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Gravity Gradiometer Integrated Inertial Navigation

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Abstract—**In high precision inertial navigation, gravity field modeling error becomes a limiting factor. Granted that high precision accelerometers are used, airborne gravity gradiometry can be employed in a self-contained way to accurately estimate the gravity field on the fly and eliminate the gravity field modeling error. The local acceleration of gravity will be estimated using the onboard accelerometer measurements, provided that the acceleration measurements are very accurate, as is the case in high precision INS using cold atom interferometry-based accelerometers. An autonomous free-inertial gravity gradiometer integrated aircraft navigation system is promulgated. The accurate mapping of the gravity field along the aircraft's flight path is an added benefit.**

I. Introduction

The ability to navigate precisely is a critical enabler on the modern battlefield. Most modern military aircraft and munitions rely on GPS for position updates as they navigate. GPS uses a set of signals from navigation satellites to determine position and velocity. It is a proven system and provides the needed accuracy for most mission objectives in a cost-effective way. However, these satellite signals can be unavailable due to physical blockage (inside a cave or underwater), denied by jamming, or subjected to interference, as in spoofing.

There is currently a significant amount of research into methods to navigate precisely in a GPS denied environment. Some of these include, but are not limited to, the use of stellar tracking [11], terrain referenced navigation (TRN) such as the Sandia Inertial Terrain Aided Navigation (SITAN) system [4], Terrain Contour Matching (TERCOM) [5], Terrain Profile Matching (TERPROM) [16], and INSs which can be provided with position updates from the aforementioned navigation methods to correct errors that accumulate over time. Finally, a method of aircraft and submarine navigation that has been suggested in the recent literature is by the use of a device known as a Gravity Gradient Instrument (GGI), or gravity gradiometer.

This paper discusses the feasibility of using an airborne GGI in a high precision inertial navigation system. The benefit of a gravity gradient instrument is that the measurement is completely passive and not subject to interference from external sources. This integrated system relies solely on passive measurements exclusively using inertial sensors, making it ideal for missions in which GPS is unavailable or unusable. The envisioned navigation system will operate passively and autonomously, without emitting or receiving signals in the electromagnetic spectrum. As designed, the GGI-augmented high precision INS will enable navigation over long durations and distances, suitable for strategic airlift, reconnaissance, or attack missions, without being subject to detection or denial.

There are two schools of thought concerning the employment of gravity gradiometry in high precision, fully autonomous INSs. The first approach is to use the best available gravity field model on board an aircraft to aid the INS with GGI measurements, using a Kalman filter arrangement. The second is particularly suited to high precision inertial navigation where gravity field error becomes a limiting factor: it is suggested to employ gravity gradiometry to estimate the gravity field on the fly to a high degree of accuracy and thus reduce the gravity modeling error in high accuracy free-inertial navigation. This paper focuses on the latter in the context of high precision inertial navigation where cold atom interferometry based accelerometers are used. The development of a strategy to integrate gravity gradiometry into free inertial navigation is undertaken.

II. The Use of Gravity Gradiometry

Gravitation and gravity are distinct terms, as the latter accounts for the centrifugal force due to Earth's rotation and is what is sensed by an observer using a plumb on the surface of the Earth. In the following discussion, the terms gravitation and gravity—and gravitational gradient and gravity gradient—are used interchangeably. However, it is important to note that a GGI measures specific force due to gravitational attraction and is more precisely described as a *gravitational* gradient instrument. Spatial changes in the gravitational field are a result of the nonhomogeneity of the Earth, oblateness, the Earth's rotation, and variations in the Earth's terrain. The magnitudes of these differences are very small, and their measurement requires sensitive instruments. A gravity gradiometer is an instrument that measures the difference in the Earth's gravitational acceleration across a known, fixed baseline. Development of these instruments is an area of active research which has already demonstrated practical utility. Use of airborne gravity gradiometry is becoming more prevalent in surveys for mining and natural resource prospecting, as aircraft have inherent advantages of covering more ground than their terrestrial-based counterparts and can survey areas that cannot easily be reached over land [13].

A relatively recent body of research into using the measurement of variations in the Earth's gravitational field for the purpose of navigation exists. In 1990, Affleck and Jircitano

†The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the United States Government.

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discussed an INS that was aided by a gravity gradient-based map-matching algorithm [2]. In 1995, Gleason presented a terrain avoidance method using optimal generation of gravity gradient maps and GGI sensor filtering [9]. In 2006, Jekeli investigated a future high-accuracy inertial measurement unit that could provide near-GPS accuracy by using an onboard integrated gravity gradiometer to provide in situ compensation [10]. In 2008, Richeson investigated airborne gravity gradiometry in a high-altitude hypersonic cruise application [15]. It is claimed to be "the first thorough derivation and investigation of a map-matching gravity gradiometer aided inertial navigation system" and explicitly includes reference to navigation in a GPS-denied environment using an extended Kalman filter. The sensitivity required for "GPS-like performance" from the onboard gradiometer in this application, if the simulated GGI is assumed to be giving the full gravitational gradient tensor at an update rate of 1 Hz, was determined from a Monte Carlo simulation to be 10^{-3} E (Eötvös, an acceleration gradient unit named for Loránd Eötvös, a nineteenth-century Hungarian physicist: $1\,\mathrm{E} = 10^{-9}/\mathrm{s}^2$). An M.S. thesis was presented by Rogers in 2009 that investigated the inclusion of terrain variation effects on a standard Earth gravitational model and used the augmented model to compensate an INS through use of a Bayesian map-matching technique [16]. DeGregoria continued this work in 2010 [8] and found that the map-matching technique stabilized INS errors if the GGI had a sufficiently low noise level.

III. Accelerometer Measurements

We begin by presenting the formulae for measurement of acceleration. Accelerometers measure specific force f:

$$\boldsymbol{f} = \boldsymbol{a} - \boldsymbol{g}$$

where $\boldsymbol{a}$ is the kinematic acceleration and $\boldsymbol{g}$ is the acceleration of gravity. Thus, in order to integrate the accelerometer measurement and obtain our position, the effect of gravity must be removed from the accelerometer's specific force measurements. Gravity is a function of position. Reference the measurement setup in Figure 1, where six non-collocated three-axis accelerometers numbered $(1-6)$ are shown. Eighteen single-axis accelerometers are needed for navigation in three dimensions. The angular acceleration can be measured with accelerometers only for a gyro-free inertial navigation system [14]. The position of the origin of the space-stabilized platform is (x, y, z). The acceleration of the space-stabilized platform is (a_x, a_y, a_z). The position-dependent gravitation vector components are G_x, G_y and G_z.

In this paper we are solely interested in the proof of concept. Thus, precision inertial navigation in a two-dimensional "vertical" plane (x, z) is considered, and the gravity field is expressed as a function of position, $\boldsymbol{g} = (G_x(x,z), G_z(x,z))$. Furthermore, we assume a perfect gyroscope on board the space stabilized platform in order to focus exclusively on gravity measurement errors. With the addition of a ring laser gyro or fiber optic gyro to the inertial navigation system,

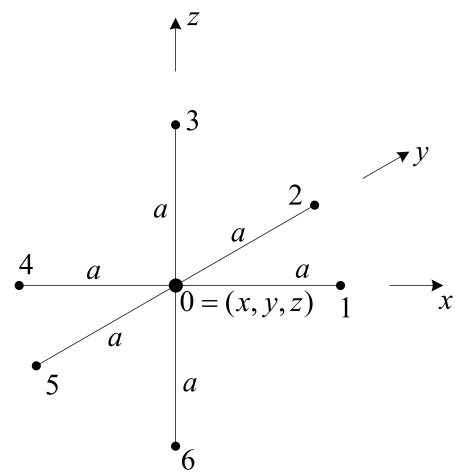

Fig. 1. Distributed accelerometer arrangement.

gyroscopic errors will obviously add to the error budget, and gyro dynamics would need to be included in the system mechanization. Thus, in [14] we consider a complete INS with an accelerometers-only inertial measurement unit that uses precise accelerometers such as those in cold atom interferometry based instruments. A gimballed, space-stabilized platform controlled by an ideal gyroscope which maintains the pitch angle $\theta \equiv 0$ is shown in Figure 2. On that platform are mounted four dual-axis accelerometers: in two dimensions, eight single-input axis accelerometers, distributed at the four locations $(1, 2, 3, 4)$, are needed. The accelerometer input axes are aligned with the platform axes, which, in principle, are rotated by an angle θ relative to the inertial frame. But, since an ideal gyroscope is considered, the angle $\theta \equiv 0$. The accelerometers are cold atom interferometry-based precision instruments, and a non-rotating Earth is assumed. In the simplified two-dimensional case, and since the angle $\theta \equiv 0$, the specific force measurement equations are

$$\begin{aligned} f1_x &= a_x - G_x(x+a, z) & f1_z &= a_z - G_z(x+a, z) \\ f2_x &= a_x - G_x(x, z+b) & f2_z &= a_z - G_z(x, z+b) \\ f3_x &= a_x - G_x(x-a, z) & f3_z &= a_z - G_z(x-a, z) \\ f4_x &= a_x - G_x(x, z-b) & f4_z &= a_z - G_z(x, z-b) \end{aligned}$$

where (a_x, a_z) is the acceleration at the center of the platform. Using a first-order Taylor series expansion, we obtain

$$f1_x \approx a_x - G_x(x,z) - G_{x,x}a \qquad f1_z \approx a_z - G_z(x,z) - G_{z,x}a \tag{1}$$

$$f2_x \approx a_x - G_x(x,z) - G_{x,z}b \qquad f2_z \approx a_z - G_z(x,z) - G_{z,z}b \tag{2}$$

$$f3_x \approx a_x - G_x(x,z) + G_{x,x}a \qquad f3_z \approx a_z - G_z(x,z) + G_{z,x}a \tag{3}$$

$$f4_x \approx a_x - G_x(x,z) + G_{x,z}b \qquad f4_z \approx a_z - G_z(x,z) + G_{z,z}b \tag{4}$$

The eight unknown variables are

$$a_x,\ a_z,\ G_x(x,z),\ G_z(x,z),\ \left.\frac{\partial G_x}{\partial x}\right|_{x,z},\ \left.\frac{\partial G_x}{\partial z}\right|_{x,z},\ \left.\frac{\partial G_z}{\partial x}\right|_{x,z},\ \left.\frac{\partial G_z}{\partial z}\right|_{x,z}$$

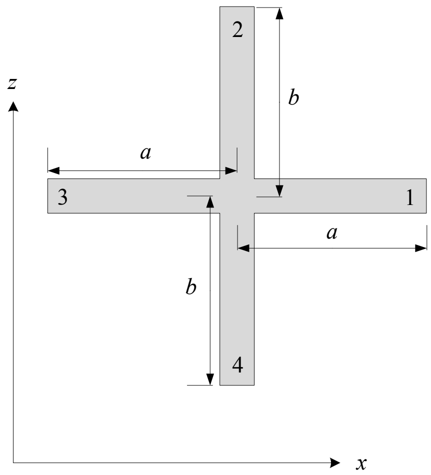

Fig. 2. Distributed accelerometer arrangement for navigation in two dimensions.

Equations (1)–(4) are differenced to obtain

$$\frac{\partial G_x}{\partial x} = \frac{f3_x - f1_x}{2a} \quad \frac{\partial G_x}{\partial z} = \frac{f4_x - f2_x}{2b}$$
$$\frac{\partial G_z}{\partial x} = \frac{f3_z - f1_z}{2a} \quad \frac{\partial G_z}{\partial z} = \frac{f4_z - f2_z}{2b}$$

In addition, Equations (1) and (3) yield

$$a_x - G_x(x,z) = \frac{1}{2}(f1_x + f3_x)$$

and similarly Equations (2) and (4) yield

$$a_x - G_x(x,z) = \frac{1}{2}(f2_x + f4_x)$$

By using all the available measurements, $a_x - G_x(x,z)$ can be expressed as

$$a_x - G_x(x,z) = \frac{1}{2}\cdot\left[\frac{1}{2}(f1_x + f3_x) + \frac{1}{2}(f2_x + f4_x)\right]$$

Thus,

$$a_x - G_x(x,z) = \frac{1}{4}(f1_x + f3_x + f2_x + f4_x)$$

which is the same measurement that would be recorded by an accelerometer placed at the center of the space-stabilized platform.

Note that the measurement's error variance is

$$\sigma_{(a_x - G_x)} = \sqrt{\left(\frac{1}{4}\right)^2 \times 4\sigma^2} = \frac{1}{2}\sigma$$

The same result is obtained for the z-axis.

IV. Estimation of Local Gravity by Dead Reckoning

One benefit of the integration of gravity gradient measurements into the inertial navigation is that the gravity field along the aircraft's flight path can be isolated from the body acceleration and recorded. Based on the high precision accelerometer measurements from cold atom interferometry based instruments, an accurate estimate of gravity can be made. Assume that at time $t = 0$, at a known starting position (x_0, z_0), the gravity vector $\mathbf{G}(x_0, z_0) = (G_x(x_0, z_0), G_z(x_0, z_0))$ is known. Also assume that the platform is kept in perfect alignment with the inertial frame, that is, $\theta \equiv 0$. We exclusively focus on the navigation accuracy afforded by cold atom interferometry-based highly accurate accelerometers—to this end, we assume a perfect gyroscope. It is necessary to determine the time history of the gravity vector

$$G_x(x(t), z(t)),\ G_z(x(t), z(t)),\ \forall\ 0 \le t$$

Taking the total derivative gives

$$\frac{\mathrm{d}}{\mathrm{d}t}G_x(x(t), z(t)) = \left.\frac{\partial G_x}{\partial x}\right|_{x,z} V_x + \left.\frac{\partial G_x}{\partial z}\right|_{x,z} V_z$$

where (V_x, V_z) is the inertial velocity. Given the initial condition/initial gravity measurement

$$G_x(x(0), z(0)) = G_x(x_0, z_0)$$

the following expression for the x-component of local gravity along the flight trajectory is obtained:

$$G_x(x(t), z(t)) = G_x(x_0, z_0) + \int_0^t \left(\left.\frac{\partial G_x}{\partial x}\right|_{x,z} V_x + \left.\frac{\partial G_x}{\partial z}\right|_{x,z} V_z \right) \mathrm{d}t$$

Similarly:

$$G_z(x(t), z(t)) = G_z(x_0, z_0) + \int_0^t \left(\left.\frac{\partial G_z}{\partial x}\right|_{x,z} V_x + \left.\frac{\partial G_z}{\partial z}\right|_{x,z} V_z \right) \mathrm{d}t$$

Substituting the accelerometer measurements of the gravity gradients gives

$$G_x(x(t), z(t)) = G_x(x_0, z_0) + \frac{1}{2}\left[\frac{1}{a}\int_0^t (f3_x - f1_x)V_x\,\mathrm{d}t + \frac{1}{b}\int_0^t (f4_x - f2_x)V_z\,\mathrm{d}t\right] \tag{5}$$

$$G_z(x(t), z(t)) = G_z(x_0, z_0) + \frac{1}{2}\left[\frac{1}{a}\int_0^t (f3_z - f1_z)V_x\,\mathrm{d}t + \frac{1}{b}\int_0^t (f4_z - f2_z)V_z\,\mathrm{d}t\right] \tag{6}$$

Hence, as long as the initial gravity vector is known, the local acceleration of gravity $G(x,z)$ can be autonomously estimated on the fly using onboard accelerometer measurements in a GGI arrangement, and the estimate is good if the acceleration measurements are very accurate, as is the case in cold atom interferometry.

V. Mechanization and Error Equations in the Inertial Frame

We now present the INS mechanization and error equations for the distributed accelerometer arrangement. The mechanization equations for a gimballed space-stabilized INS, assuming a perfect gyroscope so that $\theta \equiv 0$, are

$$
\begin{aligned}
\dot{x} &= V_x \\
\dot{z} &= V_z \\
\dot{V}_x &= f_x(x,z) - G_x(x,z) \\
\dot{V}_z &= f_z(x,z) + G_z(x,z) \\
\dot{G}_x &= \frac{1}{2a}(f3_x - f1_x)V_x + \frac{1}{2b}(f4_x - f2_x)V_z \\
\dot{G}_z &= \frac{1}{2a}(f3_z - f1_z)V_x + \frac{1}{2b}(f4_z - f2_z)V_z
\end{aligned}
$$

where:

$$
\begin{aligned}
f_x(x,z) &= \frac{1}{4}(f1_x + f3_x + f2_x + f4_x) \\
f_z(x,z) &= \frac{1}{4}(f1_z + f3_z + f2_z + f4_z)
\end{aligned}
$$

These are the specific forces in each axis that putatively would be measured by accelerometers at the center of the platform. The specific force measurements of the distributed accelerometers are $f1_x, f2_x, f3_x, f4_x$ and $f1_z, f2_z, f3_z, f4_z$, and the gravitational accelerations $G_x(x,z)$ and $G_z(x,z)$ are given by Equations (5) and (6), respectively. In our two-dimensional scenario and under the assumption of an inertial space stabilized gimballed platform using a perfect gyroscope, the augmented navigation state is x, z, V_x, V_z, G_x and G_z; the gravity vector components G_x and G_z are new non-traditional navigation states.

We assume that a perfect gyroscope is used, i.e., the platform axes are aligned with the i-frame at all times; this is done so that we can focus exclusively on the achievable navigation accuracy afforded by cold atom interferometry-based high precision accelerometers. By construction, the navigation state error dynamics are linear. Thus, superposition can be used to add the various sources of error to obtain the total error in the navigation system. The error equations in the gimballed, space stabilized INS are then expressed as follows:

$$
\begin{aligned}
\dot{\delta x} &= \delta V_x \\
\dot{\delta z} &= \delta V_z \\
\dot{\delta V}_x &= \frac{1}{4}(\delta f1_x + \delta f2_x + \delta f3_x + \delta f4_x) - \delta G_x(x,z) \\
\dot{\delta V}_z &= \frac{1}{4}(\delta f1_z + \delta f2_z + \delta f3_z + \delta f4_z) - \delta G_z(x,z) \\
\dot{\delta G}_x &= \frac{1}{2a}(\delta f3_x - \delta f1_x)V_x + \frac{1}{2b}(\delta f4_x - \delta f2_x)V_z \\
&\quad + \frac{1}{2a}(f3_x - f1_x)\delta V_x + \frac{1}{2b}(f4_x - f2_x)\delta V_z \\
\dot{\delta G}_z &= \frac{1}{2a}(\delta f3_z - \delta f1_z)V_x + \frac{1}{2b}(\delta f4_z - \delta f2_z)V_z \\
&\quad + \frac{1}{2a}(f3_z - f1_z)\delta V_x + \frac{1}{2b}(f4_z - f2_z)\delta V_z
\end{aligned}
$$

We use the notation: the navigation state error vector $\boldsymbol{\delta x}$ is

$$\boldsymbol{\delta x} = (\delta x, \delta z, \delta V_x, \delta V_z, \delta G_x, \delta G_z)^{\mathrm{T}} \in \mathbb{R}^6$$

and the disturbance vector $\boldsymbol{d}$, consisting of the eight accelerometers' random biases, is:

$$\boldsymbol{d} = (\delta f1_x, \delta f2_x, \delta f3_x, \delta f4_x, \delta f1_z, \delta f2_z, \delta f3_z, \delta f4_z) \in \mathbb{R}^8$$

Thus, the navigation state error dynamics are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta x} = \mathbf{F}(t)\boldsymbol{\delta x} + \boldsymbol{\Gamma}(t)\boldsymbol{d}, \;\; \boldsymbol{\delta x}(0) = \boldsymbol{\delta x}_0 \tag{7}$$

with initial conditions (for an aircraft at rest)

$$\boldsymbol{\delta x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \delta G_x(0) & \delta G_z(0) \end{bmatrix}^{\mathrm{T}} \tag{8}$$

where the initial uncertainty of the gravity field measurement is the same as the accuracy of the INS accelerometers [7], [14], that is, $\delta G(0) \approx 2 \times 10^{-12}\ \mathrm{m/s^2}$. Substituting the gravity gradient terms, the dynamics matrix $\mathbf{F}$ and the disturbance input matrix $\boldsymbol{\Gamma}$ are

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \left.\frac{\partial G_x}{\partial x}\right|_{x,z} & \left.\frac{\partial G_x}{\partial z}\right|_{x,z} & 0 & 0 \\ 0 & 0 & \left.\frac{\partial G_z}{\partial x}\right|_{x,z} & \left.\frac{\partial G_z}{\partial z}\right|_{x,z} & 0 & 0 \end{bmatrix} \tag{9}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{-V_x}{2a} & \frac{-V_z}{2b} & \frac{V_x}{2a} & \frac{V_z}{2b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-V_x}{2a} & \frac{-V_z}{2b} & \frac{V_x}{2a} & \frac{V_z}{2b} \end{bmatrix} \tag{10}$$

VI. INS Error Dynamics

Suppose an aircraft starts at latitude $L = L_0$ at a constant airspeed $V = const.$ and flies due North. Its latitude and its velocity components V_x and V_z are then functions of time. This allows us to calculate the necessary time-dependent terms in the $\mathbf{F}$ and $\boldsymbol{\Gamma}$ matrices

$$
\begin{aligned}
V_x &= -V\sin\left(L_0 + \frac{V}{R_e + h}\cdot t\right) \\
V_z &= V\cos\left(L_0 + \frac{V}{R_e + h}\cdot t\right) \\
\frac{\partial G_x}{\partial x} &= \omega_s^2\left(2\cos^2\left(L_0 + \frac{V}{R_e+h}\cdot t\right) - \sin^2\left(L_0 + \frac{V}{R_e+h}\cdot t\right)\right) \\
\frac{\partial G_x}{\partial z} &= 3\omega_s^2\cos\left(L_0 + \frac{V}{R_e+h}\cdot t\right)\sin\left(L_0 + \frac{V}{R_e+h}\cdot t\right) \\
\frac{\partial G_z}{\partial x} &= 3\omega_s^2\cos\left(L_0 + \frac{V}{R_e+h}\cdot t\right)\sin\left(L_0 + \frac{V}{R_e+h}\cdot t\right) \\
\frac{\partial G_z}{\partial z} &= \omega_s^2\left(2\sin^2\left(L_0 + \frac{V}{R_e+h}\cdot t\right) - \cos^2\left(L_0 + \frac{V}{R_e+h}\cdot t\right)\right)
\end{aligned}
$$

In the above calculations a spherical Earth was assumed. One could also use the closed form gravity formulae of

Pizzetti and Somigliana [12] along with the parameters of the WGS-84 ellipsoid [1] to account for the ellipsoidal geometry of the Earth, although this makes little difference in the calculations. ω_s is the Schuler frequency: $\omega_s = \sqrt{g/R_e}$. With the above equations, we can simulate the development of the error in the integrated navigation system by propagating the navigation state error dynamics from Equations (7) and (8), wherein we randomly select constant accelerometer biases such that the disturbance input vector $\boldsymbol{d}_{8\times1} \sim \mathcal{N}(\mathbf{0}_{8\times1}, \sigma_a^2 \cdot \mathbf{1}_{8\times1})$. The results of one such simulation are depicted in Figures 3 and 4. The plots are smooth because the measurement errors are random, but constant, biases.

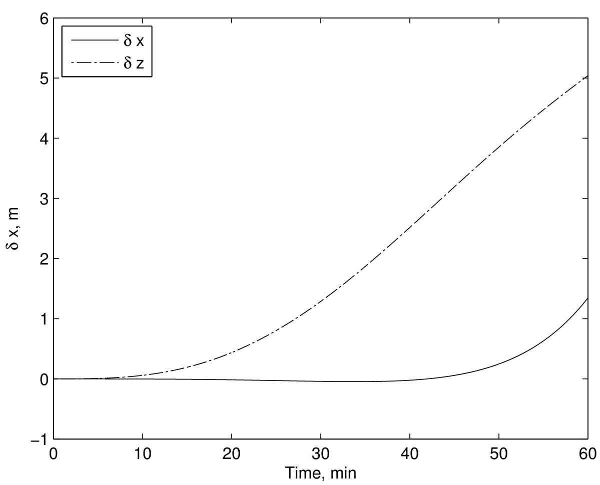

Fig. 3. Position error propagation for a GGI integrated INS, $V = 300\,\mathrm{m/s}$, one hour simulation.

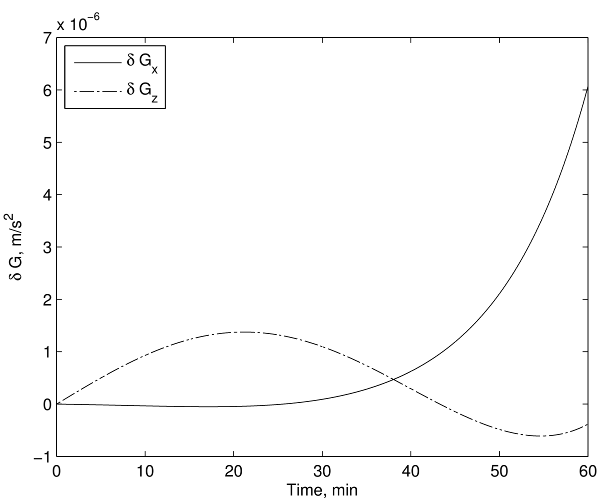

Fig. 4. Gravity error propagation for a GGI integrated INS, $V = 300\,\mathrm{m/s}$, one hour simulation.

To ascertain the performance of the gravity gradiometer integrated system, a covariance analysis is performed for the navigation state mechanization equations presented in Section V. The $\mathbf{F}$ and $\mathbf{\Gamma}$ matrices in Equations (9) and (10) are

TABLE I
One-sigma position, velocity, and gravity errors (random accelerometer bias model) after one hour.

δx	41.41 m	δv_x	7.37 cm/s	δG_x	0.1291 mm/s^2
δz	5.08 m	δv_z	0.98 cm/s	δG_z	0.0208 mm/s^2

used to generate an augmented dynamics matrix $\mathbf{F}_a = \begin{bmatrix} \mathbf{F} & \mathbf{\Gamma} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ whereupon the Lyapunov equation in the predicted navigation state's error covariance matrix $\mathbf{P}$ is the solution of the Lyapunov equation,

$$\dot{\mathbf{P}} = \mathbf{F}_a\mathbf{P} + \mathbf{P}\mathbf{F}_a^{\mathrm{T}}$$

where

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{F}_{6\times6} & \mathbf{\Gamma}_{6\times8} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{14\times14}, \quad \mathbf{P}(0) = \begin{bmatrix} \mathbf{0}_{6\times6} & \mathbf{0} \\ \mathbf{0} & \sigma_a^2 \cdot \mathbf{I}_{8\times8} \end{bmatrix} \tag{11}$$

for $0 \le t \le T$, with $T = 3600\,\mathrm{s}$.

For the covariance analysis that follows, we set $V = 300\,\mathrm{m/s}$ and $L_0 = 0$. The covariance matrix $\mathbf{P}$ is initialized at a known position with the accelerometer bias statistic $\sigma_a = 2 \times 10^{-12}\,\mathrm{m/s^2}$. The integration is carried out for one hour. The propagations of the one-sigma position, velocity, and gravity errors for the given aircraft trajectory are depicted in Figure 5. The position, velocity, and gravity errors for $t = 3600\,\mathrm{s}$ are shown in Table I. Evidently, an accelerometer bias causes the position error to grow $\propto t^2$.

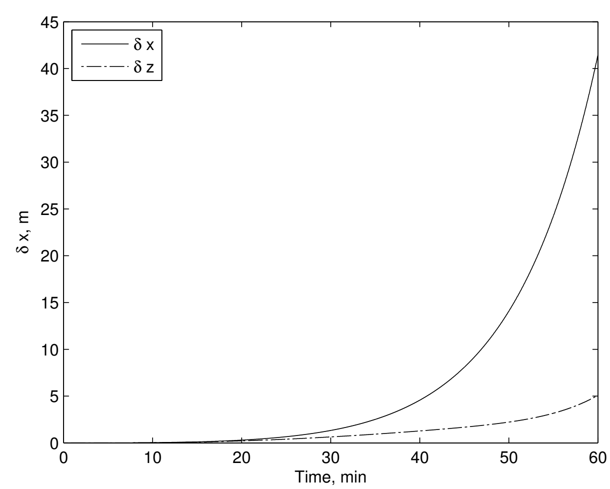

Fig. 5. Position error propagation for GGI integrated INS (random accelerometer bias model), $V = 300\,\mathrm{m/s}$, one hour.

Consider now the accumulated error over time in the calculation of gravitation $\boldsymbol{g}$ in Equations (5) and (6). For a cold atom interferometry precision instrument, say, $\delta f \approx 10^{-12}\,\mathrm{m/s^2}$, the accumulated velocity error $\delta V \approx 2\,\mathrm{cm/s}$ after one hour [14]. The question to answer is: assuming level flight at 300 m/s, does an accelerometer measurement error (δf) of 10^{-12} m/s^2 result in position errors of less than $5\,\text{m}/\sqrt{\text{h}}$?

Since

$$f_x = 0$$
$$\delta G_x \approx \delta G_x(0) + \int_0^t \frac{\delta f_x}{1} V_x dt$$

$$\Rightarrow \delta G \approx \int_0^t \delta f_x V_x dt \approx 10^{-12}/\text{s}^2 \times 300\,\text{m/s} \times 3600\,\text{s}$$
$$\approx 10^{-6}\,\text{m/s}^2$$

Now, for a five meter per hour CEP precision navigation system,

$$5\,\text{m} = \frac{1}{2}\delta a \times (3600\,\text{s})^2$$
$$\Rightarrow \delta a = \frac{10\,\text{m}}{(3600\,\text{s})^2}$$
$$\approx 7.7 \times 10^{-7}\,\text{m/s}^2$$
$$\delta a = \delta f + \delta G = 10^{-12}\,\text{m/s}^2 + \delta G$$

Then, the required accuracy is on the order of $\delta G = 10^{-6}\,\text{m/s}^2$, and so in order for the accumulated error in the calculated gravity estimate to be $\approx 10^{-6}\,\text{m/s}^2$, the specific force measurement error must be on the order of $\delta f = 10^{-12}\,\text{m/s}^2$. This is compatible with reported gravity gradient measurement accuracies of 0.004 E in the Gravity field and steady-state Ocean Circulation Explorer (GOCE) satellite, assuming they were taken over a baseline of 0.5 m [7]. It is also noteworthy that a required measurement accuracy on this order was calculated in a Russian text on inertial systems in 1966, well before such accuracy was achievable [3].

When gravity gradiometry is used, the dynamics matrix $\mathbf{F}_r$ of the reduced error state has the eigenvalues of $\pm\sqrt{2}\omega_s$ and $\pm j\omega_s$, where ω_s is the Schuler frequency. There remains the unstable mode which is always a problem with standalone INS and requires an independent altitude measurement for stability. This problem has been properly addressed by Britting in [6], but sadly is not addressed in most of the current literature [17]. In this respect, a pressure altitude measurement from an altimeter would be sufficient to aid the INS in order to "dampen" the vertical channel while also maintaining the passivity of the inertial navigation system.

VII. Conclusion

We have shown that using cold atom interferometry-based accelerometers whose accuracy is $\delta f \approx 10^{-12}\,\text{m/s}^2$, in a free INS integrated into a gravity gradient instrument, and under ideal conditions (no gyro error), a positional accuracy of $5\,\text{m}/\sqrt{\text{h}}$ is possible; including gyro errors would of course increase the error budget. Interestingly, the error dynamics of an INS using gravity gradiometry are more benign than the error dynamics for a standalone INS—assuming perfect alignment, both position and velocity errors are identically zero in a gravity gradiometry integrated navigation system despite the accelerometer measurement errors, whereas with a standalone INS the navigation state error diverges, even with perfect alignment. Admittedly, we have confined our attention to an academic two-dimensional scenario and, even more importantly, have assumed error-free gyroscope measurements. In this respect, the measurement of aircraft attitude exclusively using accelerometer measurements is addressed in a separate paper [14]. However, the objective of this work is a proof of concept: by using highly accurate accelerometers, one can, in a self-contained way, do away with gravity modeling in an inertial navigation system and achieve high precision autonomous navigation. In future work, the mechanization equations will be developed and an error analysis will be performed for a full-blown INS.

References

[1] "Department of Defense World Geodetic System 1984," National Imagery and Mapping Agency, Tech. Rep. NIMA TR8350.2, Third Edition, Amendment 1, Jan. 2000.

[2] Affleck, C. and Jircitano, A., "Passive gravity gradiometer navigation system," in *IEEE Position Location and Navigation Symposium, Record, The 1990's—A Decade of Excellence in the Navigation Sciences*, Mar. 1990, pp. 60–66.

[3] Andreev, V. D., *Теория инерциальной навигации. Автономные системы (Theory of inertial navigation. Autonomous systems).* Moscow: Наука (Nauka), 1966, pp. 228–252, in Russian, via private communication from anonymous reviewer.

[4] Baird, C., Snyder, F., and Beierle, M., "Terrain-aided altitude computations on the AFTI/F-16," in *IEEE Position Location and Navigation Symposium, Record, The 1990's—A Decade of Excellence in the Navigation Sciences*, Mar. 1990, pp. 474–481.

[5] Baird, C. A., "Map-aided navigation system employing TERCOM-SITAN signal processing," Patent 4 829 304, May, 1989.

[6] Britting, K. R., *Inertial Navigation Systems Analysis.* Norwood, Massachusetts: Artech House, 2010.

[7] Canuto, E., Bona, B., Calafiore, G., and Indri, M., "Drag free control for the European satellite GOCE. Part I: modelling," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, Dec. 2002, pp. 1269–1274.

[8] DeGregoria, A., "Gravity gradiometry and map matching: An aid to aircraft inertial navigation systems," Master's thesis, Air Force Institute of Technology, Mar. 2010.

[9] Gleason, D. M., "Passive airborne navigation and terrain avoidance using gravity gradiometry," *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 6, pp. 1450–1458, 1995.

[10] Jekeli, C., "Precision free-inertial navigation with gravity compensation by an onboard gradiometer," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 3, pp. 704–713, May-Jun. 2006.

[11] Lichtenstein, B., "Testing of stellar-inertial guidance systems," *IEEE Transactions on Military Electronics*, vol. 7, no. 1, pp. 29–35, Jan. 1963.

[12] Moritz, H., "Geodetic reference system 1980," *Journal of Geodesy*, vol. 74, pp. 128–162, 2000.

[13] Murphy, C. A., "The Air-FTG airborne gravity gradiometer system," in *Abstracts from the ASEG-PESA Airborne Gravity Workshop: Geoscience Australia Record*, 2004, pp. 7–14.

[14] Pachter, M., Welker, T. C., and Huffman, R. E., Jr., "Gyro-free INS theory," *NAVIGATION: Journal of the Institute of Navigation*, 2012, accepted for publication.

[15] Richeson, J. A., "Gravity gradiometer aided inertial navigation within non-GNSS environments," Ph.D. dissertation, University of Maryland, College Park, 2008.

[16] Rogers, M. M., "An investigation into the feasibility of using a modern gravity gradient instrument for passive aircraft navigation and terrain avoidance," Master's thesis, Air Force Institute of Technology, Mar. 2009.

[17] Titterton, D. H. and Weston, J. L., *Strapdown Inertial Navigation Technology*, 2nd ed. Stevenage, Hertfordshire, United Kingdom: The Institution of Electrical Engineers, 2004.