Y_t)=\mathbf{V}(\mathcal{I})\setminus(Z_1\cup\cdots\cup Z_s)$, which implies that

$$\begin{aligned}&\mathbf{V}(\mathbf{I}(\mathbf{V}(\mathcal{I})\setminus(Y_1\cup\cdots\cup Y_t)))\\&=\mathbf{V}(\mathbf{I}(\mathbf{V}(\mathcal{I})\setminus(Z_1\cup\cdots\cup Z_s))).\quad(19)\end{aligned}$$

We consider the right hand side of (19). From Proposition 3 4) in Appendix, we have

$$\begin{aligned}&\mathbf{I}(\mathbf{V}(\mathcal{I})\setminus(Z_1\cup\cdots\cup Z_s))\\&=\mathbf{I}(\mathbf{V}(\mathcal{I})):\mathbf{I}(Z_1\cup\cdots\cup Z_s).\quad(20)\end{aligned}$$

Since from the assumption, $\mathbf{V}(\mathcal{I})\supsetneq Z_1\cup\cdots\cup Z_s$ holds, Propositions 2 2) and 3) in Appendix leads to

$$\mathbf{I}(\mathbf{V}(\mathcal{I}))\supsetneq\mathbf{I}(Z_1\cup\cdots\cup Z_s).\quad(21)$$

Proposition 7 in Appendix and Lemma 2 2) imply that $\mathbf{I}(\mathbf{V}(\mathcal{I}))$ is prime. Thus, from (21) and Proposition 13 in Appendix, we obtain

$$\mathbf{I}(\mathbf{V}(\mathcal{I})):\mathbf{I}(Z_1\cup\cdots\cup Z_s)=\mathbf{I}(\mathbf{V}(\mathcal{I})).\quad(22)$$

Therefore, (19), (20), (22) and Proposition 4 in Appendix imply that

$$\mathbf{V}(\mathbf{I}(\mathbf{V}(\mathcal{I})\setminus(Z_1\cup\cdots\cup Z_s)))=\mathbf{V}(\mathbf{I}(\mathbf{V}(\mathcal{I})))=\mathbf{V}(\mathcal{I})$$

holds. ■

Remark 1: Condition (12) is a necessary and sufficient condition for local observability at initial states $\sigma\in\mathbf{R}^n$ satisfying (14). Moreover, from Theorem 3 3), (12) is a necessary and sufficient condition for local observability for almost all initial states.

To check condition (12), it suffices to check whether $\mathbf{V}(\mathcal{J}_{\mathcal{I}})$, which is the common zeros of a finite set of polynomials, equals $\mathbf{V}(\mathcal{I})$ or not. To test local observability of nonlinear systems, the observability rank condition (1) is commonly used, and that is a sufficient condition for all initial states and necessary condition for almost all initial states. In contrast, our condition is a necessary condition for all initial states and guarantees sufficiency for almost all initial states.

Theorem 3 leads to the following corollary.

Corollary 1: A discrete-time polynomial system Σ_D is locally observable if and only if the followings hold:

1) (12) holds, and
2) The system is locally observable at $\sigma\in\mathbf{R}^n$ satisfying

$$(\sigma,\sigma)\in W:=\mathbf{V}(\mathcal{I})\cap(Y_1\cup\cdots\cup Y_t),\quad(23)$$

where Y_i $(i=1,\ldots,t)$ is defined in (13).

Proof: The necessity is obvious. We show the sufficiency. If (12) holds, Theorem 3 2) implies that the system is locally observable at $\sigma\in\mathbf{R}^n$ satisfying (14), i.e., not satisfying (23). Therefore, if condition 2) holds, the system is locally observable. ■

Remark 2: It is not always easy to check condition 2) of Corollary 1 because we need to check local observability for an infinite set of initial states in general. However, Corollary 1 helps testing local observability of polynomial systems not satisfying the observability rank condition. Actually, based on Corollary 1, we can test local observability of a polynomial system as shown in Section IV.

IV. Example

We give an example in which local observability of a discrete-time polynomial system is tested. Consider

$$\begin{aligned}x[t+1]&=\begin{bmatrix}x_2^2[t]+x_3[t]\\x_1[t]\\x_3[t]\end{bmatrix},\\y[t]&=x_1[t].\end{aligned}$$

The observability rank condition does not hold when $x_2[0] = 0$. Thus, we test local observability with the proposed criterion.

To check local observability, we generate an ideal $\mathcal{I} \subset \mathbf{R}[\xi,\eta](:= \mathbf{R}[\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3])$ as

$$\mathcal{I} = \langle \xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3 \rangle,$$

and $J_i \subset \mathbf{R}[\xi,\eta]$ $(i \in \mathbf{N})$ as:

$$\begin{aligned} J_0 &= \langle \xi_1 - \eta_1 \rangle, \\ J_1 &= J_0 + \langle \xi_2^2 + \xi_3 - \eta_2^2 - \eta_3 \rangle, \\ J_2 &= J_1 + \langle \xi_1^2 + \xi_3 - \eta_1^2 - \eta_3 \rangle, \\ J_3 &= J_2 + \cdots . \end{aligned}$$

Here, $J_3 = J_2$ is obtained. Thus, we define $\mathcal{J}$ as J_2, which is generated as

$$\mathcal{J} = \langle \xi_1 - \eta_1, \xi_2^2 - \eta_2^2, \xi_3 - \eta_3 \rangle,$$

and we have $\mathcal{J}_{\mathcal{I}} \cap \mathbf{R}[\xi,\eta] = \mathcal{I}$. Thus, $\mathbf{V}(\mathcal{J}_{\mathcal{I}}) = \mathbf{V}(\mathcal{I})$ holds. The minimal decomposition of $\mathbf{V}(\mathcal{J})$ can be computed as follows.

$$\begin{aligned} &\mathbf{V}(\mathcal{J}) = \mathbf{V}(\mathcal{I}) \cup Y, \\ &Y = \left\{ (\xi,\eta) \in \mathbf{R}^3 \times \mathbf{R}^3 : \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ -\eta_2 \\ \eta_3 \end{bmatrix} \right\}, \end{aligned}$$

and the intersection between $\mathbf{V}(\mathcal{I})$ and Y is obtained as

$$\begin{aligned} W &:= \mathbf{V}(\mathcal{I}) \cap Y \\ &= \left\{ (\xi,\eta) \in \mathbf{R}^3 \times \mathbf{R}^3 : \begin{bmatrix} \xi_1 \\ 0 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ 0 \\ \eta_3 \end{bmatrix} \right\}, \end{aligned}$$

Thus, Theorem 3 2) implies that the system is locally observable at initial states $\sigma \in \mathbf{R}^3$ satisfying $\sigma_2 \neq 0$.

Next, we test local observability at $\sigma \in \hat{W} \subset \mathbf{R}^3$:

$$\hat{W} := \{\sigma \in \mathbf{R}^3 : \sigma_2 = 0\}.$$

Let $\psi : \mathbf{R}[\xi,\eta] \to \mathbf{R}[\xi_1,\xi_3,\eta]$ be a mapping that substitutes $\xi_2 = 0$. Then, we have

$$\varphi(\mathcal{J}) = \langle \xi_1 - \eta_1, \eta_2^2, \xi_3 - \eta_3 \rangle,$$

and $\mathbf{V}(\varphi(\mathcal{J})) \subset \mathbf{R}^2 \times \mathbf{R}^3$ represents the initial states indistinguishable from $\sigma_2 = 0$. Since $\mathbf{V}(\varphi(\mathcal{J})) = \mathbf{V}(\varphi(\mathcal{I}))$ holds, each initial state $[\sigma_1\ 0\ \sigma_3]^{\mathrm{T}} \in \mathbf{R}^3$ is only indistinguishable from itself. That is, the system is locally observable at $\sigma \in \hat{W}$. In summary, the system is locally observable while the observability rank condition does not hold.

V. Conclusion

In this paper, we showed that a necessary condition for local observability of discrete-time polynomial systems is a sufficient condition for almost all initial states. Since the necessary condition can be checked simply by solving a set of algebraic equations, local observability for almost all initial states can be checked by solving it. Moreover, based on the necessary condition, we derived a necessary and sufficient condition for local observability, and we evaluated local observability of a polynomial system not satisfying the observability rank condition by our condition. However, it is not always easy to test the necessary and sufficient condition because it is necessary to check local observability for an infinite set of initial states in general. Therefore, it is necessary to derive another local observability condition which can be checked simply by solving a set of algebraic equations, which is a future work.

Appendix

A. Basic Proposition

We show basic definitions and propositions of commutative algebra and algebraic geometry described in [9]–[12].

An ideal I of the polynomial ring $\mathbf{R}[\xi]$ has the following properties for any $a, b \in I$ and $c \in \mathbf{R}[\xi]$:

1) $a + b \in I$,
2) $ca \in I$.

Let $p_1, p_2, \ldots, p_r$ be in $\mathbf{R}[\xi]$. Then the set $\langle p_1, \ldots, p_r \rangle$, defined as

$$\langle p_1, \ldots, p_r \rangle = \left\{ \sum_{i=1}^{r} a_i p_i : a_i \in \mathbf{R}[x] (i = 1, 2, \ldots, r) \right\},$$

is called the ideal generated by $p_1, \ldots, p_r$.

If I and J are ideals of $\mathbf{R}[\xi]$, then the sum of I and J, denoted $I + J$, is the set

$$I + J := \{a + b : a \in I \text{ and } b \in J\},$$

which is also an ideal in $\mathbf{R}[\xi]$. The intersection $I \cap J$ of two ideals I and J in $\mathbf{R}[\xi]$ is the set of polynomials which belong to both I and J and an ideals in $\mathbf{R}[\xi]$. Next, the ideal quotient of I by J is the following set

$$I : J := \{a \in \mathbf{R}[\xi] : ab \in I \text{ for all } b \in J\},$$

which is also an ideal in $\mathbf{R}[\xi]$.

An ideal $p \subset \mathbf{R}[\xi]$ is said to be prime if $ab \in p$ implies $a \in p$ or $b \in p$.

The affine variety of polynomials $p_1, \ldots, p_r \in \mathbf{R}[\xi]$ is defined as

$$\mathbf{V}(p_1, \ldots, p_r) = \{\xi \in \mathbf{R}^n : p_i(\xi) = 0, i = 1, \ldots, r\}.$$

The affine variety of an ideal $I \subset \mathbf{R}[\xi]$ is defined as

$$\mathbf{V}(I) = \{\xi \in \mathbf{R}^n : p(\xi) = 0, \forall p \in I\},$$

and thus $\mathbf{V}(I)$ represents the set of common zeros of all polynomials in I. If $p_1, \ldots, p_r \in \mathbf{R}[\xi]$ are generators of the ideal $I \subset \mathbf{R}[\xi]$, we have

$$\mathbf{V}(I) = \mathbf{V}(p_1, \ldots, p_r).$$

An affine variety $V \subset \mathbf{R}^n$ is said to be irreducible if $V = V_1 \cup V_2$ implies $V = V_1$ or $V = V_2$, where $V_1, V_2 \subset \mathbf{R}^n$ are affine algebraic varieties. Let $V \subset \mathbf{R}^n$ be an affine variety. A decomposition $V = V_0 \cup V_1 \cup \cdots \cup V_m$ $(m \in \mathbf{N})$, where each V_i is an irreducible variety, is called a minimal decomposition (or, sometimes, an irredundant union) if $V_i \not\subset$

V_j for $i \neq j$. It is shown that every affine variety has the minimal decomposition.

Let $V \subset \mathbf{R}^n$ be an affine variety. We define the following set:

$$\mathbf{I}(V) = \{p \in \mathbf{R}[\xi] : p(\xi) = 0, \forall \xi \in V\},$$

which is an ideal, and we call $\mathbf{I}(V)$ the ideal of V.

Next, we state some technical lemmas used in this paper.

Proposition 2: Let $I, J \subset \mathbf{R}[\xi]$ be ideals, and let $V, W \subset \mathbf{R}^n$ be affine varieties. The followings hold:

1) If $I \supset J$, then $\mathbf{V}(I) \subset \mathbf{V}(J)$.
2) If $V \supset W$, then $\mathbf{I}(V) \subset \mathbf{I}(W)$.
3) If $V \not\supset W$, then $\mathbf{I}(V) \not\subset \mathbf{I}(W)$.

Proposition 3: Let $I, J \subset \mathbf{R}[\xi]$ be ideals, and let $V, W \subset \mathbf{R}^n$ be affine varieties. The followings hold:

1) $\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$.
2) $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$.
3) $\mathbf{I}(V \cup W) = \mathbf{I}(V) \cap \mathbf{I}(W)$.
4) $\mathbf{I}(V \setminus W) = \mathbf{I}(V) : \mathbf{I}(W)$.

Proposition 4: For an affine variety $W \subset \mathbf{R}^n$, $\mathbf{V}(\mathbf{I}(W)) = W$ holds.

Proposition 5: Let $\varphi : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\xi]$ be a ring homomorphism. Suppose that ideals $I, J \subset \mathbf{R}[\xi, \eta]$ contain $\mathrm{Ker}\varphi$. Then, $I \subset J$ holds if and only if $\varphi(I) \subset \varphi(J)$ holds.

Proposition 6: Let $\varphi : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\xi]$ be a ring homomorphism, and let $\{I_t\}_{t \in T}$ be a family of ideals in $\mathbf{R}[\xi, \eta]$. If $I_t \supset \mathrm{Ker}\varphi$ holds for all $t \in T$, we have $\varphi\left(\bigcap_{t \in T} I_t\right) = \bigcap_{t \in T} \varphi(I_t)$.

Proposition 7: Let $V \subset \mathbf{R}^n$ be an affine variety. V is irreducible if and only if $\mathbf{I}(V) \subset \mathbf{R}[\xi]$ is prime.

B. Some Technical Results

The following propositions are given in [7], [8].

Proposition 8: Let $p \subset \mathbf{R}[\xi]$ be a prime ideal and $I \not\subset p, J \subset p$ be ideals in $\mathbf{R}[\xi]$. Then, $(I \cap J)_p = J_p$ holds.

Proposition 9: Let $p \subset \mathbf{R}[\xi]$ be a prime ideal, and $q \subset \mathbf{R}[\xi]$ be a primary ideal. If $q \subset p$ holds, $q_p \cap \mathbf{R}[\xi] = q$ holds.

Proposition 10: Suppose that $I \subset \mathbf{R}[\xi]$ is an ideal and $p \subset \mathbf{R}[\xi]$ is a prime ideal satisfying $p = \mathbf{I}(\mathbf{V}(p))$. Then, $\mathbf{V}(I_p) = \mathbf{V}(p)$ holds if and only if $\mathbf{V}(p) \cup X_1 \cup \cdots \cup X_t$ is the minimal decomposition of $\mathbf{V}(I) \subset \mathbf{R}^n$.

Proposition 11: Let $I \subset \mathbf{R}[\xi]$ be an ideal. If a prime ideal $p \subset \mathbf{R}[\xi]$ satisfies $p = \mathbf{I}(\mathbf{V}(p))$, then the following holds

$$\mathbf{V}(\mathbf{I}(\mathbf{V}(I_p))) = \mathbf{V}(\mathbf{I}(\mathbf{V}(I_p))_p) = \mathbf{V}(I_p). \tag{24}$$

Proposition 12: Let $\varphi_\sigma : \mathbf{R}[\xi, \eta] \to \mathbf{R}[\xi]$ be a mapping that substitutes $\eta = \sigma \in \mathbf{R}^n$. For an ideal $I \subset \mathbf{R}[\xi, \eta]$, we have

$$\varphi_\sigma(\mathbf{I}(\mathbf{V}(I + \mathrm{Ker}\varphi_\sigma))) = \mathbf{I}(\mathbf{V}(\varphi_\sigma(I)))). \tag{25}$$

Proposition 13: Let $I \subset \mathbf{R}[\xi]$ be an ideal, and let $p \subset \mathbf{R}[\xi]$ be a prime ideal. If $p \subsetneq I$, then $p : I = p$.

Proof: Suppose that $f \in p : I$. Then, $fg \in p$ holds for all $g \in I$. Thus, $fg \in p$ holds for all $g \in I \setminus p$. Since p is prime, $f \in p$ holds. The converse is proved in [9]. ■

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