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Linear Machine: A Novel Approach to Point Location Problem

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Abstract: In recent literature, explicit model predictive control (e-MPC) has been proposed to facilitate implementation of the popular model predictive control (MPC) approach to fast dynamical systems. e-MPC is based on multi-parametric programming. The key idea in e-MPC is to replace the online optimization problem in MPC by a point location problem. After locating the current point, the control law is simply computed as an appropriate linear function of the states. A variety of approaches have been proposed in literature for the point location problem. In this work, we present a novel approach based on linear machines for solving this problem. Linear machines are widely used in multi-category pattern classification literature for developing linear classifiers given representative data from various classes. The idea in linear machines is to associate a linear discriminant function with each class. A given point is then assigned to the class with the largest discriminant function value. In this work, we develop an approach for identifying such discriminant functions from the hyperplanes characterizing the given regions as in multi-parametric programming. Apart from being an elegant solution to the point location problem as required in e-MPC, the proposed approach also links two apparently diverse fields namely e-MPC and multi-category pattern classification. To illustrate the utility of the approach, it is implemented on a hypothetical example as well as on a quadruple tank benchmark system taken from literature.

Keywords: explicit model predictive control, linear machine classifier, multi-parametric quadratic programming

1. INTRODUCTION

Model predictive control (MPC) requires solving an optimization problem online. This restricts its applicability typically to systems with sampling periods of the order of few seconds or minutes as commonly encountered in the process industry. In order to extend the range of MPC to applications with sampling periods in the milli/micro-second range as encountered in fast dynamical systems, it is imperative to employ efficient optimization algorithms.

A recent approach for fast solutions of online optimization problem, based on multi-parametric quadratic programming (mp-QP) (Bemporad et al., 2000a), has emerged as a promising tool. Multi-parametric programming solves optimization problems by computing a parameter-dependent solution offline and subsequently integrating the pre-computed solution with parameters whose values become available online. In the context of MPC, these parameters include past inputs, measurements and reference values. The offline solution consists of determining, i) a set of critical regions that exhaustively partitions the parameter space, and ii) a set of functions corresponding to each critical region, whose evaluation yields the optimized decision variables (that is, the optimal input profile) (Bemporad et al., 2002b, Gupta et al. 2011). The online implementation requires, i) a search algorithm to determine the critical region, corresponding to the current parameter values, and ii) a subsequent evaluation of the optimized decision variables using the corresponding function. Such an approach for solving the MPC optimization problem using multi-parametric programming has been referred to as explicit-MPC or e-MPC (Bemporad et al., 2002a). A well-known impediment in a successful implementation of e-MPC for medium to large-scale problems is that the number of critical regions may become prohibitively large and retrieving the region corresponding to the current parameter values itself becomes computationally challenging. Various efforts have been presented in literature to overcome this so-called point location problem. Tondel et al. (2003) used binary search tree as a tool to solve this problem. Recently work has been done by Bayat et al. (2011) and Monnigmann and Kastsian (2011) to further reduce the online computation time. The complexity of these algorithms depends on the number of hyperplanes that divide the parametric space into polyhedral regions.

In this work we propose a fundamentally different algorithm for solving the point location problem. The complexity of the proposed algorithm depends on the number of regions instead of the number of hyperplanes in the parametric space. Our novel approach is based on linear machines that are commonly used in pattern recognition for multi-category classification (Duda et al., 2001). Linear machines are widely used in multi-category pattern classification literature for developing linear classifiers given representative data from various classes. The idea in linear machines is to associate a linear discriminant function with each class. A given point is then assigned to the class with the largest discriminant function value. In context of eMPC, these classes can be conceptualized as the critical regions in the parametric space. Then, the point location problem in eMPC is analogous to the classification problem in pattern classification. However, unlike pattern classification problems, where finite samples corresponding to each class are available, in the case of linear eMPC, the critical regions are polytopes, each of which is represented by a set of linear inequalities. Designing a linear machine for eMPC then involves identifying discriminant functions that can classify these well-defined, contiguous critical regions. In this paper, we propose a novel approach for obtaining these discriminant functions using linear algebra and optimization methods. Our approach results in discriminant functions that exactly correspond to the underlying critical regions. Apart from being an elegant solution to the point location problem as required in e-MPC, the proposed approach also links two apparently diverse fields namely e-MPC and multi-category pattern classification. This opens up exciting possibilities of utilizing the vast amount of pattern classification literature to understand and solve a variety of challenging eMPC problems.

This paper is organized as follows: Section 2 briefly reviews theoretical concepts in mp-QP and linear machine; Section 3 presents the proposed algorithm using linear machine; Section 4 presents the results using two examples; Section 5 discusses computational issues and finally Section 6 presents the conclusion.

2. THEORETICAL BACKGROUND

2.1 Multi-parametric quadratic programming

It has been shown in Bemporad et al. (2000b) that the linear MPC problem can be written as a mp-QP problem of the following form:

$$\begin{aligned} \min_{z} \;\; & \frac{1}{2} z^T H z + c^T z \\ \text{s.t} \;\; & \mathcal{A} z \leq b + \mathcal{F} x \end{aligned} \tag{1}$$

where $z \subset \mathbb{R}^p$ is the decision variable and $x \in \mathbf{X} \subset \mathbb{R}^d$ represents the parameter. The parameter dependent solution is based on the knowledge of optimal active constraints. Let $\mathbb{N}_c = \{1, 2, \ldots n_c\}$ refer to the set of indices of n_c constraints of the QP and $\mathbb{N}_c^i$ refer to set of indices of active constraints contained in the i^{th} optimal active set $\mathbb{A}^i(x), i = 1, ..., n_r$ as follows,

$$\mathbb{A}^i(x) = \{ j \in \mathbb{N}_c^i \mid \mathcal{A}_j z - b_j - \mathcal{F}_j x = 0 \} \tag{2}$$

where, $\mathcal{A}_j$ denotes the j^{th} row of matrix $\mathcal{A}$ and n_r is the number of optimal active sets. The set of indices which are not in the optimal active set $\mathbb{A}^i$ are members of the inactive set $\mathbb{I}^i = \mathbb{N}_c \backslash \mathbb{N}_c^i$. Fiacco (1983) provided the necessary condition for parameter dependent optimality. In particular, the parametric variation of the optimal pair of the decision variable and the dual variable (i.e. the Lagrangian multiplier) (z, λ) in the neighborhood of x', a parameter vector in convex mp-QP in Eq. 1, is given by the Basic Sensitivity Theorem as follows:

$$\begin{bmatrix} z(x) \\ \lambda(x) \end{bmatrix} = -M^{-1} N (x - x') + \begin{bmatrix} z' \\ \lambda' \end{bmatrix} \tag{3}$$

where z' and λ' are the optimal values at x',

$$M = \begin{bmatrix} H & \mathcal{A}_1^T & \ldots & \mathcal{A}_{n_c}^T \\ -\lambda_1' \mathcal{A}_1 & -V_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda_{n_c}' \mathcal{A}_{n_c} & 0 & \ldots & -V_{n_c} \end{bmatrix}$$

$$N = \begin{bmatrix} Y & -\lambda_1' \mathcal{F}_1^T & \ldots & -\lambda_{n_c}' \mathcal{F}_{n_c}^T \end{bmatrix}^T$$

$$V_j = \mathcal{A}_j z' - b_j - \mathcal{F}_j x' \;\text{ and }\; Y = [0]_{d \times p}$$

Note that the parametric solution of Eq. 1 is obtained as a set of piecewise affine functions of parameters as shown in Eq. 3, where each function is valid in a closed polyhedron called as critical region. All optimal active set $\mathbb{A}^i$ and the corresponding optimal inactive sets obtained from the Karush-Kuhn-Tucker (KKT) conditions can be characterized as follows,

$$\begin{aligned} \mathcal{A}_j z(x) \leq b_j + \mathcal{F}_j x, j \in \mathbb{I}^i(x') \\ \lambda_j(x) \geq 0, j \in \mathbb{A}^i(x') \end{aligned} \tag{4}$$

where, λ_j represents the Lagrange multipliers that correspond to the j^{th} constraint. These inequalities along with the original parameter bounds $\mathbf{X}$, after removal of redundant constraints, represent a polyhedron in the parameter space, termed as Critical Region (CR) and correspond to the active set $\mathbb{A}^i$,

$$\begin{aligned} CR_{\mathbb{A}^i} = \{ x \in \mathbf{X} \subset \mathbb{R}^d : \mathcal{A}_j z(x) - b_j - \mathcal{F}_j x \leq 0, j \in \mathbb{I}^i; \\ \lambda_j \geq 0, j \in \mathbb{A}^i \} \end{aligned} \tag{5}$$

For the i^{th} set of active constraints $\mathbb{A}^i, i = 1, ..., n_r$, the optimal solution of Eq. 1 can be represented as a piecewise affine function of x.

$$z_{\mathbb{A}^i} = \Omega_{\mathbb{A}^i} x + \omega_{\mathbb{A}^i} \tag{6}$$

where $\Omega_{\mathbb{A}^i}$ and $\omega_{\mathbb{A}^i}$ are constant matrices for the i^{th} critical region. Together the set of critical regions characterized by Eq. 5 and the optimal solution given by Eq. 6 represent the analytical solution of Eq. 1. The online component consists of determining the current critical region i.e. solving point location problem followed by evaluation of Eq. 6 to yield the optimal control action.

As discussed earlier various techniques have been proposed to solve the point location problem, which is the most critical step of e-MPC implementation online. In this paper we propose a novel approach based on linear machine to solve the point location problem.

2.2 Linear Machine

Linear machine is based on linear discriminant analysis where the boundary separating two adjacent classes is linear in the variables (Duda et al., 2001). Given a set of n_r classes, with each class represented by a set of known points, the problem in multicategory classification is to develop a classifier that can assign a point to one of the n_r classes. The linear machine based solution to this classification problem is to associate a linear discriminant function $g_i(x)$ with each class i as (Duda et al., 2001):

$$g_i(x) = \alpha_i^T x + \alpha_{i,0}; i = 1, 2, ..., n_r \tag{7}$$

In the above α_i is the weight vector and $\alpha_{i,0}$ is the bias or threshold weight associated with the i^{th} class. These linear discriminant functions should satisfy the following property:

$$g_i(x) \geq g_k(x) \; \forall \; k \neq i \tag{8}$$

whenever point $x \in$ class i. Thus, a linear machine divides the space in n_r regions with $g_i(x)$ being the largest whenever x is in class i. The boundary shared by contiguous classes i and k is given by a hyperplane as (Duda et al., 2001)

$$g_i(x) - g_k(x) = 0 \tag{9}$$

Given a set of points for each of the n_r classes, several approaches for obtaining a linear machine are available in the pattern classification literature. These include the Kessler's construction and minimum squared error approaches (Duda et al., 2001).

We now propose to extend the above ideas for the point location problem in e-MPC. In principle, we can sample points from each critical region obtained by mp-QP and then apply the known algorithms (Duda et al., 2001) for obtaining a linear machine. However, the resulting linear machine will then be a function of the sample (and its size) chosen for this purpose and thus may not be able to assign the correct critical region when solving the point location problem. In the next section, we propose a novel approach for obtaining a linear machine directly from the knowledge of the hyperplanes defining each critical region. The approach utilizes tools from linear algebra and optimization to design a linear machine whose partition of the parameter space exactly coincides with the critical regions in e-MPC.

3. LINEAR MACHINE ALGORITHM FOR POINT LOCATION PROBLEM

Consider the parametric space $\mathbf{X}$ that is partitioned into n_r polyhedra $P_1, P_2, \ldots P_{n_r}$. The i^{th} polyhedral region P_i can be represented by n_{f_i} facets enclosing that region as:

$$A^i x \leq b^i; \forall i = 1, 2, ..., n_r \tag{10}$$

where $A^i \in \mathbb{R}^{n_{f_i} \times d}, b^i \in \mathbb{R}^{n_{f_i}}$ are obtained from Eq. 5 as discussed in Section 2.1. The j^{th} facet of polyhedron P_i is represented by the hyperplane $H^i_j(x) = A^i_j x - b^i_j = 0$. Let n_f denote the total number of facets in the parametric space. Note that since neighboring polyhedra share common facets, $n_f \neq \sum_{i=1}^{n_r} n_{f_i}$.

Designing a linear machine for the point location problem now involves identifying linear discriminant function, $g_i(x) = \alpha_i^T x + \alpha_{i,0} = \alpha_{i,1}x_1 + \alpha_{i,2}x_2 + \ldots \alpha_{i,d}x_d + \alpha_{i,0}$ for each region P_i. The proposed approach for designing such discriminant functions involves two steps:

- Determination of neighboring regions, i.e. regions sharing a common facet.
- Determination of the discriminant functions based on the identified neighboring regions.

3.1 Determination of Neighboring Regions

Determination of neighboring regions is a non-trivial problem due to the fact that regions sharing a common hyperplane may not necessarily share a common facet. The various possibilities for regions sharing common hyperplane are illustrated in Fig. 1. Fig. 1(a) represents regions

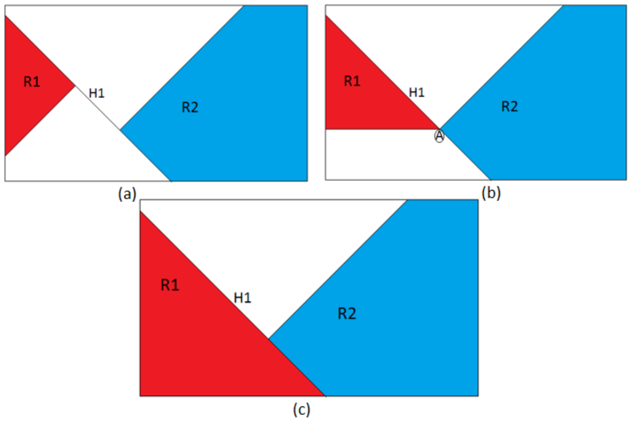

Fig. 1. Regions R1 and R2 sharing common hyperplane H1

R1 and R2 that share a common hyperplane H1 but do not intersect at any point. Fig. 1(b) represents regions R1 and R2 sharing a common hyperplane H1 and intersecting at a single point A. Fig. 1(c) represents regions R1 and R2 sharing a common facet which is represented by the common hyperplane H1. Thus sharing a hyperplane does not necessarily imply that the regions are neighboring regions.

To determine if region P_i shares its j^{th} facet, represented by hyperplane H^i_j, with region P_k, we first determine if hyperplane H^i_j represents a facet of region P_k. If H^i_j is a facet of both the regions, then to determine whether H^i_j is a common facet of region P_i and P_k, a point x on hyperplane H^i_j is obtained by solving the optimization problem given by $FORMULATION I$,

FORMULATION I:

$$\begin{aligned} &\text{if } (P_i \cap P_k) \neq \phi \\ &\{\max \;\; t \\ &\;\text{s.t. } x \in P_i \\ &\qquad x \in P_k \\ &\qquad t \leq d(x, H^i_l) \;\; \forall \;\; l \in \{1, \ldots, n_{f_i}\} - \{j\}\} \end{aligned} \tag{11}$$

where t is the minimum distance between x and all facets of region P_i except j^{th} facet, $d(x, H^i_l)$ is the distance of point x from l^{th} facet of i^{th} region. The optimization problem (Eq. 11) can be either infeasible or feasible with $t \geq 0$. When the solution is feasible with $t > 0$, region P_i and P_k share a common facet given by hyperplane H^i_j. In this case the common facet between region P_i and P_k can be represented as $H^*_{i,k} = A^*_{i,k}x - b^*_{i,k}$ which is equal to $H^i_j(x)$.

The results of implementation of $FORMULATION I$ are now discussed for the three cases in Fig 1. For case (a) Eq. 11 will be infeasible as $(\text{R1} \cap \text{R2}) = \phi$ since the regions R1 and R2 do not share a facet. For case (b) Eq. 11 is feasible with $t = 0$ In this case x is point A which is the point of intersection of the two regions. The distance of Point A from H2 (another facet of R1) is zero thus implying that the two regions do not share a common facet. The solution to Eq. 11 for case (c) is feasible with $t > 0$ as x is a point B on facet H1 obtained by maximizing the minimum of its distance from H2 and H3 which are the other facets of R1. The distance t is greater than 0 in this case thus implying that the two regions share a common facet.

Once neighboring regions are determined, a linear machine is constructed as discussed next.

3.2 Determination of Linear Discriminant Functions

Given the set of n_r critical regions and information about neighboring regions we intend to identify linear discriminant functions $g_i(x) = \alpha_i^T x + \alpha_{i,0} = \alpha_{i,1}x_1 + \alpha_{i,2}x_2 + \ldots \alpha_{i,d}x + \alpha_{i,0}$ associated with each region P_i, $\forall i \in \{1, \ldots, n_r\}$ The key idea in the proposed approach is that hyperplanes $H^*_{i,k}$ dividing neighboring regions P_i and P_k correspond to decision boundaries for these regions. In other words, the discriminant functions $g_i(x)$ and $g_k(x)$ should be equal whenever $H^*_{i,k}(x) = 0$. This then leads to an equality constraint on discriminant function pair $(g_i(x), g_k(x))$ as:

$$\begin{aligned} &- g_i(x) + g_k(x) = \beta_{i,k}(A^*_{i,k}x - b^*_{i,k}) \\ \text{or } &- \alpha_i^T x + \alpha_k^T x - \alpha_{i,0} + \alpha_{k,0} - \beta_{i,k}(A^*_{i,k}x - b^*_{i,k}) = 0 \end{aligned} \tag{12}$$

In writing the above, we have assumed that region P_i corresponds to $H^*_{i,k}(x) \le 0$ while region P_k corresponds to $H^*_{i,k}(x) \ge 0$. The choice of signs (-1 for $g_i(x)$ and +1 for $g_k(x)$) in the above equation then ensures that $g_i(x) > g_k(x)$ whenever $H^*_{i,k}(x) < 0$. Further, $\beta_{i,k}$ is a positive constant to be determined along with the parameters of the discriminant functions $g_i(x)$ and $g_k(x)$. The parameter $\beta_{i,k}$ is included since the discriminant functions can be identified only upto a scaling factor. On matching coefficients term by term in Equation 12 leads to:

$$\begin{aligned} -\alpha_{i,1} + \alpha_{k,1} - \beta_{i,k}A^*_{i,k}(1) &= 0 \\ &\vdots \\ -\alpha_{i,d} + \alpha_{k,d} - \beta_{i,k}A^*_{i,k}(d) &= 0 \\ -\alpha_{i,0} + \alpha_{k,0} + \beta_{i,k}b^*_{i,k} &= 0 \end{aligned} \tag{13}$$

These equations are written for all pairs of neighboring regions as identified using the approach discussed in Section 3.1. This then leads to the following homogeneous system of linear equations of the form:

$$Gy = 0 \tag{14}$$

where $y = [\alpha_{1,1} \ldots \alpha_{1,d}\ \alpha_{1,0}\ \alpha_{2,1} \ldots \alpha_{n_r,d}\ \alpha_{n_r,0}\ \beta_1 \ldots \beta_{n_f}]^T$ is the vector of parameters of linear discriminant functions and scale factors that need to be determined. Since each pair of neighbouring regions is separated by a unique facet there are a total of n_f scale factors and $\beta_{i,k}$ can be represented as β_j where j^{th} facet of n_f facets set is a common facet between region P_i and P_k. G is a matrix of size $(n_f(d+1) \times (n_r(d+1) + n_f))$. Any vector in the null space of G will be a solution to Eq. 14. The general solution of Eq. 14 can be written as

$$y = \sum_{u=1}^{N_{NS}} c_u y^u \tag{15}$$

where N_{NS} is the dimension of the null space of matrix G and $y^u, u = 1, ..., NS$ are the basis vectors of the null space of G. While any choice of the coefficients c_u of the linear combination will ensure that y is a solution to Eq. 15, it will not necessarily lead to positive values for all the scale factors $\beta_{i,k}$. We thus propose the following optimization problem to select the coefficients $c_u, u = 1, ..., N_{NS}$. Apart from ensuring that the scale factors are positive, this formulation will also lead to well separated discriminant functions.

FORMULATION II:

$$\begin{aligned} \min \quad & s \\ \text{s.t} \quad & s \ge \frac{|g_i(x).g_k(x)|}{||g_i(x)||||g_k(x)||} \\ & \forall i \in (1, \ldots, n_{r-1}) \text{ and } k \in (i+1, \ldots, n_r) \\ & \sum_{u=1}^{N_{NS}} c_u y^u_\beta \ge [0]_{n_f \times 1} \end{aligned} \tag{16}$$

where y^u_β contains the last n_f elements of y^u (corresponding to scale factors). In $Formulation\ II$, in order to obtain well separated linear discriminant functions the minimum angle between all pairs of linear discriminant functions is maximized. This is done by minimizing the maximum of absolute value of cosine of angle between pairs of linear discriminant functions. Here s is the maximum of absolute value of cosine of angle between all pairs of linear discriminant functions. Further, positivity of scale factors $\beta_{i,k}$ is also explicitly ensured in the above formulation.

The overall approach for the point location problem as proposed in this work can then be summarized as:

- Given the facets defining each critical region, the linear machine is constructed as discussed in this section. This is an offline activity.
- The region in which the given parameter vector x lies is computed by first evaluating the linear discriminant functions for each region, $g_i(x)\ \forall i \in n_r$ and then identifying the discriminant function which achieves the maximum value. In other words the index of the critical region containing the parameter x is given by $\{i \mid g_i(x) = \max\{g_1(x), \ldots, g_{n_r}(x)\}\}$.

While it is known in pattern classification literature that the classification regions corresponding to a linear machine are convex (Duda et al., 2001), the converse is not always true. We are currently investigating this issue further.

4. EXAMPLES

In this section two examples showing implementation of linear machine algorithm are presented

4.1 Implementation for a 2-dimensional polyhedral region

The linear machine algorithm is implemented for a hypothetical 2-dimensional region shown in Fig. 2. The polyhedral region contains 8 convex contiguous regions and 12 facets. The linear machine was obtained as discussed is Section 3.2 and the corresponding linear discriminant functions are given in Table 1. The value of s obtained by solving $FORMULATION\ II$ comes out to be 0.9918. A plot of these linear discriminant functions values for the 2-d polyhedral region is given in Fig. 3. From this figure,

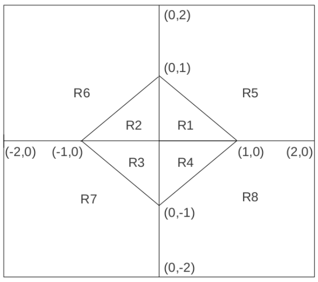

Fig. 2. 2-dimensional polyhedral region for implementing linear machine algorithm

Table 1. Linear Discriminant functions for 2-dimensional polyhedral region (Fig. 2)

Region	Linear Discriminant Function $g_i(x)$
R_1	$0.04583x_1 - 0.15674x_2 - 0.03396$
R_2	$-0.04583x_1 - 0.15674x_2 - 0.03396$
R_3	$-0.04583x_1 - 0.29053x_2 - 0.03396$
R_4	$0.04583x_1 - 0.29053x_2 - 0.03396$
R_5	$0.19037x_1 - 0.01220x_2 - 0.17850$
R_6	$-0.19037x_1 - 0.01220x_2 - 0.17850$
R_7	$-0.19037x_1 - 0.43507x_2 - 0.17850$
R_8	$0.19037x_1 - 0.43507x_2 - 0.17850$

it can be seen that for any region R_i, the corresponding linear discriminant function $g_i(x)$ has the maximum value.

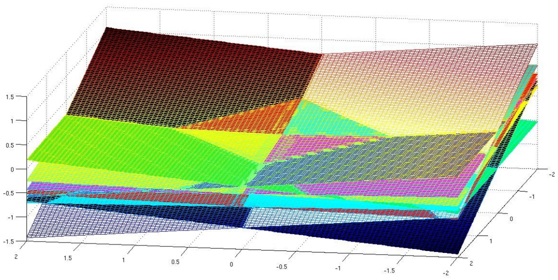

Fig. 3. Linear discriminant functions value corresponding to the 2-dimensional polyhedral region shown in Fig. 2. The linear discriminant functions corresponding to different regions are represented by the following colors: g_1-yellow, g_2-magenta, g_3-cyan, g_4-red, g_5-green, g_6-blue, g_7-white, g_8-black.

4.2 Simulation Case Study: Quadruple Tank System

The linear machine algorithm was also used for the quadruple tank system (Johansson, 2000) control problem. The system consists of four interconnected tanks as shown in Fig. 4. The objective is to control the water level in the lower two tanks with the two pumps. The process inputs, (ν_1, ν_2) represent input voltages to the two pumps and the states (h_1, h_2, h_3, h_4) represent the level of water in the four tanks. Outputs h_1 and h_2 (water level in the two bottom tanks) represent the variables to be controlled. At nominal operation, with input voltages in both pumps being $3V$, the corresponding values of h_1 and h_2 are 12.4 and 12.7 cm, respectively. The nonlinear state space model is linearized and simulated for a sampling time of 5 seconds, with constraints on deviation variables for level

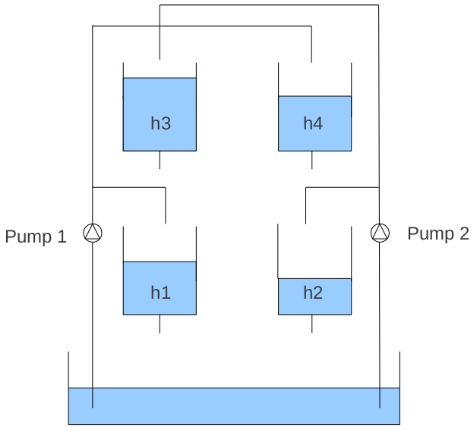

Fig. 4. Quadruple Tank System (Johansson, 2000)

h as $\pm[5, 5, 1, 1]cm$, input voltage ν as $\pm[1, 1]V$ and $\Delta\nu$ as $\pm[0.1, 0.1]V$.

Critical regions were generated using the MPT Toolbox (Kvasnica et al., 2004) for this four tank system control problem with prediction and control horizons of 5 and 1 samples, respectively. Control action was obtained as a function of states for the various critical regions. For the given parameter ranges, the state space was partitioned into 15 critical regions by 22 facets. Fig 5 shows the polyhedral partition representing state space $\mathbf{X}$ for $x_4 = 0$, where $x_i = h_i - h_i^0$ represents deviation of state from steady state. For each region a linear discriminant function was

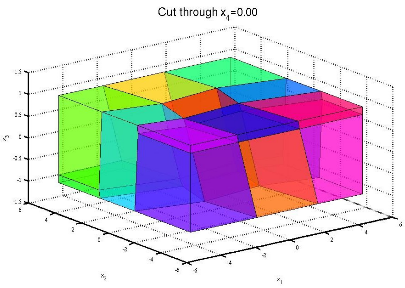

Fig. 5. State space $\mathbf{X}$ partitioned into critical regions for $x_4 = 0$

determined using the linear machine approach presented in our work. Control action was then implemented for the system with initial condition $(h_1, h_2, h_3, h_4) = (15.2630, 15.7832, 2.3339, 2.1090)$ using linear machine algorithm. The resulting control actions and the corresponding system response are shown in Figures 6 and 7, respectively. To compare our approach with some existing approaches in terms of the computational requirements, the control action is also implemented using: (i) the inbuilt algorithm in the MPT toolbox (Kvasnica et al., 2004), and (ii) the binary search tree algorithm (Tondel et al., 2003). The MPT toolbox algorithm uses a sequential search which involves searching in each region one at a time, until the region that contains the current point is found. For this system, the average time taken for implementing control action online by linear machine algorithm was 0.05 seconds while the corresponding times taken by binary search tree algorithm and the sequential search of MPT toolbox were 0.04 and 0.14 seconds, respectively.

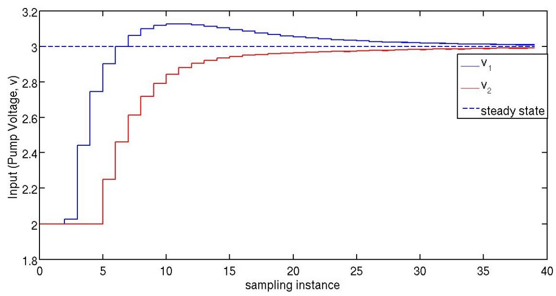

Fig. 6. Manipulated variables for Quadruple Tank System (Johansson, 2000)

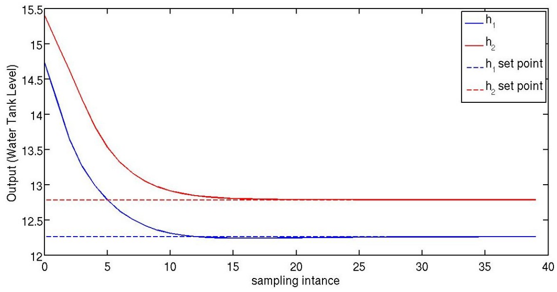

Fig. 7. Controlled variables for Quadruple Tank System (Johansson, 2000)

5. COMPLEXITY ANALYSIS

The point location problem that is solved using linear machine algorithm during online e-MPC implementation consists of two steps: 1) computing linear discriminant functions values for current state $x(t)$, and 2) determining the maximum linear discriminant function value. The first step involves multiplying a matrix containing the coefficients of n_r linear discriminant functions with current state vector $x(t)$. The second step requires $n_r - 1$ comparisons (Cormen et al., 2001) to compute the maximum linear discriminant function and will largely determine the computational performance of our approach. Thus the complexity of our proposed algorithm is $O(n_r)$.

Complexity of other algorithms given in the literature such as binary tree algorithm (Tondel et al., 2003) and multiway tree algorithm (Monnigmann and Kastsian, 2011) depends on the number of hyperplanes that divide the polyhedral space into critical regions. For binary tree approach, this complexity is $O(\log_2 n_f)$ (Tondel et al., 2003). Currently we are investigating use of search trees in the proposed linear machine approach to reduce its complexity.

6. CONCLUSION

In this paper we have presented a novel approach to solve the point location problem in e-MPC using the idea of linear machine adapted from the multicategory pattern classification literature. The approach for building a linear machine for the given critical regions was discussed. The approach was implemented on a hypothetical example and the well known four tank system to demonstrate its potential. Apart from being an elegant solution to the point location problem as required in e-MPC, the proposed approach also links two apparently diverse fields namely e-MPC and multi-category pattern classification. This opens up exciting possibilities of utilizing the vast amount of pattern classification literature to understand and solve a variety of challenging eMPC problems.

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