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Multi-Rate Dissipative Control of Large-Scale Systems

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Abstract: An approach to distributed multi-rate control for large-scale systems, and in particular process networks, is presented. Where the local measurements, local control and controller communication are allowed to operate at different sampling rates. Dissipative systems theory is used to facilitate stability and performance analysis of the process network, based upon dynamic supply rates which have been lifted into a global sampling rate. Quadratic difference forms are used as supply rates and storage functions, which facilitate less conservative stability and performance conditions as compared to classical types of supply rates.

Keywords: Dissipative Systems Theory, Multi-Rate Control

1. INTRODUCTION

Control of chemical process networks is characterized by their scale, interactions and differing dynamics of each process unit Skogestad [2004]. The scale of the problem means that centralized control approaches, whilst potentially offering high performance, may be impractical or even infeasible. A logical alternative may be decentralized control. A difficulty of this approach, however, is that the interaction effects between process units are not explicitly captured, thus leading to potentially conservative results. These deficiencies motivate distributed control strategies, wherein local controllers communicate with one another to improve performance of the process network.

The focus of the current work is to address the issue of distributed multi-rate control of process networks in a discrete-time setting. The motivation for this development is that many process units may have different time constants, thus requiring multiple sampling rates to avoid over or under sampling. Additionally, it may be preferable to sample critical variables at a higher rate than non-critical ones to decrease capital costs. Sensor limitations may require such an approach, for example, concentration measurement by online chromatography may only be capable of sampling rates in the tens of minutes, whilst thermocouples are capable of sub-second sampling rates.

Dissipativity-based control design approaches include Moylan and Hill [1978], Xu and Bao [2011], the latter of which considers multi-rate control. These approaches are scalable, as the dissipativity properties of the process network may be determined as a linear combination of that of the individual processes and controllers. As dissipativity is fundamentally an input-output property of systems, this approach allows for interaction effects to accounted for. The proposed approach is based on dissipativity theory, and as such, shares these advantages with aforementioned approaches. It differs, however, in that a more general type of dissipativity is considered, that is, using quadratic difference forms (QdFs) as supply rates and storage functions, Thus providing sharper stability results, Tippett and Bao [2011].

Some notation used in the remainder of this paper is introduced. $\mathrm{diag}(A_1, \ldots, A_n)$ denotes the formation of a block diagonal matrix with A_i as its ith block diagonal entry. $\phi(\zeta, \eta) \in \mathbb{R}^{n\times m}(\zeta, \eta)$ denotes an $n \times m$ dimensional two variable polynomial matrix in the indeterminates ζ and η, with real coefficients. The degree of such a matrix, denoted by $\deg(\phi)$ is defined as the maximum power of ζ or η. The inertia of a matrix is (q_-, q_0, q_+), the number of negative, zero and positive eigenvalues. ρ denotes the forward shift operator.

2. MOTIVATING EXAMPLE

As way of motivation, we consider a linearized and discretized version of the heat exchanger network given in Rojas et al. [2009]. The heat exchanger network is shown diagrammatically in Figure 1, the objective of the network is to cool down a hot process stream, stream 1, using two other process streams, streams 2 and 3. To facilitate this, an additional utility stream is also used (the external input to HEX 2), the temperature of this utility stream is treated as a disturbance to the process network. The heat exchanger network is designed to maximize steady state efficiency, however, due to the recycle/feedback structure of the process network, the disturbance will be propagated to all units in the network, thus motivating a distributed, as opposed to decentralized, control approach.

The heat exchangers have differing designs with different volumes and heat transfer areas (the models of the equations are given in Section 5), as such, they have different time constants, this motivates the multi-rate control

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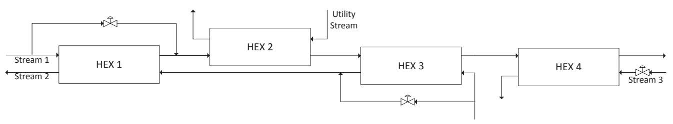

Fig. 1. Heat exchanger network under study

approach presented in this work. As the local controllers sampling rates may be tuned to that of their respective processes. HEX 1 and 3 have a sampling period of 1 sec, whilst HEX 2 has a sampling period of 3 sec and HEX 4 a rate of 2 sec. The following sections detail the proposed distributed control design methodology.

3. BACKGROUND MATERIAL

First introduced in Willems [1972], a discrete time dynamical system with input, output and state $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and $x \in \mathbb{R}^n$ respectively is dissipative if there exists a function defined on the input and output variables, the supply rate $s(u, y)$ and positive semi-definite function, the storage function $V(x(t))$ such that:

$$V(x(t+1)) - V(x(t)) \leq s(u(t), y(t)) \ \ \forall t \geq 0. \tag{1}$$

The following '(Q, S, R)' supply rate is commonly used:

$$s(u(t), y(t)) = y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t). \tag{2}$$

Quadratic *differential* forms were first introduced in Willems and Trentelman [1998] for continuous time systems, this framework was then adapted to the discrete time case in Kojima and Takaba [2005]. A quadratic *difference* form (QdF) may be written in terms of extended inputs and outputs. Defining $\hat{u}^T(t) = \left(u^T(t) \ u^T(t+1) \ \dots \ u^T(t+\tilde{n}) \right)$, and $\hat{y}^T(t) = \left(y^T(t) \ y^T(t+1) \ \dots \ y^T(t+\tilde{m}) \right)$, a QdF supply rate, denoted Q_ϕ, is defined as follows:

$$Q_\phi(y, u) = \begin{pmatrix} \hat{y}(t) \\ \hat{u}(t) \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \hat{y}(t) \\ \hat{u}(t) \end{pmatrix}. \tag{3}$$

Essentially, a QdF is a quadratic form similar to (2), extended to include future inputs and outputs. This allows for a more detailed description of the system. Defining $\hat{w}(t) = [\hat{y}(t)^T, \hat{u}(t)^T]^T$, a QdF in (3) can be written in a compact form as $Q_\phi(y, u) = \sum_{k=0}^{N} \sum_{l=0}^{N} \hat{w}^T(t+k)\phi_{kl}\hat{w}(t+l)$, where N is called the degree of supply rate, which is the maximum number of forward steps in the supply rate. In the supply rate given in (3), $N = \max\{\tilde{n}, \tilde{m}\}$. Such a QdF is induced by the symmetric two variable polynomial matrix $\phi(\zeta, \eta) = \sum_{k=0}^{N} \sum_{l=0}^{N} \phi_{kl}\zeta^k\eta^l$.

Here ϕ_{kl} is the (k, l)th coefficient matrix of $\phi(\zeta, \eta)$ and the indeterminates ζ and η represent a forward step in time on the left and right of $\phi(\zeta, \eta)$ respectively. The coefficient matrix of $\phi(\zeta, \eta)$, $\tilde{\phi}$, is a matrix with (k, l)th block ϕ_{kl}.

In this work, QdFs are used for both supply rates and storage functions. The dissipation inequality with supply rate Q_ϕ, and storage function Q_ψ, is as follows:

$$\sum_{t=0}^{\infty} Q_\phi(w(t)) \geq Q_\psi(w(t)). \tag{4}$$

If ∇ is defined as the rate of change of a QdF, $\nabla Q_\phi(w(t)) = Q_\phi(w(t+1)) - Q_\phi(w(t))$, then a useful property of QdFs is that the rate of change of a QdF is itself a QdF, *i.e.*, $\nabla Q_\phi = Q_{\nabla\phi}$. That is, the rate of change of the QdF induced by $\phi(\zeta, \eta)$ is itself a QdF induced by $\nabla\phi(\zeta, \eta)$, where $\nabla\phi(\zeta, \eta) = (\zeta\eta - 1)\phi(\zeta, \eta)$. Equation (4) may then be written as $Q_\phi(w(t)) \geq Q_{\nabla\psi}(w(t)), \ \forall t \geq 0$.

Some results underlying dissipativity and its links to stability in this framework are reproduced below.

Theorem 1. (Kojima and Takaba [2005]). A discrete time linear time-invariant (LTI) system is asymptotically stable if there exists a symmetric two variable polynomial matrix $\psi(\zeta, \eta) \geq 0$ and $\nabla\psi < 0$ for all input and output satisfying the system equations.

Proposition 1. (Tippett and Bao [2013b]). A discrete time LTI system with state space representation (A, B, C, D) is dissipative with the supply rate and (positive semidefinite) storage function pair induced by $\phi(\zeta, \eta)$ and $\psi(\zeta, \eta)$ respectively, with the corresponding coefficient matrices $\tilde{\phi}$ and $\tilde{\psi}$ partitioned as $\tilde{\phi} = \begin{pmatrix} \tilde{\phi}_Q & \tilde{\phi}_S \\ \tilde{\phi}_S{}^T & \tilde{\phi}_R \end{pmatrix}$ and $\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_X & \tilde{\psi}_Y \\ \tilde{\psi}_Y{}^T & \tilde{\psi}_Z \end{pmatrix}$, if and only if the following LMIs are satisfied:

$$\begin{pmatrix} \mathbb{T}_{11} & \mathbb{T}_{12} \\ \mathbb{T}_{12}^T & \mathbb{T}_{22} \end{pmatrix} \geq 0; \tag{5}$$

$$\tilde{\psi} \geq 0. \tag{6}$$

With

$$\hat{C} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^N \end{pmatrix} \tag{7}$$

$$\hat{D} = \begin{pmatrix} D & 0 & \dots & 0 & 0 \\ CB & D & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \dots & CB & D \end{pmatrix} \tag{8}$$

$$\begin{aligned} \mathbb{T}_{11} &= \hat{C}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{C} \\ \mathbb{T}_{12} &= \hat{C}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{D} + \hat{C}^T[\tilde{\phi}_S - \tilde{\nu}_Y] \\ \mathbb{T}_{22} &= \hat{D}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{D} + \hat{D}^T[\tilde{\phi}_S - \tilde{\nu}_Y] \\ &\quad + [\tilde{\phi}_S - \tilde{\nu}_Y]^T\hat{D} + [\tilde{\phi}_R - \tilde{\nu}_Z], \end{aligned} \tag{9}$$

where N is the degree of the supply rate and $\nu(\zeta, \eta) = \nabla\psi(\zeta, \eta)$.

4. MAIN RESULTS

The individual processes are shifted into a common sampling rate, in which the dissipativity properties of the process network are studied. Controllers are then designed in this sampling rate such that the closed-loop process network satisfies certain dissipativity properties, which in turn imply desired stability and performance bounds. In a separate step the controllers are realized individually.

Suppose that the n_p processes and n_c controllers have sampling periods given by $\left(\Delta t_1, \dots, \Delta t_{n_p+n_c}\right)$, without loss of generality the units of the sampling instants are such that $\min\left(\Delta t_1, \ldots, \Delta t_{n_p+n_c}\right) \geq 1$. Define a network sampling period as $\Delta T = \mathbf{lcm}\left(\Delta t_1, \ldots, \Delta t_{n_p+n_c}\right)$. In the ΔT sampling rate the i-th subsystem is:

$$x(k + m\Delta t_i) = A^{m\Delta t_i} x(k)+ \left(A^{m\Delta t_i -1}B \;\cdots\; AB \;\; B \right) \begin{pmatrix} u(k) \\ \vdots \\ u(k+m\Delta t_i-2) \\ u(k+m\Delta t_i-1) \end{pmatrix}, \tag{10}$$

$$\begin{pmatrix} y(k) \\ \vdots \\ y(k+m\Delta t_i-2) \\ y(k+m\Delta t_i-1) \end{pmatrix} = \begin{pmatrix} C \\ \vdots \\ CA^{m\Delta t_i-2} \\ CA^{m\Delta t_i-1} \end{pmatrix} x(k)+ \begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{m\Delta t_i-2}B & CA^{m\Delta t_i-1}B & \cdots & D \end{pmatrix} \begin{pmatrix} u(k) \\ \vdots \\ u(k+m\Delta t_i-2) \\ u(k+m\Delta t_i-1) \end{pmatrix}. \tag{11}$$

Where $\Delta T = m\Delta t_i$. Defining $\hat{y}(k) = \begin{pmatrix} y(k) \\ \vdots \\ y(k+m\Delta t_i-2) \\ y(k+m\Delta t_i-1) \end{pmatrix}$ and $\hat{u}(k) = \begin{pmatrix} u(k) \\ \vdots \\ u(k+m\Delta t_i-2) \\ u(k+m\Delta t_i-1) \end{pmatrix}$, the i-th system is then

$$x(k + \Delta T) = \hat{A}x(k) + \hat{B}\hat{u}(k) \tag{12}$$

$$\hat{y}(k) = \hat{C}x(k) + \hat{D}\hat{u}(k) \tag{13}$$

The dissipativity of the system in the ΔT sampling rate is found using Proposition 1 using the description $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$. The following result shows how a QdF may be converted from the Δt_i sampling rate to the ΔT sampling rate and visa versa.

Lemma 1. A QdF defined in the sampling rate Δt, induced by $\phi(\zeta, \eta)$, and one defined in the ΔT sampling rate, induced by $\Phi(\zeta, \eta)$, are related by:

$$\Phi_{ij} = \tilde{\phi}_{ij} = \begin{pmatrix} \phi_{(im)(jm)} & \cdots & \phi_{(im)(jm+m-1)} \\ \vdots & \ddots & \vdots \\ \phi_{(im+m-1)(jm)} & \cdots & \phi_{(im+m-1)(jm+m-1)} \end{pmatrix}, \tag{14}$$

where $m\Delta t = \Delta T$.

Proof. $Q_\Phi(\hat{u}, \hat{y})$ may be represented as:

$$Q_\phi(\hat{w}) = \sum_{k=0}^{N} \sum_{l=0}^{N} \rho^k_{\Delta T} \hat{w}^T(t) \Phi_{kl} \rho^l_{\Delta T} \hat{w}(t), \tag{15}$$

where $\rho_{\Delta T}$ is the forward shift operator in the ΔT sampling rate. Consider the QdF induced by Φ_{kl}, $Q_{\Phi_{kl}}(\hat{w}) = \rho^k_{\Delta T} \hat{w} \Phi_{kl} \rho^l_{\Delta T} \hat{w}$. By the definition of $\hat{w}$ and w, along with $\rho_{\Delta T} = \rho^m_{\Delta t}$ it is clear that this may be written as:

$$Q_{\Phi_{kl}}(w) = \sum_{i}^{m-1} \sum_{j}^{m-1} \rho^{km+i}_{\Delta t} w(t) \Phi_{kl} \rho^{lm+j}_{\Delta t} w(t), \tag{16}$$

$$Q_{\phi^{kl}}(w) = \sum_{i}^{m-1} \sum_{j}^{m-1} \sum_{k}^{N} \sum_{l}^{N} \rho^{km+i}_{\Delta t} w(t) \phi^{kl}_{ij} \rho^{lm+j}_{\Delta t} w(t), \tag{17}$$

where

$$\Phi_{kl} = \begin{pmatrix} \phi^{kl}_{00} & \cdots & \phi^{kl}_{0N} \\ \vdots & \ddots & \vdots \\ \phi^{kl}_{N0} & \cdots & \phi^{kl}_{NN} \end{pmatrix}. \tag{18}$$

Applying (17) and (18) $\forall k, l$, The QdF $Q_\phi(w)$ is given by:

$$Q_\phi(w) = \sum_{k}^{N} \sum_{l}^{N} Q_{\phi^{kl}}(w). \tag{19}$$

Upon expanding:

$$\Phi_{ij} = \tilde{\phi}_{ij} = \begin{pmatrix} \phi_{(im)(jm)} & \cdots & \phi_{(im)(jm+m-1)} \\ \vdots & \ddots & \vdots \\ \phi_{(im+m-1)(jm)} & \cdots & \phi_{(im+m-1)(jm+m-1)} \end{pmatrix} \quad \forall i, j. \tag{20}$$

Following Tippett and Bao [2013a], once all processes are transformed into the plant-wide sampling rate, the process network is decomposed into individual process units. Consider a network with n subsystems. Partitioning the input to each process unit into: the input from interconnected processes u_p, external disturbances d and manipulated input u_c. Similarly, the inputs and outputs of the controllers are partitioned into local u_L, y_L and remote u_R, y_R inputs and outputs, respectively. Define $\tilde{P}$ as the diagonal stacking of each process, with inputs and outputs $\mathbf{u_c} = (u_{c_1}, \ldots, u_{c_n})$, $\mathbf{u_p} = (u_{p_1}, \ldots, u_{p_n})$, $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, respectively. $\tilde{C}$ may be defined analogously, the closed-loop process network is shown in Figure 2. The matrices H_p and H_c capture the process and controller network topologies, respectively. F_p and F_I are matrices which select which signals are interconnecting flows and measured variables, respectively.

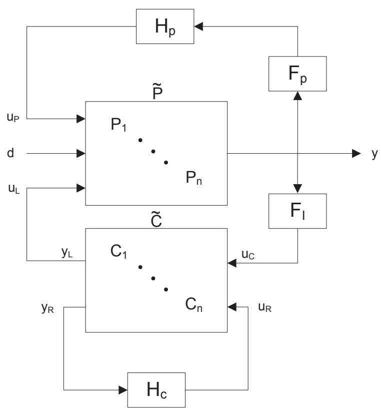

Fig. 2. Closed-loop process network

Suppressing the dependence on ζ and η, the i-th process has supply rate induced by:

$$\phi_i = \begin{pmatrix} Q_i & S_{c_i} & S_{p_i} & S_{d_i} \\ \star & R_{cc_i} & R_{cp_i} & R_{cd_i} \\ \star & \star & R_{pp_i} & R_{pd_i} \\ \star & \star & \star & R_{dd_i} \end{pmatrix}, \tag{21}$$

and the i-th controllers supply rate is induced by:

$$\phi_{c_i} = \begin{pmatrix} Q_{c_i} & S_{c_i} \\ \star & R_{c_i} \end{pmatrix}. \tag{22}$$

Where $Q_{c_i} = \begin{pmatrix} Q^{ll}_{c_i} & Q^{lr}_{c_i} \\ \star & Q^{rr}_{c_i} \end{pmatrix}$, and Sc_i, R_{c_i} are similarly partitioned. The supply rates of $\tilde{P}$ and $\tilde{C}$ are induced by Φ and Θ respectively, and are structured in a similar way to (21) and (22) with $\mathcal{Q} = \mathrm{diag}(Q_1, \ldots, Q_n)$ and other submatrices similarly defined. The storage functions of $\tilde{P}$ and $\tilde{C}$ are induced by Ψ and Σ, respectively, and are formed from that of the individual processes and controllers by diagonal stacking in an analogous manner. The supply rate of the closed-loop process network is given below.

Lemma 2. (Tippett and Bao [2013a]). Consider the interconnected system as shown in Figure 2. If the collection of process units $\tilde{\mathcal{P}}$ is dissipative with respect to supply rate Q_Φ and storage function Q_Ψ, and the collection of controllers $\tilde{\mathcal{C}}$ is dissipative with respect to the supply rate Q_Θ and storage function Q_Σ, then the process network from all disturbances $\mathbf{d}$ to all process output and controller output $\mathbf{y_{pw}} = \left[\mathbf{y}^T, \mathbf{y_L}^T, \mathbf{y_R}^T\right]^T$, is dissipative with storage function $Q_\nu = Q_{\Psi+\Sigma}$ and supply rate Q_μ and induced by

$$\mu(\zeta,\eta) = \begin{pmatrix} \Gamma_{11}(\zeta,\eta) & \Gamma_{12}(\zeta,\eta) \\ \Gamma_{12}^T(\zeta,\eta) & \Gamma_{22}(\zeta,\eta) \end{pmatrix}, \tag{23}$$

$$\Gamma_{11}(\zeta,\eta) = \begin{pmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} & \mathbb{X}_{13} \\ \star & \mathbb{X}_{22} & \mathbb{X}_{23} \\ \star & \star & \mathbb{X}_{33} \end{pmatrix},$$

$$\Gamma_{12}(\zeta,\eta) = \begin{pmatrix} \mathcal{S}_d + F_p^T H_p^T \mathcal{R}_{pd} \\ \mathcal{R}_{pd}^T \\ 0 \end{pmatrix},$$

$$\Gamma_{22}(\zeta,\eta) = \mathcal{R}_{dd},$$

with

$$\begin{aligned}
\mathbb{X}_{11} &= \mathcal{Q} + \mathcal{S}_p H_p F_p + F_p^T H_p^T \mathcal{S}_p^T + F_p^T H_p^T \mathcal{R}_{pp} H_p F_p, \\
&\quad + F_I^T \mathcal{R}_c^{ll} F_I, \\
\mathbb{X}_{12} &= \mathcal{S}_L + F_p^T H_p^T \mathcal{R}_{cp}^T + F_I^T \mathcal{S}_c^{ll^T}, \\
\mathbb{X}_{13} &= F_I^T \left(\mathcal{S}_c^{rl^T} + \mathcal{R}_c^{lr} H_c\right), \\
\mathbb{X}_{22} &= \mathcal{R}_{cc} + \mathcal{Q}_c^{ll}, \\
\mathbb{X}_{23} &= \mathcal{Q}_c^{lr} + \mathcal{S}_c^{lr} H_c, \\
\mathbb{X}_{33} &= \mathcal{Q}_c^{rr} + \mathcal{S}_c^{rr} H_c + H_c^T \mathcal{S}_c^{rr^T} + H_c^T \mathcal{R}_c^{rr} H_c.
\end{aligned}$$

Using this supply rate formulated in the ΔT sampling rate, the following gives conditions for plant-wide stability.

Theorem 2. Consider the process network with control as described above. Assume that the process network with external disturbances $\mathbf{d}$ and all process and controller outputs $\mathbf{y_{pw}} = \left[\mathbf{y}^T, \mathbf{y_L}^T, \mathbf{y_R}^T\right]^T$ is dissipative with respect to the supply rate $Q_\mu(w)$ induced by the polynomial matrix $\mu(\zeta,\eta) = \left(\begin{smallmatrix} \Gamma_{11}(\zeta,\eta) & \Gamma_{12}(\zeta,\eta) \\ \Gamma_{12}^T(\zeta,\eta) & \Gamma_{22}(\zeta,\eta) \end{smallmatrix}\right)$ (as per Lemma 2) and storage function $Q_\nu(w)$. Then the process network is asymptotically stable with finite $\mathcal{L}_2$ gain from the extended variable of the disturbance, $\hat{\mathbf{d}}$, to the extended variable of the process network output, $\hat{\mathbf{y}}_{\mathbf{pw}}$, i.e.

$$\|\hat{\mathbf{y}}_{\mathbf{pw}}\|_2 \leq \gamma \|\hat{\mathbf{d}}\|_2 \tag{24}$$

with $\gamma = \|\bar{\Gamma}_{11}^{-\frac{1}{2}}\|_2 \left(\|\bar{\Gamma}_{11}^{-\frac{1}{2}}\tilde{\Gamma}_{12}\| + \alpha\right)$ and α being a positive constant satisfying $\alpha^2 I \geq \tilde{\Gamma}_{22} + \tilde{\Gamma}_{12}^T \bar{\Gamma}_{11}^{-1} \tilde{\Gamma}_{12}$ and $\bar{\Gamma}_{11} = -\tilde{\Gamma}_{11}$ if the following conditions are satisfied:

(1) $\tilde{\Gamma}_{11}$, the coefficient matrix of $\Gamma_{11}(\zeta,\eta)$, is negative definite and
(2) $\nu(\zeta,\eta) \geq 0$.

Proof. The dissipativity of the process network implies that:

$$\sum_{t=0}^{k\Delta T} Q_\mu(\mathbf{d}(t), \mathbf{y_{pw}}(t)) \geq Q_\nu(\mathbf{d}(t), \mathbf{y_{pw}}(t)). \tag{25}$$

For vanishing disturbance

$$\sum_{t=0}^{k\Delta T} Q_\mu(0, \mathbf{y_{pw}}(t)) \geq Q_\nu(0, \mathbf{y_{pw}}(t)), \tag{26}$$

$$\sum_{t=0}^{k\Delta T} Q_{\Gamma_{11}}(\mathbf{y_{pw}}(t)) \geq Q_\nu(0, \mathbf{y_{pw}}(t)), \tag{27}$$

$$Q_{\Gamma_{11}}(\mathbf{y_{pw}}(t)) \geq Q_\nu(0, \mathbf{y_{pw}}(t)) - Q_\nu(0, \mathbf{y_{pw}}(t-1)),\ \forall t \geq 1, \tag{28}$$

$$Q_{\Gamma_{11}}(\mathbf{y_{pw}}(t)) \geq \nabla Q_\nu(0, \mathbf{y_{pw}}(t-1)). \tag{29}$$

Using $\tilde{\Gamma}_{11} < 0 \implies \Gamma_{11}(\zeta,\eta) < 0$, this implies Q_ν acts as a Lyapunov function for the process network in the ΔT sampling rate, thus, implying asymptotic stability of the outputs. asymptotic stability of the states is ensured by the assumption of zero state detectability. This alone does not imply Lyapunov stability of the i-th sub-system in its Δt_i sampling rate, as if $\Delta t_i < \Delta T$ the storage function of the i-th sub-system may increase within the longer ΔT time step. This is rectified as follows, from the dissipation inequality of the process network,

$$\sum_{t=0}^{k\Delta T} Q_\mu(\mathbf{d}(t), \mathbf{y_{pw}}(t)) \geq Q_\nu(\mathbf{d}(t), \mathbf{y_{pw}}(t)). \tag{30}$$

Suppressing time dependence, for vanishing disturbance

$$\sum_{t=0}^{k\Delta T} Q_\mu(0, \mathbf{y_{pw}}) \geq Q_\nu(0, \mathbf{y_{pw}}), \tag{31}$$

as $Q_\nu = Q_{\psi_1} + \cdots + Q_{\psi_n}$, and $Q_{\psi_j} \geq 0$ for all j.

$$\sum_{t=0}^{k\Delta T} Q_\mu(0, \mathbf{y_{pw}}) \geq Q_{\psi_i}(0, \mathbf{y_{pw}}), \tag{32}$$

$$\sum_{t=0}^{k\Delta T} Q_\mu(0, \mathbf{y_{pw}}) - \sum_{t=0}^{k\Delta T - \Delta t_i} Q_\mu(0, \mathbf{y_{pw}}) \geq \nabla Q_{\psi_i}(0, \mathbf{y_{pw}}). \tag{33}$$

Now, $\Gamma_{11} < 0$ implies that $Q_\mu(0, \mathbf{y_{pw}}(t)) < 0$, and therefore that $\sum_{t=0}^{N} Q_\mu(0, \mathbf{y_{pw}}(t))$ is non-increasing for all N. As such $0 > \nabla Q_{\psi_i}(0, \mathbf{y_{pw}}(t))$. Therefore, Q_{ψ_i} does not increase within the ΔT time step.

Performance bounds on the process network, in the form of weighted norm bounds, can be determined in a similar way to the above stability result.

Theorem 3. Consider a process network $\tilde{\mathcal{P}}$ with a supply rate of $\Psi \geq 0$ with control. If the process network from external disturbances, $\mathbf{d}$, to combined process and controller outputs, $\mathbf{y_{pw}} = \left(y^T\ y_L^T\right)^T$, is dissipative in the ΔT sampling rate with respect to supply rate Q_μ, and $\mu = \left(\begin{smallmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{smallmatrix}\right)$ with $\Gamma_{11} < 0$, then the minimum process network performance level

$$\|W\mathbf{y_{pw}}\|_2 \leq \|\mathbf{d}\|_2 \tag{34}$$

is guaranteed. Choosing

$$W(z) = \frac{1}{p(z)}\hat{\Gamma}_{11}^{\frac{1}{2}}(z),$$

with $\underline{\sigma}(W(j\omega)) \geq \frac{1}{\gamma}\ \forall\omega \in [0, 2\pi]$ and a scalar $p(\eta)$ such that its coefficient column vector, p, satisfies $p^T p \geq \max\left(\bar{\sigma}(\Gamma_{22} + \Gamma_{12}^T\hat{\Gamma}_{11}\Gamma_{12}), \bar{\sigma}(\Gamma_{12}^T\hat{\Gamma}_{11}\Gamma_{12})\right)$, we have:

$$\|\mathbf{y_{pw}}\|_2 \leq \gamma\|\mathbf{d}\|_2 \tag{35}$$

where $\bar{\sigma}$ and $\underline{\sigma}$ denote the maximum and minimum singular values respectively.

Proof. The proof follows the same lines as Theorem 3 in Tippett and Bao [2013b].

Suitable controller dissipativity properties may be formulated in the ΔT sampling rate to meet stability and performance criteria. To realize the controllers their dissipativity properties must be shifted back into the 'native' sampling rate, this may be achieved using Lemma 1.

To realize a controller which is dissipative with respect to the required supply rate, we perform a J-spectral factorization of the controller supply rate, and subsequently use the parameterization of J-dissipative systems presented in Pendharkar and Pillai [2009] to augment a seed system. A general method of performing discrete J-spectral factorization is presented in Stefanovski [2004] based on the solution of an associated discrete algebraic Riccati equation, a sufficient condition for the existence of such a J-spectral factorization is that $\partial\phi_c(e^{-j\omega}, e^{j\omega})$ has constant inertia $\forall\omega \in [0, 2\pi)$. The following result gives a LMI condition for $\partial\phi_c(e^{-j\omega}, e^{j\omega})$ to have constant inertia.

Lemma 3. Consider a QdF induced by $\phi(\zeta, \eta)$ which (perhaps after pre- and post-multiplication by a permutation matrix P) may be represented as:

$$P^T\phi(\zeta,\eta)P = \begin{pmatrix} X(\zeta,\eta) & 0 \\ 0 & Z(\zeta,\eta) \end{pmatrix}. \tag{36}$$

It has constant inertia if

$$\sum_{i=0}^{n} X_{ii} \geq 2k_1\sum_{i=1}^{n} X_{i,i-1} + 2k_2\sum_{i=2}^{n} X_{i,i-2} + \ldots + 2k_n X_{n0} \tag{37}$$

$$\sum_{i=0}^{n} Z_{ii} \leq 2k_1\sum_{i=1}^{n} Z_{i,i-1} + 2k_2\sum_{i=2}^{n} Z_{i,i-2} + \ldots + 2k_n Z_{n0}. \tag{38}$$

and $X_{ij} = X_{ji}$, $Z_{ij} = Z_{ji}$. Where k_i cycle through every value of ± 1 and $n = \deg\phi(\zeta,\eta)$.

Proof. Omitted due to space constraints.

The above results may be combined to design distributed controllers in two distinct steps: (1) Determination of required controller dissipativity properties. (2) Synthesis of suitably dissipative controllers. The special case of decentralized controllers are realized by setting $H_c = 0$, that is no communication between controllers.

Theorem 4. Consider a process network with external disturbances, $\mathbf{d}$, and outputs consisting of all process and controller outputs $\mathbf{y_{pw}} = [\mathbf{y}^T, \mathbf{y_L}^T, \mathbf{y_R}^T]^T$ in the ΔT sampling rate. The i-th process and i-th controller are dissipative with positive semidefinite storage with supply rates $\phi_i(\zeta,\eta) = \begin{pmatrix} Q_i(\zeta,\eta) & S_i(\zeta,\eta) \\ S_i^T(\zeta,\eta) & R_i(\zeta,\eta) \end{pmatrix}$ and $\phi_{ci}(\zeta,\eta) = \begin{pmatrix} Q_{ci}(\zeta,\eta) & S_{ci}(\zeta,\eta) \\ S_{ci}^T(\zeta,\eta) & R_{ci}(\zeta,\eta) \end{pmatrix}$ respectively for all i. There exists an internally stabilizing distributed multi-rate controller, which ensures the process network satisfies the norm bound $\|W\mathbf{y_{pw}}\|_2 \leq \gamma\|\mathbf{d}\|_2$ if the following conditions are satisfied:

$$\tilde{\Gamma}_{11} \leq -\tilde{N}^T\tilde{N} \tag{39}$$

$$\tilde{\Gamma}_{12} = 0 \tag{40}$$

$$\tilde{\Gamma}_{22} \geq \gamma^2\tilde{d}^T\tilde{d}I \tag{41}$$

Stability and performance bounds

$$\sum_{i=0}^{n} X_{ii_i} \geq 2k_1\sum_{i=1}^{n} X_{i,i-1_i} + 2k_2\sum_{i=2}^{n} X_{i,i-2_i} + \ldots + 2k_n X_{n0_i} \tag{42}$$

$$\sum_{i=0}^{n} Z_{ii_i} \leq 2k_1\sum_{i=1}^{n} Z_{i,i-1_i} + 2k_2\sum_{i=2}^{n} Z_{i,i-2_i} + \ldots + 2k_n Z_{n0_i} \tag{43}$$

Controller existence conditions.

Where $\phi_{ci}(\zeta,\eta) = \begin{pmatrix} X_i(\zeta,\eta) & 0 \\ 0 & Z_i(\zeta,\eta) \end{pmatrix} + D_i^T(\zeta)D_i(\eta)$ and $W(z) = \frac{1}{d(z)}N(z)$.

Proof. The assumption that each process and controller has positive semidefinite storage ensures that so to does the entire process network. Then conditions (39) to (41) imply that the process network is internally asymptotically stable, and satisfies the norm bound $\|W\mathbf{y_{pw}}\|_2 \leq \gamma\|\mathbf{d}\|_2$ by Theorem 3. The final of the above conditions ensure that $\begin{pmatrix} X_i(\zeta,\eta) & 0 \\ 0 & Z_i(\zeta,\eta) \end{pmatrix}$ admits a J-spectral factorization by Lemma 3. As $D_i^T(\zeta)D_i(\eta) \geq 0$, then, if a controller is synthesized to be dissipative with respect to supply rate induced by $\begin{pmatrix} X_i(\zeta,\eta) & 0 \\ 0 & Z_i(\zeta,\eta) \end{pmatrix}$, then it will also be dissipative with respect to the supply rate induced by $\phi_{ci}(\zeta,\eta)$.

Theorem 5. (Controller Synthesis). Consider a QdF supply rate induced by $\phi_{ci}(\zeta,\eta)$ which admits a J-spectral factorization as

$$\phi_{ci}(\zeta,\eta) = F_i^T(\zeta)J_iF_i(\eta). \tag{44}$$

If a 'seed' system, $\left(\begin{smallmatrix} v \\ z \end{smallmatrix}\right) = M(\xi)l$, is dissipative with supply rate induced by J. Then the system

$$\left(\begin{smallmatrix} y \\ u \end{smallmatrix}\right) = L_i(\xi)M(\xi)l \tag{45}$$

is dissipative with supply rate induced by $\phi_{ci}(\zeta,\eta)$. Where $L_i(\xi) = \mathbf{adj}(F_i(\xi))$.

Proof. The proof for the discrete time case follows analogously as for the continuous time case presented in Pendharkar and Pillai [2009].

In this way the problem of synthesizing a ϕ_{ci}-dissipative system is reduced to that of finding a J_i-dissipative 'seed' system, and then augmenting it. This reduced to finding a 'seed' system satisfying either a small gain or minimum phase condition.

5. ILLUSTRATIVE EXAMPLE

The discrete-time models of the heat exchangers are given below *in their native sampling periods.* Note that HEX 1

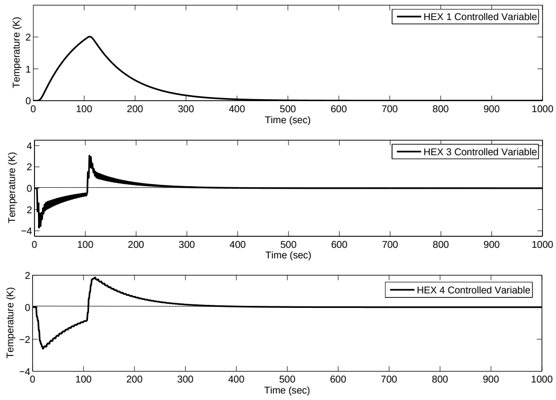

Fig. 3. Controlled Variables

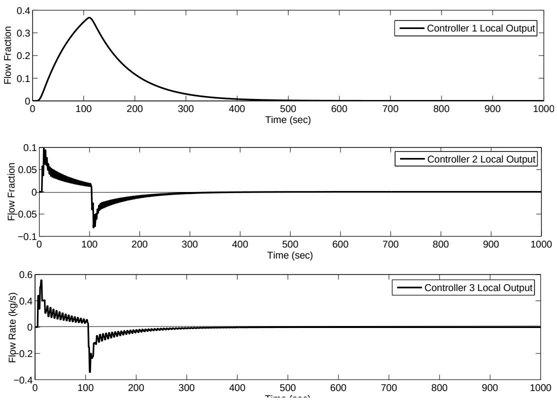

Fig. 4. Manipulated Variables

and 3 have relative degree zero due to the presence of the bypass, which is used as a manipulated variable.

$$\left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) = \left(\begin{array}{cc|cc} 0.8152 & 1.335e{-4} & 16.15 & 1.267e{-5} \\ 2.628e{-4} & 0.4096 & 2.812e{-3} & 0.1048 \\ \hline 5 & 0 & -437.5 & 0 \\ 0 & 5.63 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right) = \left(\begin{array}{cc|cc} 0.5521 & 1.93e{-4} & 0.07159 & 6.015e{-5} \\ 4.303e{-4} & 0.171 & 3.05e{-5} & 0.1958 \\ \hline 6.25 & 0 & \multicolumn{2}{c}{0_{2\times 2}} \\ 0 & 4.23 & & \end{array}\right)$$

$$\left(\begin{array}{c|c} A_3 & B_3 \\ \hline C_3 & D_3 \end{array}\right) = \left(\begin{array}{cc|cc} 0.8 & 7.774e{-5} & 12.49 & 7.535e{-6} \\ 4.915e{-4} & 0.3705 & 4.183e{-3} & 0.1006 \\ \hline 4.504 & 0 & -281.5 & 0 \\ 0 & 6.25 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|c} A_4 & B_4 \\ \hline C_4 & D_4 \end{array}\right) = \left(\begin{array}{cc|cc} 0.8009 & 2.77e{-4} & 12.49 & 7.535e{-6} \\ 4.634e{-4} & 0.476 & 4.183e{-3} & 0.1006 \\ \hline 6.25 & 0 & \multicolumn{2}{c}{0_{2\times 2}} \\ 0 & 6.34 & & \end{array}\right)$$

The controlled variables are the outlet temperatures of the hot sides (stream 1) of Heat exchangers 1, 3 and 4. The manipulated variables are the fractions of the cold streams passing through heat exchangers 1 and 3 and the flow rate of the utility stream (stream 3) on the cold side of heat exchanger 4. A distributed multi-rate controller was synthesized using the approach outlined above. The individual controllers are designed to have the same sampling rate as their local processes, the controller network is chosen to have the same topology as the process network. The simulation was carried out with a pulse external disturbance of magnitude 20K from $t = 0$ to 100 seconds. The controlled and manipulated variables are shown in Figures 3 and 4 respectively, *all variables are deviation variables.*

6. CONCLUSIONS

An approach to distributed multi-rate control design has been presented, which ensures global stability and worst case performance bounds. In contrast to existing dissipativity-based multi-rate control approaches (i.e. Xu and Bao [2011]), the proposed approach enjoys sharper results due to the use of QdFs as supply rates and storage functions and does not require the use of controllers with switching gain. Thus, leading to simpler analysis.

The control synthesis method that has been presented consists of two steps. In the first step the dissipativity requirements of each local controller are determined. This is facilitated by lifting the processes and (to be designed) controllers into a global sampling rate. This 'global planning' step is posed as an LMI optimization problem. In the second step, local controllers are synthesized independently by a discrete algebraic Riccati equation based approach to realize controllers which achieve the desired closed-loop process network properties.

REFERENCES

Kojima, C. and Takaba, K. (2005). A generalized lyapunov stability theorem for discrete-time systems based on quadratic difference forms. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, Seville, Spain, 2911-2916.*

Moylan, P.J. and Hill, D.J. (1978). Stability criteria for large-scale systems. *IEEE Transactions on Automatic Control*, 23, 143–149.

Pendharkar, I. and Pillai, H.K. (2009). A parametrisation for dissipative behaviours - the matrix case. *International Journal of Control*, 82(6), 2009.

Rojas, O.J., Setiawan, R., Bao, J., and Lee, P. (2009). Dynamic operability analysis of nonlinear process networks based on dissipativity. *American Institute of Chemical Engineers Journal*, 55(4), 963–982.

Skogestad, S. (2004). Control structure design for complete chemical plants. *Computers & Chemical Engineering*, 28, 219–234.

Stefanovski, J. (2004). Discrete J-spectral factorization of possibly singular polynomial matrices. *Systems & Control Letters*, 53, 127–140.

Tippett, M.J. and Bao, J. (2011). Dissipativity based analysis using dynamic supply rates. In *Proceedings of the 18th IFAC World Congress, Milano, Italy, 1319-1325.*

Tippett, M.J. and Bao, J. (2013a). Dissipativity based distributed control synthesis. *Journal of Process Control*, 23, 755–766.

Tippett, M.J. and Bao, J. (2013b). Plant-wide dissipative model predictive control. *AIChE Journal*, 59, 787–804.

Willems, J.C. (1972). Dissipative dynamical systems, part I: General theory. *Archive for Rational Mechanics and Analysis*, 45, 321–351.

Willems, J.C. and Trentelman, H.L. (1998). On quadratic differential forms. *SIAM Journal on Control and Optimisation*, 36(5), 1703–1749.

Xu, S. and Bao, J. (2011). Plantwide process control with asynchronous sampling and communications. *Journal of Process Control*, 21, 927–948.