Preprints of the 10th IFAC International Symposium on Dynamics and Control of Process Systems
The International Federation of Automatic Control
December 18-20, 2013. Mumbai, India

On the Estimation of Time-varying Parameters in Continuous-time Nonlinear Systems ⋆

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Abstract: The estimation of time-varying parameters in continuous-time nonlinear systems is considered under the framework of the modulating functions method. The parameter is approximated as a finite Fourier series, which is reconstructed from the estimated Fourier spectral coefficients. Unlike the popular polynomial approximation, this approach is general enough for piecewise smooth parameter changes. The locations of abrupt jumps are accurately identified by the presence of Gibbs phenomenon. The global Fourier spectral coefficients are then used to extract local finite Gegenbauer polynomial series to recover smooth parameter variations between the jumps. This method of resolution of the Gibbs phenomenon avoids the necessity of estimating a large number of Fourier coefficients for series convergence. A van der Pol oscillator simulation example is included to demonstrate the performance of the approach.

Keywords: Continuous-time systems; Parameter estimation; Nonlinear systems; Least squares estimation; Time-varying systems.

1. INTRODUCTION

The identification of dynamic systems deals with mathematical descriptions of the input-output behavior by the selection of appropriate model structures and estimation of the model parameters using measured data. This paper is focused on nonlinear continuous-time models and the modulating functions method for estimating time-varying parameters. Since its original introduction as the method of moment functionals (Shinbrot, 1957), the modulating functions method has gained prominence as an efficient parameter estimation approach for a wide class of continuous-time nonlinear systems. Several comprehensive reviews are available, detailing a variety of modulating functions and problem descriptions (Unbehauen and Rao, 1990; Preisig and Rippin, 1993a; Patra and Unbehauen, 1995; Unbehauen, 1996). The main advantage of the approach is that it avoids the estimation of initial conditions and state derivatives from potentially noisy data.

The modulating functions method is applicable to systems that are linear in combinations of constant parameters, which may be obtained by inverting the estimates of the combinations (Preisig and Rippin, 1993a). The general framework is also used for time-varying parameters. Puchkov and Chinayev (1973) used an estimate of Gregory's interpolation polynomial to recover the trajectory of the parameter. Preisig and Rippin (1993b) explicitly solved for a state dependent parameter as a function of the modulated states. A moving window formulation can track slowly changing parameters (Co and Ungarala, 1997). In general, polynomial approximations are straightforward and easy to estimate (Braiek and Rotella, 1990; Ungarala and Co, 2000).

This paper retains the accepted practice of estimating a time-varying parameter as a series expansion on a chosen set of basis functions, which has been limited to polynomials so far. The approach is based on the fact that the product of a modulating function and a sufficiently smooth basis function is also a modulating function (Braiek and Rotella, 1990). However, polynomials, while convenient, are not general enough since they are not suitable for parameters that exhibit abrupt changes at previously unknown locations. In this paper we submit that a Fourier series is more general to estimate gradual changes as well as a finite number of discontinuities inside the estimation interval. It is shown that a partial Fourier spectrum of the parameter can be estimated using the well known Pearson and Lee (1985) formulation with trigonometric modulating functions.

A consequence of using Fourier series is poor convergence due to series truncation and the Gibbs phenomenon. We note that the presence of Gibbs effect identifies the location of an abrupt change and the piece-wise smooth sections in between can be readily recovered as finite Gegenbauer polynomial series. An explicit formula is available to relate the Gegenbauer coefficients to the estimated Fourier spectral coefficients (Gottlieb et al., 1992). It is noteworthy that this approach of post processing requires only a few low frequency Fourier coefficients to reconstruct the time-varying parameters as compared to a full Fourier reconstruction, which also suffers from artifacts introduced by the Gibbs phenomenon.

The van der Pol oscillator is an extensively studied nonlinear model for a wide range of dynamic behaviors observed in physical, chemical and biological systems (Besancon et al., 2010; Quaranta et al., 2010). Examples of models derived from the van der Pol oscillator include the solar cycle (Pontieri et al., 2003), chemical kinetics of periodic and chaotic dynamics of reactive species concentrations (Samardzija et al., 1989), rhyth-

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mic contractions of the heart muscle (Karreman and Prood, 1995; Kaplan et al., 2008) and wave forms of electrical activity in intestines (Robertson-Dunn and Linkens, 1974) to name a few. In this paper we demonstrate the estimation of time-varying parameters, including the location of abrupt changes, using simulated noisy measurements from a van der Pol oscillator.

The paper is organized as follows. A brief review of the Pearson-Lee modulating functions method is provided in Sections 2 and 3. It is followed by Section 4 with details of the proposed time-varying parameter estimation as reconstruction using finite Fourier series. The resolution of Gibbs effects is discussed in Section 5 and the simulated example of a van der Pol oscillator is included at the end to demonstrate the effectiveness of the approach.

2. PRELIMINARIES

A modulating function $\phi(t) \in \mathbf{C}^K$ is defined over $[0,T]$ with terminal conditions:

$$D^k\phi(0) = D^k\phi(T) = 0, \qquad k = 0, 1, \ldots, K-1, \tag{1}$$

where $D^k = d^k/dt^k$. Modulation of $f(t)$ is the inner product:

$$\langle \phi, f \rangle = \int_0^T \phi f \, dt. \tag{2}$$

The terminal conditions of $\phi(t)$ allow arbitrary initial and final values of the signal $f(t)$. The adjointness property transfers any differentiation operation on $f(t)$, to $\phi(t)$:

$$\left\langle \phi, D^k f \right\rangle = (-1)^k \left\langle D^k \phi, f \right\rangle. \tag{3}$$

A set of r trigonometric modulating functions $\Phi(t)$, is a linear combination of sinusoids of the first $1+L$ frequencies:

$$\Phi(t) = CF(\omega,t), \tag{4}$$

$$F(\omega,t) = [1, \cos\omega t, -\sin\omega t, \ldots, \cos L\omega t, -\sin L\omega t]^t, \tag{5}$$

where $\omega = 2\pi/T$ and the matrix $C_{[r\times(1+2L)]}$, enforces the terminal conditions and linear independence. A simple block diagonal operator performs differentiation on $\Phi(t)$:

$$D^k\Phi = (-1)^k C\mathbf{D}^k F, \tag{6}$$

where

$$\mathbf{D} = \omega \operatorname{diag}[0, \mathbf{d}, \ldots, L\mathbf{d}], \qquad \mathbf{d} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{7}$$

3. PARAMETER ESTIMATION

Modulation can be used to estimate the parameters of a nonlinear differential equation of the following affine structure:

$$\sum_{j=1}^{p} D^{k_j}\zeta_j(u,y) = \sum_{j=1}^{q} \alpha_j D^{k_j}\psi_j(u,y), \tag{8}$$

where the constant parameters α_j multiply derivatives of computable nonlinear functions of inputs u and outputs y. This class of systems is known as *integrable nonlinear systems*. Equation (8) may be written in matrix notation as:

$$PW = QA, \tag{9}$$

where

$$P = \left[D^{k_1}\zeta_1, \ldots, D^{k_p}\zeta_p\right], \tag{10}$$

$$Q = \left[D^{k_1}\psi_1, \ldots, D^{k_q}\psi_q\right], \tag{11}$$

$$A = [\alpha_1, \ldots, \alpha_q]^t, \tag{12}$$

and W is a column vector of ones. Modulation transforms the nonlinear ODE into a set of r algebraic equations:

$$\langle \Phi, P \rangle W = \langle \Phi, Q \rangle A \qquad \text{or} \qquad CG_\zeta W = CG_\psi A, \tag{13}$$

where

$$G_\zeta = \left[\mathbf{D}^{k_1} Z_{\zeta_1}, \ldots, \mathbf{D}^{k_p} Z_{\zeta_p}\right], \tag{14}$$

$$G_\psi = \left[\mathbf{D}^{k_1} Z_{\psi_1}, \ldots, \mathbf{D}^{k_q} Z_{\psi_q}\right], \tag{15}$$

$$Z_f = [C_0(f), C_1(f), S_1(f), \ldots, C_L(f), S_L(f)]^t, \tag{16}$$

$$C_m(f) = \int_0^T f(t) \cos m\omega t \, dt, \tag{17}$$

$$S_m(f) = -\int_0^T f(t) \sin m\omega t \, dt. \tag{18}$$

The integrals $C_m(f)$ and $S_m(f)$, are obtained by DFT and A is estimated using least squares on Equation (13).

The modulating functions method can be extended to the following general class of systems:

$$\sum_{i=1}^{p'} \sum_{j=1}^{p} \zeta_i D^{k_j} \zeta_j = \sum_{i=1}^{q'} \sum_{j=1}^{q} \alpha_j \psi_i D^{k_j} \psi_j, \tag{19}$$

known as *convolvable nonlinear systems*, where the parameters multiply products of a function and a derivative. This is the most general form of nonlinear continuous-time system used in parameter estimation (Patra and Unbehauen, 1995). The following identity transforms the convolvable form to the integral form:

$$\psi_i D^k \psi_j = \sum_{l=0}^{k} (-1)^l \binom{k}{l} D^{k-l} \left(\psi_j D^i \psi_i\right). \tag{20}$$

4. TIME-VARYING PARAMETER ESTIMATION

Consider an integrable nonlinear system with time-varying parameters:

$$\sum_{j=1}^{p} D^{k_j}\zeta_j(u,y) = \sum_{j=1}^{q} \alpha_j(t) D^{k_j}\psi_j(u,y). \tag{21}$$

If the parameter's functionality with time is previously determined, the embedded coefficients belonging to the parameter model may be estimated from the resulting convolvable form (Preisig and Rippin, 1993a). In the absence of such *a priori* knowledge, a series expansion over a suitable family of basis functions may be used to approximate the parameter trajectory in a finite time interval:

$$\alpha(t) = \sum_{n=0}^{N} \mu_n(t)\hat{\mu}_n, \qquad 0 \le t \le T, \tag{22}$$

where μ_n is a sufficiently smooth basis function and $\hat{\mu}_n$ is an unknown coefficient. The coefficients in the resulting convolvable form can be estimated and the parameter is reconstructed with desired accuracy depending on N.

4.1 Finite Fourier series approximation

Let μ_n be the trigonometric basis functions vector and $\hat{\mu}_n$ is the spectral coefficient vector for $n = 1, \ldots, N$,

$$\mu_n(t) = [\cos n\omega t,\ \sin n\omega t], \qquad \hat{\mu}_n = \begin{bmatrix} a_n \\ b_n \end{bmatrix}, \tag{23}$$

with $\mu_0 = 1$ and $\hat{\mu}_0 = a_0$. A time-varying parameter $\alpha(t)$, assuming Dirichlet's conditions, is written as a finite Fourier series. The following lemma shows the transformation of the convolvable form to integral form.

Lemma 1. Let $\alpha(t)$, be represented by an Nth order finite Fourier series approximation over $[0, T]$,

$$\alpha(t) = a_0 + \sum_{n=1}^{N} (a_n \cos n\omega t + b_n \sin n\omega t), \tag{24}$$

then, modulation is applied to a new set of terms of the form $D^{k-i}\{\psi D^i(a_n \cos n\omega t + b_n \sin n\omega t)\}$ instead of $a_n \cos n\omega t D^k\psi + b_n \sin n\omega t D^k\psi$.

Proof: The ith order derivative of the basis vector is:

$$D^i \mu_n = (n\omega)^i \mu_n \mathbf{d}^i. \tag{25}$$

Using Equation (20) on $\alpha D^k\psi$,

$$\begin{aligned}
\alpha D^k\psi &= \sum_{i=0}^{k} (-1)^i \frac{k!}{i!(k-i)!} D^{k-i}\left(\psi D^i\alpha\right), \\
&= \sum_{i=0}^{k} (-1)^i \frac{k!}{i!(k-i)!} D^{k-i}\left\{\psi D^i\left(\hat{\mu}_0 + \sum_{n=1}^{N} \mu_n\hat{\mu}_n\right)\right\}, \\
&= \hat{\mu}_0 D^k\psi + \sum_{i=0}^{k}\sum_{n=1}^{N} \beta_{k,i,n} D^{k-i}\left(\psi\mu_n\mathbf{d}^i\right)\hat{\mu}_n,
\end{aligned} \tag{26}$$

where

$$\beta_{k,i,n} = (-1)^i \frac{k!(n\omega)^i}{i!(k-i)!}. \tag{27}$$

Modulating $\alpha D^k\psi$ with ϕ,

$$\begin{aligned}
\left\langle \phi,\ \alpha(t)D^k\psi \right\rangle = \left\langle \phi,\ D^k\psi \right\rangle \hat{\mu}_0 + \\
\sum_{i=0}^{k}\sum_{n=1}^{N} \beta_{k,i,n} \left\langle \phi,\ D^{k-i}\left(\psi\mu_n\mathbf{d}^i\right)\right\rangle \hat{\mu}_n.
\end{aligned} \tag{28}$$

□

4.2 Estimation of spectral coefficients

In general, each of the q time varying parameters of the integral nonlinear system in Equation (21) is modeled as a finite Fourier series. The nonlinear functions can be regrouped into row vectors $\bar{P}$ and $\bar{Q}$ as follows:

$$\bar{P} = \left[D^{k_1}\zeta_1,\ D^{k_2}\zeta_2, \ldots,\ D^{k_p}\zeta_p\right], \tag{29}$$

$$\bar{Q} = [\bar{\psi}_1,\ \bar{\psi}_2, \ldots,\ \bar{\psi}_q], \tag{30}$$

where

$$\begin{aligned}
\bar{\psi}_j = \Big[&D^{k_j}\psi_j,\ D^{k_j}(\psi_j\mu_1),\ D^{k_j-1}(\psi_j\mu_1\mathbf{d}) \ldots,\ \psi_j\mu_1\mathbf{d}^{k_j}, \\
&\ldots,\ D^{k_j}(\psi_j\mu_{N_j}),\ D^{k_j-1}(\psi_j\mu_{N_j}\mathbf{d}), \ldots,\ \psi_j\mu_{N_j}\mathbf{d}^{k_j}\Big]
\end{aligned} \tag{31}$$

Define a block diagonal matrix R as below:

$$R = \mathrm{diag}[B_1, \ldots, B_q], \qquad B_j = \mathrm{diag}[1, b_1, \ldots, b_{M_j}], \tag{32}$$

$$b_n = [\beta_{k_j,0,n}, \ldots, \beta_{k_j,k_j,n}]^t. \tag{33}$$

Hence, Equation (21) with Fourier series models for parameters is written in matrix notation as follows:

$$\bar{P}W = (\bar{Q}R)\bar{A}, \tag{34}$$

where $\bar{A}$ contains the Fourier spectral coefficients of the time-varying parameters

$$\bar{A} = (\mu_{1,0}, \ldots,\ \mu_{1,N_1}, \ldots,\ \mu_{q,0}, \ldots,\ \mu_{q,N_q})^t. \tag{35}$$

The spectral coefficient vector can now be estimated using trigonometric modulation as shown in Section 3. The parameter trajectories are reconstructed as Fourier series. The choice of the order of the Fourier series plays a crucial role in the accuracy of the estimation, which is discussed next.

5. RESOLUTION OF THE GIBBS PHENOMENON

Given a piecewise smooth $\alpha(t)$, the series constructed with $2N + 1$ Fourier coefficients offers slow convergence. No matter how large N may be, the error persists in the form of overshoots, undershoots and spurious oscillations around the discontinuities. Similarly, for non-periodic $\alpha(t)$, the series fails to converge at 0 and T, a case of jump discontinuity. This behavior of the Fourier series representing non-periodic piecewise smooth functions is known as the Gibbs phenomenon. Traditionally, the error is reduced with the use of Fejer or Lanczos smoothing factors, which suppress the influence of spectral coefficients corresponding to high frequency modes (Acton, 1990). Although these methods reduce the levels of oscillations and overshoots, they do not eliminate the Gibbs effects.

More powerful techniques are now available due to Gottlieb and coworkers in a series of papers during 1990s. The reader is referred to a review paper (Gottlieb and Shu, 1997). The Gottlieb method showed that the first $2N + 1$ Fourier coefficients contain enough information to obtain an exponentially convergent Gegenbauer polynomial series approximation of $\alpha(t)$ (Gottlieb et al., 1992).

The Gottlieb method is employed here as a post processing step in parameter estimation using finite Fourier series and modulation. Although it is desirable to eliminate the spurious oscillations and overshoots due to the Gibbs phenomenon, its very presence is an accurate indicator of the location of a discontinuity. Thus, a finite Fourier series is general enough to estimate parameters with abrupt changes at unknown locations.

The main result of the Gottlieb method is included here for completeness. A finite Gegenbauer polynomial series for an analytic and non-periodic square integrable function $\alpha(x)$ is defined over $x \in [-1, 1]$ as:

$$\alpha(x) = \sum_{m=0}^{M} g_m^\lambda(x)\hat{g}_m^\lambda, \tag{36}$$

where $\hat{g}_m^\lambda$ is the Gegenbauer spectral coefficient and $g_m^\lambda(x)$ is the Gegenbauer polynomial of degree m with a parameter λ. A three-term recurrence formula is available to generate the family of Gegenbauer polynomials:

$$g_{m+1}^\lambda(x) = \frac{2(m+\lambda)x}{m+1} g_m^\lambda(x) - \frac{m+2\lambda-1}{m+1} g_{m-1}^\lambda(x), \tag{37}$$

with $g_0^\lambda(x) = 1$ and $g_1^\lambda(x) = 2\lambda x$.

If the first $2N+1$ complex Fourier spectral coefficients c_n of $\alpha(x)$ are known, then the Gegenbauer spectral coefficients $\hat{g}_m^\lambda$, are explicitly related to c_n by the following equation:

$$\hat{g}_m^\lambda = \delta_{0m} c_0 + \Gamma(\lambda) \mathrm{i}^m (m+\lambda) \sum_{0<|n|\leq N} J_{m+\lambda}(\pi n) \left(\frac{2}{\pi n}\right)^\lambda c_n, \quad (38)$$

where δ is the Kronecker delta and J is the Bessel function of the first kind.

The proposed method of parameter estimation contains three tuning parameters that affect both the accuracy of estimated parameter trajectories as well as numerical stability. They are the number of frequencies in the Fourier series N, the order of the Gegenbauer polynomial series m, and the Gegenbauer parameter λ. The choice of N is related to the computational cost of DFT, which may be chosen to keep the cost reasonable. Strategies for choosing m and λ regarding susceptibility to numerical round-off errors are available in literature (Gelb, 2004; Jackiewicz and Park, 2009).

6. SIMULATION EXAMPLE

Consider the three parameter Van der Pol oscillator with unknown parameter $a(t)$, $b=3$ and $c=1$:

$$\frac{d^2y}{dt^2} = a(t)\frac{dy}{dt}(1-by^2) - cy, \quad (39)$$

which is rearranged into the following affine form:

$$D^2y + y = \alpha(t)D(y-y^3). \quad (40)$$

Three types of changes in α are investigated:

Case 1: linear function

$$\alpha(t) = 1 + 0.05t. \quad (41)$$

Case 2: logistic function

$$\alpha(t) = 1 + \frac{2}{1+e^{-0.5(t-10)}}. \quad (42)$$

Case 3: double logistic function with an abrupt change

$$\alpha(t) = \begin{cases} 2 - \dfrac{1}{1+e^{-(t-5)}} & t \leq 10, \\ 2 - \dfrac{1}{1+e^{-(t-15)}} & t > 10. \end{cases} \quad (43)$$

The system is simulated for $T=20$ with $y(0)=1$ and $Dy(0)=0.4$ and sampled at $\Delta t = 0.01$. Zero mean Gaussian noise with $\sigma = 0.05$ is added to the sampled data. Fig. 1 shows the simulated noisy data for the three cases.

The coordinate transformation $x = 2t/T - 1$, is necessary such that $x \in [-1,1]$, which requires a shift in the Fourier basis functions to be centered on zero:

$$\mu_n = [\cos(n\omega t - n\pi),\ \sin(n\omega t - n\pi)]. \quad (44)$$

The linear change in the parameter is estimated as a fourth order ($N=4$) Fourier series, shown in Fig. 2. Using higher order series can result in smoother middle section but would not eliminate the undershoot and overshoot near the edges of the data window. Furthermore, the estimate will always converge to the middle of the jump discontinuity between the edges of the data window, in this case 1.5. Using the Gottlieb method, the parameter is recovered from the Fourier coefficients by a first order ($M=1$) Gegenbauer polynomial series with parameter $\lambda = 1$. This example demonstrates that it is not necessary to estimate a large number of Fourier coefficients to reconstruct the parameter. Larger N is feasible for more accuracy because the estimate reveals that the parameter change is an odd function with $a_{1,\ldots,N} = 0$, which reduces the problem size from $2N+1$ down to $N+1$.

In the second case, a ninth order ($N=9$) Fourier series is estimated as shown in Fig. 3. The trend of the logistic function is broadly captured by the series but suffers from oscillations, which may be damped out by the use of Lanczos sigma factors. However, the the error will remain at the edges. A seventh order ($M=7$) Gegenbauer series with parameter $\lambda = 3$ adequately reconstructs the parameter. It should be noted that the first two cases can be readily estimated as polynomials directly since the parameter is analytical inside the time interval.

The third case is a more challenging parameter estimation problem, it involves an abrupt change in the parameter value at $t=10$. Fig. 4 shows a Fourier series estimate with $N=12$.

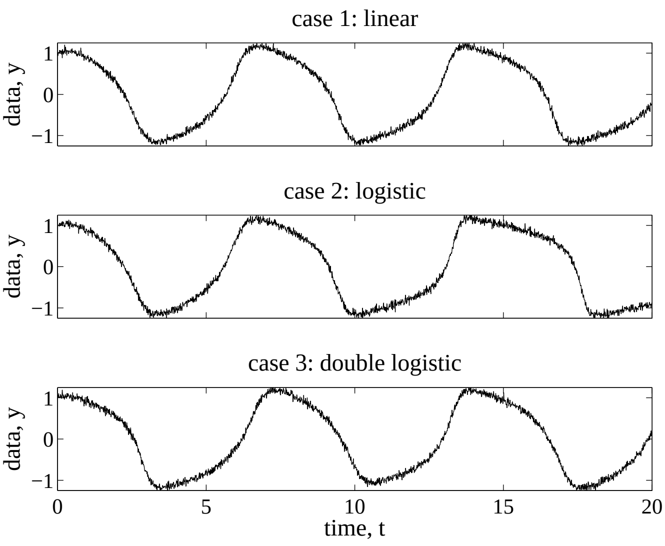

Fig. 1. Simulated data for three cases of parameter change.

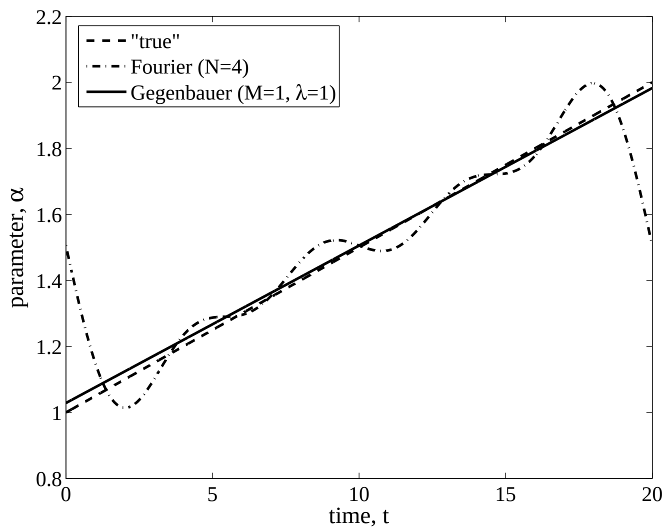

Fig. 2. Estimation of linear parameter change.

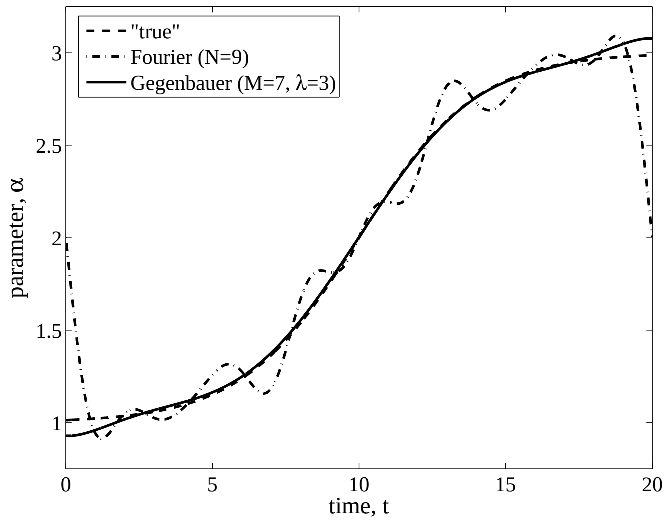

Fig. 3. Estimation of gradual parameter change.

The estimated series follows the analytical trends of both halves and indicates the existence of a jump discontinuity due to the presence of an undershoot followed by sharp rise and an overshoot. The midpoint between the peaks occurs at $t = 10$. Thus, the signature of Gibbs phenomenon can be effectively employed to find the location of abrupt changes. This feature alone gives the Fourier series a versatility not possible with the use of polynomial models. In the subsequent step, the time interval is split into two halves and a local Gegenbauer series ($M = 5$ and $\lambda = 5$) is reconstructed for each window.

Since the Gegenbauer polynomials are defined on $x \in [-1,1]$, focusing on a subinterval $[x_1,x_2] = [-1,0]$ and $[0,1]$ requires another coordinate transformation, $x = \varepsilon z + \rho$, where $\varepsilon = (x_2 - x_1)/2$ and $\rho = (x_2 + x_1)/2$, such that $z \in [-1,1]$. The explicit formula for Gegenbauer coefficients in Equation (38) is adapted for the local coordinate system as follows:

$$\hat{g}_m^\lambda = \delta_{0m} c_0 + \Gamma(\lambda) \mathrm{i}^m (m+\lambda) \times \sum_{0<|n|\leq N} J_{m+\lambda}(\pi n \varepsilon) \left(\frac{2}{\pi n \varepsilon}\right)^\lambda c_n e^{in\pi\rho}. \tag{45}$$

The above equation is derived from the construction proposed in Gottlieb and Shu (1996), and is limited to cases where the jump discontinuities at the edge and inside the window are the same.

7. CONCLUSIONS

The modulating functions method has been previously extended to time-varying parameter estimation in continuous-time nonlinear systems. The parameters are usually approximated as polynomials in the estimation time interval. The approach is limited to analytical parameter variations due to the limitations of polynomials as basis functions. This paper generalizes the approach to piecewise smooth changes by estimating the finite Fourier series approximation of parameter trajectories. It is shown that the presence of Gibbs phenomenon acts as an indicator of discontinuities. The spurious oscillations, overshoots and undershoots due to series truncation and Gibbs phenomenon are eliminated by recovering the analytical portions of the parameter trajectory as local finite Gegenbauer series.

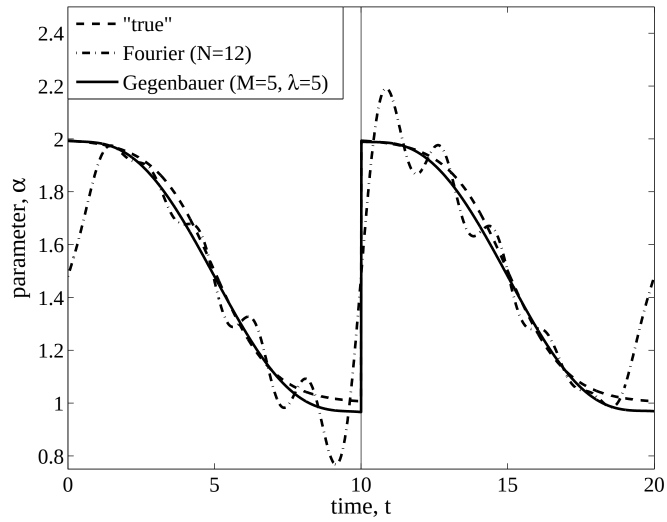

Fig. 4. Estimation of abrupt parameter change.

Several guidelines are available for the choice of the tuning parameters N in the Fourier series and M and λ in the Gegenbauer series, which is also the subject of ongoing research. Presented with the advantage of arbitrary initial conditions and direct use of noisy signals, the generalization is a useful tool for time-varying system identification.

REFERENCES

Acton, F.S. (1990). *Numerical Methods That Work*. Math. Assoc. Amer., Washington DC.

Besancon, G., Voda, A., and Jouffroy, G. (2010). A note on stet and parameter estimation in a van der Pol oscillator. *Automatica*, 46, 1735–1738.

Braiek, E.B. and Rotella, F. (1990). Identification of non-stationary continuous systems using modulating functions. *J. Franklin Inst.*, 327, 831–840.

Co, T.B. and Ungarala, S. (1997). Batch scheme recursive parameter estimation of continuous-time systems using the modulating functions method. *Automatica*, 33, 1185–1191.

Gelb, A. (2004). On the reduction of round-off errors for the Gegenbauer reconstruction method. *J. Sci. Comput.*, 20, 433–459.

Gottlieb, D. and Shu, C. (1996). On the Gibbs phenomenon III: recovering exponential accuracy in a sub-interval from a spectral partial sum of a piecewise analytic function. *SIAM J. Numer. Anal.*, 33, 280–290.

Gottlieb, D. and Shu, C. (1997). On the Gibbs phenomenon and its resolution. *SIAM Rev.*, 39, 644–668.

Gottlieb, D., Shu, C., Solomonoff, A., and Vandeven, H. (1992). On the Gibbs phenomenon I: recovering exponential accuracy from the Fourier partial sum of a nonperiodic analytic function. *J. Comp. Appl. Math*, 43, 81–98.

Jackiewicz, Z. and Park, R. (2009). A strategy for choosing Gegenbauer reconstruction parameters for numerical stability. *App. Math. Comput.*, 212, 418–434.

Kaplan, Z.Z., Gabay, I., Sarafian, G., and Sarafian, D. (2008). Biological applications of the "Filtered" Van der Pol oscillator. *J. Franklin Inst.*, 345, 226–232.

Karreman, G. and Prood, C. (1995). Heart muscle contraction oscillation. *Int. J. Bio-Med. Comp.*, 38, 49–53.

Patra, A. and Unbehauen, H. (1995). Identification of a class of nonlinear continuous-time systems using Hartley modulating functions. *Int. J. Control*, 62, 1431–1451.

Pearson, A.E. and Lee, F.C. (1985). On the identification of polynomial input-output differential equations. *IEEE Trans. Autom. Control*, AC-30, 778–782.

Pontieri, A., Lepreti, F., and Sorriso-Valvo and A. Vecchio and V. Carbone, L. (2003). A simple model for the solar cycle. *Solar Physics*, 213, 195–201.

Preisig, H.A. and Rippin, D.W.T. (1993a). Theory and applications of the modulating functions method–I. Review and theory of the method and theory of the spline–type modulating functions. *Comput. Chem. Eng.*, 17, 1–16.

Preisig, H.A. and Rippin, D.W.T. (1993b). Theory and applications of the modulating functions method–II. Algebraic representation of Maletinsky's spline–type modulating functions. *Comput. Chem. Eng.*, 17, 17–28.

Puchkov, V.F. and Chinayev, P.I. (1973). The method of sliding modulation functions for identifying plants with delay. *Sov. Autom. Control*, 6, 5–11.

Quaranta, G., Monti, G., and Marano, G.C. (2010). Parameters identification of Van der Pol-Duffing oscillators via particle swarm optimization and differential evolution. *Mech. Sys. Sig. Proc.*, 25, 2076–2095.

Robertson-Dunn, B. and Linkens, D.A. (1974). A mathematical model of the slow-wave electrical activity of the human small intestine. *Med. Biol. Eng.*, (Nov.), 750–758.

Samardzija, N., Greller, L.D., and Wasserman, E. (1989). Nonlinear chemical kinetic schemes derived from mechanical and electrical dynamical systems. *J. Chem. Phys.*, 90(4), 2296–2304.

Shinbrot, M. (1957). On the analysis of linear and nonlinear systems. *Trans. ASME*, 79, 547–552.

Unbehauen, H. (1996). Some new trends in identification and modeling of nonlinear dynamic systems. *Appl. Math. Comput.*, 78, 279–297.

Unbehauen, H. and Rao, G.P. (1990). Continuous-time approaches to system identification—a survey. *Automatica*, 26, 23–35.

Ungarala, S. and Co, T.B. (2000). Time-varying system identification using modulating functions and spline models with application to bio-processes. *Comput. Chem. Eng*, 24, 2739–2753.