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An Iterative Eigenvector Tangential Interpolation Algorithm for Large-Scale LTI and a Class of LPV Model Approximation

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***Abstract*— In this paper, a new Iterative Eigenvector Tangential Interpolation Algorithm (IETIA) is proposed to approximate large-scale LTI models, allowing to achieve $\mathcal{H}_2$ optimality-like conditions while preserving some user-defined eigenvalues. The proposed algorithm is also shown to be applicable to a certain class of LPV models. Numerical applications on standard model reduction benchmarks assess the effectiveness of the approach.**

***Index Terms*— Model reduction, large-scale linear dynamical systems, eigenvectors, linear parameter varying.**

I. INTRODUCTION

A. Forewords and projection-based approximation problem

In many applied engineering fields, dedicated computer-based dynamical modelling software, allowing to enhance the "traditional" mathematical engineer modelling, are being more and more used by practitioners and researchers to capture the complex dynamical phenomena and to enhance model accuracy. This faithful modelling, usually characterized by a large set of differential equations - either Linear Time Invariant (LTI), Linear Time Varying (LTV) or Linear Parameter Varying (LPV)[1] - offers the possibility for a better understanding of the system and claims in favour of more simulation-based development. On the other side it often leads to models with a prohibitively large number of variables to manage, rendering the application of modern numerical *analysis* and *synthesis* tools (*e.g.* μ-analysis, LQ, $\mathcal{H}_2$, $\mathcal{H}_\infty$-norms computation and control...) practically impossible. This is especially the case for *(i)* highly critical systems such as aircrafts [22], [8], satellites, civilian structures (buildings / bridges / windmills), and/or *(ii)* strongly optimized ones, such as integrated industrial electrical systems [7], [24], [25].

To overpass this limitation, as grounded on the Petrov-Galerkin projection-based approximation, a model reduction step is usually performed. The root idea is the following: given a dynamical system $\dot{x}(t) = f(x(t), u(t))$ (where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^{n_u}$), the problem consists in finding the right projector *(i)* $V \in \mathbb{C}^{n\times r}$ ($r \ll n$) forming a basis of a subspace $\mathcal{V}$, and the left projector *(ii)* $W = (\overline{W}^* V)^{-1}\overline{W}$ (where $\overline{W} \in \mathbb{C}^{n\times r}$ is a basis of a subspace $\mathcal{W}$), such that $\overline{W}^* V$ is invertible, $\Pi_{V,W} = VW^*$ is a projector, and

$$\hat{x}(t) \in \mathcal{V} \quad \text{and} \quad \overline{W}^*\Big(V\dot{\hat{x}}(t) - f(V\hat{x}(t), u(t))\Big) = 0 \qquad (1)$$

where $x(t) \approx V\hat{x}(t)$ and $\hat{x}(t) \approx W^* x(t)$ are the full original and reduced order states, respectively.

B. Some literature background

Within this research field, one of the the main challenges is to propose numerically stable and robust procedures allowing to construct the projector $\Pi_{V,W}$, spanning $\mathcal{V}$ and $\mathcal{W}$ subspaces that achieve some approximation properties such as stability [25], [12], [27], structure or passivity preservation [15] and minimal mismatch error [31], [13], [26]. For additional details, reader is invited to refer to the very complete book of [1] and works of [11], [2], [10].

Without loss of generalities, when considering the approximation of *LTI models*, a lot of attention has been - and still is - devoted to the development of fast and robust techniques reaching the so-called $\mathcal{H}_2$ optimal model approximation using Krylov-like schemes (see *e.g.* the MIMO IRKA, also denoted ITIA of [13], [9], [26] and [6]), or Sylvester ones (see *e.g.* the TSIA of [32], the ISRKA of [12] or the ISTIA of [21]) or structure properties (see [14] and references therein). In the *LTV model case*, very appealing results have been proposed in [19], allowing to handle linear time varying discrete time models using a finite time criterion (see also [20]). When considering the *multi-LTI model case*, different methods have been proposed by [17], [18], involving a local reduction followed by a common basis generation, or, when model parametrization is a-priori unknown, by [22], where a local reduction with a dedicated constraint on the pole location, before an interpolation step - to generate a reduced order LPV model - is proposed. Finally, very recent techniques to approximate *LPV models* has been developed using the relation between LPV and bilinear models (see *e.g.* [4], [5]), or using a "gridding version" of the tangential interpolation algorithm (*e.g.* [3]).

C. Paper contribution

Most of the above approximation techniques generally lead to models where *states or modes* have no meaning any-more, which can be a problem for practitioners. As an example, within the aeronautical industry, some modes are of particular interest and it is often required to identically reproduce them after the reduction step (*e.g.* the Dutch mode, the pitch mode...). This need for *mode preservation* is motivated by the fact that *(i)* their parameter dependency

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[1]Note that the large-scale nonlinear case is much more complex to handle and is, to the best of authors' knowledge, rarely considered in an industrial context for both control and analysis purpose; consequently, linear-like approaches are practically preferred.

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might be, in a practical context, well known and understood and *(ii)* their physical meaning is well known by engineers, control designers and specialists. To this end, the traditional approach consist either in structure preserving (if system can be structured) or in separating the states and approximating the non physical ones only. However this procedure may be complex due to coupling phenomena and approximation results might be non optimal.

This paper is devoted to the approximation *(i)* of LTI models, where some eigenvalues have to be kept; *(ii)* moreover it is also shown that it is appropriate for approximation of a certain class of LPV models as well (*i.e.* models that present a known parameter dependency on a finite number of eigenvalues). Formally, the problem is stated as follows:

Problem 1 (Eigenvalues preserving model approximation): Let us consider a stable and strictly proper n_u inputs n_y outputs MIMO LTI (*resp.* LPV) dynamical model of the form $\mathbf{H} := (A, B, C)$ (*resp.* $\mathbf{H}(\theta) := (A(\theta), B, C)$) of order n, defined as

$$\begin{array}{rcl}\mathbf{H} & : & \dot{x}(t) = Ax(t) + Bu(t)\ ,\ y(t) = Cx(t) \\ \mathbf{H}(\theta) & : & \dot{x}(t) = A(\theta)x(t) + Bu(t)\ ,\ y(t) = Cx(t)\end{array} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ (*resp.* $A(\theta) \in \mathbb{R}^{n\times n}$ a block diagonal matrix function of the parameter $\theta \in \Theta \subset \mathbb{R}^p$, where θ only affects a finite number of eigenvalues only), $B \in \mathbb{R}^{n\times n_u}$ and $C \in \mathbb{R}^{n_y\times n}$. The projection-based approximation problem consists of finding a projector $\Pi_{V,W} = VW^* \in \mathbb{C}^{n\times n}$ (with $V, W \in \mathbb{C}^{n\times r}$, $W^*V = I_r$), such that $\hat{\mathbf{H}} := (\hat{A}, \hat{B}, \hat{C})$ (*resp.* $\hat{\mathbf{H}}(\theta) := (\hat{A}(\theta), \hat{B}, \hat{C})$), a reduced model of order $r \ll n$, defined as

$$\begin{array}{rcl}\hat{\mathbf{H}} & : & \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)\ ,\ \ \hat{y}(t) = \hat{C}\hat{x}(t) \\ \hat{\mathbf{H}}(\theta) & : & \dot{\hat{x}}(t) = \hat{A}(\theta)\hat{x}(t) + \hat{B}u(t)\ ,\ \ \hat{y}(t) = \hat{C}\hat{x}(t)\end{array} \tag{3}$$

where $\hat{A} = W^*AV$ (*resp.* $\hat{A}(\theta) = W^*A(\theta)V$), $\hat{B} = W^*B$ and $\hat{C} = CV$, well approximates $\mathbf{H}$ (*resp.* $\mathbf{H}(\theta)$) in the sense of a given measure, and matches some given eigenvalues $\lambda_i^\star$, $i = 1, \ldots, q_1$ of A ($q_1 < r$).

The contribution of the paper is then to propose a mixed *Iterative Eigenvector Tangential Interpolation Algorithm (IETIA)* to solve Problem 1. The proposed procedure allows approximating large-scale (*i.e.* models with order n up to 20000 states) *(i)* LTI models while preserving some eigenvalues and *(ii)* a certain class of LPV models.

D. Paper structure & notations

The rest of the paper is organized as follows: preliminary results on the projection-based MIMO LTI approximation problem, the so-called $\mathcal{H}_2$ optimality conditions and a few very recent technical results, mainly based on the tangential approaches, are reminded in Section II. The main result, *i.e.* Theorem 2 and the Iterative Eigenvector Tangential Interpolation Algorithm (IETIA), are both presented in Section III. Illustrative examples of the performance and comparison with classical techniques using LTI and LPV model reduction benchmarks provided in [16] are presented in Section IV. Conclusion and perspectives are gathered in Section V.

Mathematical notations are standard: the original system state is denoted $x(t) \in \mathbb{R}^n$, the reduced-order system one is referred as $\hat{x}(t) \in \mathbb{R}^r$. Full order MIMO LTI state-space (*resp.* transfer) form is denoted $\mathbf{H} := (A, B, C)$ (*resp.* $H(s) := C(sI_n - A)^{-1}B$), and the reduced one $\hat{\mathbf{H}} := (\hat{A}, \hat{B}, \hat{C})$ (*resp.* $\hat{H}(s) := \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$). LPV models will be denoted with the $\theta \in \mathbb{R}^p$ parameter, and W and V denote the left and right projectors, respectively. $\mathcal{V} = \mathbf{span}(v)$ means that the space $\mathcal{V}$ is spanned by the column vectors of v. Then, $\Lambda(.)$ holds for the eigenvalue operator. Let λ, L and R be the eigenvalues, left and right eigenvectors, respectively. Finally, λ^* stands for the complex conjugate of λ.

II. Preliminary results

A. $\mathcal{H}_2$ model approximation

The main model approximation objective is to capture the most relevant system dynamics and input/output behaviour, while guaranteeing stability and achieving minimal model mismatch. As a matter of fact, the so-called $\mathcal{H}_2$ *optimal approximation problem* is often addressed. It consists of seeking an approximation $\hat{H}(s)$ of $H(s)$, such that,

$$\hat{H} = \underset{G\in\mathcal{H}_2}{\mathbf{argmin}}\, ||H - G||_{\mathcal{H}_2}, \tag{4}$$

where

$$||H||_{\mathcal{H}_2} = \Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} ||H(j\nu)||_F^2 d\nu\Big)^{1/2} \tag{5}$$

is the $\mathcal{H}_2$-norm associated to the (strictly proper and stable) rational function $H(s)$ [29]. This objective is non-convex, nevertheless, in the last years, many approaches have merged to reach a local minimum. One of the first result, due to Wilson [31], provides a characterization of the first-order optimality conditions, involving Lyapunov and Sylvester equations. Later, in two contributive papers [13], [26], another characterization is proposed using rational/tangential interpolation conditions. A link with Sylvester equation is also illustrated. Very recently, [32] shows the equivalence between both approaches, and suggests a Sylvester equation-based algorithm. Without loss of generality, as they mainly involve matrix vector operations, tangential interpolation methods appear to be appropriate for very-large scale model approximation and will thus be considered in this paper[2].

B. Results on tangential interpolation

More specifically, when considering the last tangential interpolation method class and by denoting $\mathcal{J} = ||H - \hat{H}||_{\mathcal{H}_2}$, the following theorem/corollary holds [26].

Theorem 1 (Tangential $\mathcal{H}_2$ optimality conditions): Given a full order original model $H(s)$, if $\nabla_{\hat{A}}\mathcal{J} = 0$, $\nabla_{\hat{B}}\mathcal{J} = 0$ and $\nabla_{\hat{C}}\mathcal{J} = 0$, the gradient of $\mathcal{J}$ with respect to $\hat{A}$, $\hat{B}$ and

[2] Note that a lot of attention still is devoted to the resolution of $Ax = b$, and very powerful tools exploiting the sparsity are accessible, allowing to handle very large-scale problems (*i.e.* LyaPack, M.E.S.S. and methods of P. Benner's research team).

$\hat{C}$, respectively, then the following tangential interpolation conditions are satisfied for all $\{\hat{\lambda}_1, \dots, \hat{\lambda}_r\}$, $i = 1, \dots, r$

$$\begin{aligned} \big(H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)\big)\hat{b}_i = 0 \,,\; \hat{c}_i^* \big(H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)\big) = 0 \\ \hat{c}_i^* \frac{d}{ds}\big(H(s) - \hat{H}(s)\big)\big|_{s=-\hat{\lambda}_i} \hat{b}_i = 0 \end{aligned} \tag{6}$$

where $\{\hat{b}_1, \dots, \hat{b}_r\} = \hat{B}^T L$ and $\{\hat{c}_1^*, \dots, \hat{c}_r^*\} = \hat{C} R$ (L and R are the left and right eigenvectors associated to $\hat{\lambda}$, the eigenvalues of $\hat{A}$).

Then, Corollary 1 provides the manner to construct the projectors V and W to fulfil any tangential interpolations [13], [26].

Corollary 1 (Tangential interpolation conditions): Given a full order original model $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank r such that $W^*V = I_r$. Let $\sigma_i \in \mathbb{C}$, $\hat{b}_i \in \mathbb{C}^{n_u}$ and $\hat{c}_i \in \mathbb{C}^{n_y}$, for $i = 1, \dots, r$, be given sets of interpolation points and left and right tangential directions, respectively. Assume that points σ_i are selected such that $\sigma_i I_n - A$ are invertible. If, for $i = 1, \dots, r$,

$$\begin{array}{rcl} (\sigma_i I_n - A)^{-1} B \hat{b}_i & \in & \mathbf{span}(V) \\ (\sigma_i I_n - A^T)^{-1} C^T \hat{c}_i^* & \in & \mathbf{span}(W) \end{array} \tag{7}$$

then, the reduced order system $\hat{H}(s)$, satisfies the tangential interpolation conditions,

$$\begin{aligned} H(\sigma_i)\hat{b}_i = \hat{H}(\sigma_i)\hat{b}_i \quad,\quad \hat{c}_i^* H(\sigma_i) = \hat{c}_i^* \hat{H}(\sigma_i) \\ \hat{c}_i^* \frac{d}{ds} H(\sigma_i)\hat{b}_i = \hat{c}_i^* \frac{d}{ds}\hat{H}(\sigma_i)\hat{b}_i. \end{aligned} \tag{8}$$

As rooted on the above theorem and corollary, it appears that to construct the optimal $\mathcal{H}_2$ approximation $\hat{H}(s)$ of $H(s)$, it is enough to select as interpolation points σ_i as the *mirror images of the reduced order model eigenvalues* $\hat{\lambda}_i$, and to construct the projectors V and W as in (7). As this information is obviously not a-priori available, numerically stable and robust algorithms have been developed to achieve the optimal reduced order model. To the author's feeling, the most relevant one is the IRKA of Gugercin [13]. This algorithm has then be followed by different variants (*i.e.* ISRKA of [12], the ISTIA of [21], which are available in the MORE toolbox of the authors [23]). When using the IRKA for given shift points $\sigma_i = -\hat{\lambda}_i$, it provides the best $\hat{B}$ and $\hat{C}$ matrices. The main limitation of this approach is that it leads to a procedure where no control can be done on the resulting reduced order eigenvalues. Indeed, the eigenvalues of the reduced order model are usually different (even slightly) from the original ones (see *e.g.* [22]).

III. Main result: a mixed Iterative Eigenvector Tangential Interpolation Algorithm (IETIA)

A. General idea and result

When applying the Petrov-Galerkin projection-based approximation approach, what matters are the subspaces $\mathcal{V}$ and $\mathcal{W}$ spanned by V and W. While the $\mathcal{H}_2$ optimality-oriented algorithms presented in Section II aim at constructing $V, W \in \mathbb{C}^{n\times r}$ such that they span the tangential interpolation subspaces, the ground idea of the proposed approach is to construct such a basis (V and W, *i.e.* projector $\Pi_{V,W}$), so that they *span some user-defined eigenvectors and tangential interpolation points and directions*. Thus, the objective is both to achieve $\mathcal{H}_2$ optimality conditions (as in Theorem 1) and eigenvalue recovery. The main result is given in Theorem 2.

Theorem 2 (Interpolation and eigenvectors conditions): Given a full order original model $H(s)$, let $V \in \mathbb{C}^{n\times r}$ and $W \in \mathbb{C}^{n\times r}$ be matrices of full column rank $r = q_1 + q_2$ such that $W^*V = I_r$. Let $l_i^\star \in \mathbb{C}^n$ and $r_i^\star \in \mathbb{C}^n$ be left and right eigenvectors associated to $\lambda_i^\star \in \mathbb{C}$ eigenvalues associated to A, for $i = 1, \dots, q_1$. Let $\sigma_j \in \mathbb{C}$, $\hat{b}_j \in \mathbb{C}^{n_u}$ and $\hat{c}_j \in \mathbb{C}^{n_y}$, for $j = 1, \dots, q_2$, be given sets of interpolation points and left and right tangential directions, respectively. Assume that points $\sigma_j \neq -\lambda_i^\star$ are selected such that $\sigma_j I_n - A$ are invertible. If, for $i = 1, \dots, q_1$ and $j = 1, \dots, q_2$,

$$\begin{array}{rcl} \begin{bmatrix} r_i^\star & (\sigma_j I_n - A)^{-1} B \hat{b}_j \end{bmatrix} & \in & \mathbf{span}(V) \\ \begin{bmatrix} l_i^\star & (\sigma_j I_n - A^T)^{-1} C^T \hat{c}_j^* \end{bmatrix} & \in & \mathbf{span}(W) \end{array} \tag{9}$$

then, the reduced order system $\hat{H}(s)$ satisfies the eigenvalue conditions,

$$\{\lambda_1^\star, \dots, \lambda_{q_1}^\star\} \subset \{\Lambda(\hat{A})\} \tag{10}$$

and the tangential interpolation conditions

$$\begin{aligned} H(\sigma_j)\hat{b}_j = \hat{H}(\sigma_j)\hat{b}_j \quad,\quad \hat{c}_j^* H(\sigma_j) = \hat{c}_j^* \hat{H}(\sigma_j) \\ \hat{c}_j^* \frac{d}{ds} H(\sigma_j)\hat{b}_j = \hat{c}_j^* \frac{d}{ds}\hat{H}(\sigma_j)\hat{b}_j. \end{aligned} \tag{11}$$

Proof: [of Theorem 2]

(i) First, let $\{l_i^\star, r_i^\star, \lambda_i^\star\}$ be the ith (for $i = 1, \dots q_1$) eigenvalue and eigenvectors associated to A such that,

$$A r_i^\star = \lambda_i^\star r_i^\star \text{ and } l_i^\star A = l_i^\star \lambda_i^\star \tag{12}$$

it comes that if,

$$r_i^\star \in \mathbf{span}(V_{q_1}) \text{ and } l_i^\star \in \mathbf{span}(W_{q_1}) \tag{13}$$

such that, $V_{q_1}, W_{q_1} \in \mathbb{C}^{n\times q_1}$ and $W^*V = I_{q_1}$, then $\hat{A} = W_{q_1}^* A V_{q_1}$ have eigenvalues in $\lambda_i^\star$ which proves relation (10) and thus the eigenvalue preservation.

(ii) Second, given a set of shift points $\sigma_j \neq -\lambda_i^\star$ and tangential direction $\hat{b}_j$ and $\hat{c}_j$, for $j = 1, \dots, q_2$, such that,

$$\begin{array}{rcl} (\sigma_j I_n - A)^{-1} B \hat{b}_j & \in & \mathbf{span}(V_{q_2}) \\ (\sigma_j I_n - A^T)^{-1} C^T \hat{c}_j^* & \in & \mathbf{span}(W_{q_2}) \end{array} \tag{14}$$

such that, $V_{q_2}, W_{q_2} \in \mathbb{C}^{n\times q_2}$ and $W_{q_2}^* V_{q_2} = I_{q_2}$, respectively, Corollary 1 implies relation (11).

(iii) Finally, let $\mathbf{span}(V) = \mathbf{span}(V_{q_1}) \cup \mathbf{span}(V_{q_2})$ and $\mathbf{span}(W) = \mathbf{span}(W_{q_1}) \cup \mathbf{span}(W_{q_2})$, such that $W^*V = I_r$ (with $r = q_1 + q_2$), then relation (9) holds, which concludes the proof. ■

Remark 1 (Complex and real projectors $\Pi_{V,W}$): As stated in Theorem 2, the resulting matrices V and W are both complex. However, for control and simulation purpose, it is preferable to work with real matrices only. To do so, one has to work with complex conjugate shift points σ_i and preserved eigenvalues $\lambda_i^\star$ and exploit the fact that, given $v_1 \in \mathbb{C}^n$, $\textbf{span}(v_1, v_1^*) = \textbf{span}(Re(v_1), Im(v_1))$. Therefore, when constructing V and W, one should consider either real or complex conjugated shift points and eigenvectors.

B. The IETIA

To achieve conditions of Theorem 2, the Iterative Eigenvector Tangential Interpolation Algorithm (IETIA, Algorithm 1), is proposed. It is very similar to the ITIA (or MIMO IRKA) proposed by [13], except from the fact that the basis V and W do *span the a part of the generalized multi-point tangential Krylov subspaces only, plus some eigenvalues/eigenvectors*. This allows some user-defined modes preservation and tangential interpolation.

With reference to Algorithm 1, the first step consists in finding the left and right eigenvectors related to the eigenvalues $\lambda_i^\star$. This step can be performed either with an exact eigenvalue computation or in a very cheap way, using eigenvalue approximations when A is sparse (thought Arnoldi approaches). Then, steps 2-3 constructs the basis of subspace of Theorem 2 and ensure orthogonality. Then, from step 4 to 12, the process is repeated using as new shift points σ_j, the mirror images of the eigenvalues of the reduced order model $\Lambda(\hat{A})$ (as in the IRKA), such that $\sigma_j \neq -\lambda_i^\star$. The algorithm can stop for varying criteria. As it is not the focus of the paper, here the relative variation of the shift points is used (*i.e.* $|\sigma^{(i)} - \sigma^{(i-1)}|/|\sigma^{(i)}|$). An other approach would be to involve the $\mathcal{H}_2$ error (if system is not too large, *i.e.* $n > 2000$). In this case, one should use the spectral $\mathcal{H}_2$-norm computaion approach (see [29]). The reduced order model is constructed by left/right projection, as in step 13. The algorithm satisfies Theorem 2.

Remark 2 (Initial shift points): The initial shift point selection may be done using [28] procedure or the dominant poles using subspace acceleration of [24].

Remark 3 (Algorithm convergence): The convergence of Algorithm 1 has not been proved, but, to the authors' experience, it converges in some few iterations. To be convinced of this observation, interested reader can download the MORE toolbox [23] (developped by the authors), where the IETIA (and other algorithms) is implemented.

IV. Numerical illustration

To illustrate the effectiveness of the proposed IETIA on both LTI and a specific class of LPV systems, the open access clamped beam model (SISO, $n = 348$, `'CBM'` in Compl$_e$ib [16]), is considered. Attention is of course given to the pole representation, but the relative mismatch $\varepsilon_{\mathcal{H}_2}$ will also be used for fair comparison with other approaches.

$$\varepsilon_{\mathcal{H}_2} = \frac{||H - \hat{H}||_{\mathcal{H}_2}}{||H||_{\mathcal{H}_2}}$$

A. LTI example

Here, the clamped beam model is approximated with an order $r = 18$, using the standard Balanced Truncation (BT), the ITIA (or MIMO IRKA) [13], the ISTIA [21] and the proposed IETIA (Algorithm 1). To illustrate the purpose of this paper, the IETIA is initialized with two complex conjugate eigenvalues sets, *i.e.* $\lambda^{\star(1)} = \{-37.5171 + 78.0698i, -37.5171 - 78.0698i\}$ (left frame) and $\lambda^{\star(2)} = \{-0.0066 + 0.5686i, -0.0066 - 0.5686i\}$ (right frame); consequently, $q_1 = 2$ and $q_2 = 16$ in Algorithm 1. With reference to Figure 1, the pole location of each algorithms (except the BT, for readability reasons) is plotted, illustrating the fact the proposed IETIA perfectly preserves the selected $\lambda^\star$ eigenvalues.

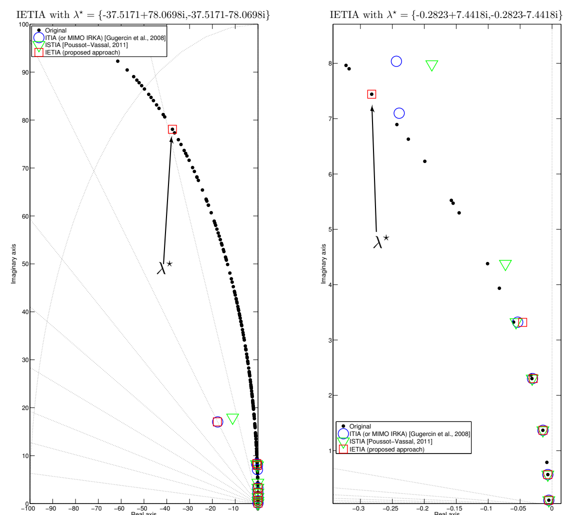

Fig. 1. Eigenvalues of the reduced order models of order $r = 18$, for different approximation methods: IRKA (rounds), ISTIA (triangles) and IETIA (squares), compared to the original one of order $n = 348$ (dots). Left, entire spectrum for IETIA initialized with $\lambda^{\star(1)}$ (in high frequency). Right, zoom around the area where the objective $\lambda^{\star(2)}$ is located.

Moreover, to illustrate the good matching as well, the relative mismatch error is also reported on Table I for all methods.

Method	$\varepsilon_{\mathcal{H}_2}$
BT	4.516895×10^{-3}
MIMO IRKA (or ITIA) [13], [26]	2.608080×10^{-3}
ISTIA [21]	2.625621×10^{-3}
IETIA (Algorithm 1, with $\lambda^{\star(1)}$)	2.615863×10^{-3}
IETIA (Algorithm 1, with $\lambda^{\star(2)}$)	8.772606×10^{-3}

TABLE I

Relative mismatch error $\varepsilon_{\mathcal{H}_2}$ for the different approximation methods, with $r = 18$.

With reference to Table I, the ITIA appears to be the best approximation, the ISTIA provides very good approximation results as well, especially compared to the BT (this is consistent theoretical results [13], [21], [30]). On the other side, the price paid by the modal preservation of the IETIA shows to depend on the selected modes. Indeed, selecting as preserved poles $\lambda^\star$, a dominant pole, will provide good results in term of mismatch error (*e.g.* $\lambda^{\star(1)}$), while selecting

Algorithm 1 Iterative Eigenvector Tangential Interpolation Algorithm (IETIA)

Require: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$, $C \in \mathbb{R}^{n_y\times n}$, $\{\lambda_1^\star,\dots,\lambda_{q_1}^\star\} \in \mathbb{C}^{q_1}$, $\{\sigma_1^{(0)},\dots,\sigma_{q_2}^{(0)}\} \in \mathbb{C}^{q_2}$, $\{\hat{b}_1,\dots,\hat{b}_{q_2}\} \in \mathbb{C}^{n_u\times q_2}$, $\{\hat{c}_1,\dots,\hat{c}_{q_2}\} \in \mathbb{C}^{n_y\times q_2}$ and $r = q_1+q_2 \in \mathbb{N}$

Compute $\{l_1^\star,\dots,l_{q_1}^\star\}$ and $\{r_1^\star,\dots,r_{q_1}^\star\}$, the left and right eigenvectors associated to $\{\lambda_1^\star,\dots,\lambda_{q_1}^*\}$, some selected eigenvalues of A (this step can be performed using a shifted Arnoldi procedure and a Schur transformation, in order to reduce the numerical complexity of eigenvalues computation).

Construct,

$$\begin{aligned}\textbf{span}(V) &= \left[l_1^\star,\dots,l_{q_1}^\star,(\sigma_1^{(0)}I_n-A)^{-1}B\hat{b}_1,\dots,(\sigma_{q_2}^{(0)}I_n-A)^{-1}B\hat{b}_{q_2}\right]\\ \textbf{span}(W) &= \left[r_1^\star,\dots,r_{q_1}^\star,(\sigma_1^{(0)}I_n-A^T)^{-1}C^T\hat{c}_1^*,\dots,(\sigma_{q_2}^{(0)}I_n-A^T)^{-1}C^T\hat{c}_{q_2}^*\right]\end{aligned} \tag{15}$$

Compute $W \leftarrow W(V^TW)^{-1}$

while Stopping criteria **do**

  $i \leftarrow i+1$

  $\hat{A} = W^TAV$, $\hat{B} = W^TB$, $\hat{C} = CV$

  Compute $\hat{A}R = \Lambda(\hat{A})R$ and $L\hat{A} = \Lambda(\hat{A})L$

  Compute $\{\hat{b}_1,\dots,\hat{b}_{q_2}\} = \hat{B}^TL$ and $\{\hat{c}_1^*,\dots,\hat{c}_{q_2}^*\} = \hat{C}R$

  Set $\sigma^{(i)} = -\Lambda(\hat{A})$

  Construct,

$$\begin{aligned}\textbf{span}(V) &= \left[l_1^\star,\dots,l_{q_1}^\star,(\sigma_1^{(i)}I_n-A)^{-1}B\hat{b}_1,\dots,(\sigma_{q_2}^{(i)}I_n-A)^{-1}B\hat{b}_{q_2}\right]\\ \textbf{span}(W) &= \left[r_1^\star,\dots,r_{q_1}^\star,(\sigma_1^{(i)}I_n-A^T)^{-1}C^T\hat{c}_1^*,\dots,(\sigma_{q_2}^{(i)}I_n-A^T)^{-1}C^T\hat{c}_{q_2}^*\right]\end{aligned} \tag{16}$$

  Compute $W \leftarrow W(V^TW)^{-1}$

end while

Construct $\hat{\mathbf{H}} := (W^TAV, W^TB, CV)$

Ensure: $V, W \in \mathbb{R}^{n\times r}$, $W^TV = I_r$, $\{\lambda_1^\star,\dots,\lambda_{q_1}^\star\} \subset \Lambda(\hat{A})$

less relevant ones will deteriorate the matching (*e.g.* $\lambda^{\star(2)}$). Nevertheless, the main objective of the IETIA is shown to be satisfied, *i.e.* preserve some modal content and ensure $\mathcal{H}_2$ optimality conditions on the rest of the approximation.

B. LPV example

Now illustration has been made on a simple LTI case, here, let us illustrate how this can be exploited in a specific LPV framework. Based on the LTI clamped beam model $\mathbf{H}$, let us construct a parametrized clamped beam model $\mathbf{H}(\theta)$ as: $\dot{x}(t) = A(\theta)x(t)+Bu(t)$, $y(t) = Cx(t)$, where $A(\theta) = A^0 + A^1\theta$. Let $\theta \in [0\ 1]$ be the varying parameter, $A^0 \in \mathbb{R}^{n\times n}$ be the original clamped beam A matrix diagonalized in a block form with eigenvalues sorted in ascending magnitude order and $A^1 \in \mathbb{R}^{n\times n}$ be a matrix with zeros everywhere except on the 5th block, where values $A^1_{9:10,9:10}$ are the copy of the A^0's ones (*i.e.* , $A^1_{9:10,9:10} = A^0_{9:10,9:10}$). Then, considering this constructed model, it is obvious that the parameter dependency only affects one single eigenvalue, *i.e.* $\lambda^\star = \{-0.0315+2.3030i, -0.0315-2.3030i\}$. This eigenvalue pair is thus used in the proposed algorithm (IETIA) in order to approximate the original model of order $n = 348$, with a $r = 4$ one (in this case, $q_1 = 2$ and $q_2 = 2$). Figure 2 shows the frequency response of the original and approximated LPV models. Note that here, the IETIA is compared to the ITIA only ($\mathcal{H}_2$ optimal), to facilitate readability.

With reference to Figure 2 it can be observed that while the ITIA focusses on well approximating the first two peaks

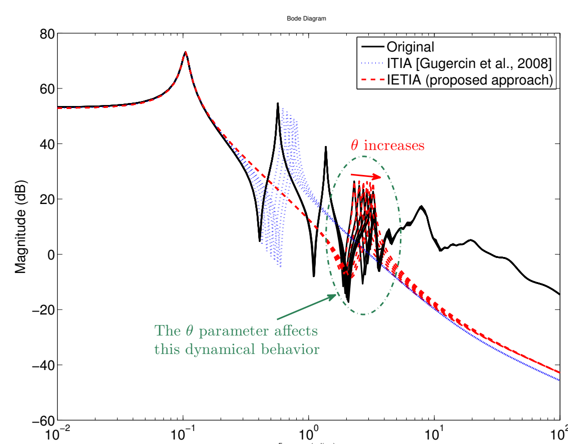

Fig. 2. Frequency response, for varying $\theta \in [0\ 1]$ values, full order model (black solid) and of the approximated model of order $r = 4$ using the ITIA (blue dotted) and the IETIA (red dashed).

(to ensure optimal mismatch error), it fails reproducing the parameter dependent part, acting on a higher frequency (or mode). On the other side the IETIA - given $\lambda^\star$ - well captures the first peak of the transfer and *the parameter dependent peak of specific interest* (linked to the parameter dependent eigenvalue). Therefore, when reconstructing the reduced LPV model (*i.e.* transform the reduced system in block diagonal for and apply the same parameter dependent function as for the original one), it well reproduces the frequency behavior in the frequency range where the θ parameter vary.

V. Conclusion and perspectives

In this paper, an extended version of the MIMO IRKA (or ITIA), namely, the Iterative Eigenvector Tangential Interpolation Algorithm (IETIA), has been proposed. This procedure allows for MIMO large-scale model approximation under the constrain of preserving some user-defined eigenvalues/eigenvectors. This algorithm is proven to approximate LTI models with optimal mismatch error, while preserving some user-defined eigenvalues. As modal content (information) usually is exploited by practitioners and engineers for understanding, analysis and control design purpose, this property makes the algorithm very appealing in many applicative fields. Indeed, modal preservation is a recursive demand from industrial partners within the aeronautic field where eigenvalues are often linked to well-known behaviours (see [22]). Due to the very-low additional numerical cost, the proposed IETIA results to be both simple and applicable to (very) large-scale MIMO models. Numerical illustrations on standard models have shown the effectiveness of the proposed algorithm for both LTI and a specific class of LPV models, which is a another contribution.

Finally, to emphasize the effectiveness of the the proposed algorithm is also made available in the MORE toolbox developed by the authors [23].

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